§4. The free ω-category on a multitopic set

We construct $\boldsymbol{Y}=\langle \boldsymbol{R} \rangle$, the free ω-cat on the arbitrary ms $\boldsymbol{R}$.

First, let $\boldsymbol{R}^+$ be the ms with the following description: $\boldsymbol{R}$ is a sub-ms of $\boldsymbol{R}^+$; calling any cell in $\boldsymbol{R}^+ - \boldsymbol{R}$ an *indeterminate*, we have:

for any $\alpha \in S_n^+$ ($= \mathtt{P}_n(\boldsymbol{R}^+)$) and $a \in R_n^+$ ($= \mathtt{C}_n(\boldsymbol{R}^+)$) which are parallel ($\mathtt{d}\alpha = \mathtt{d}a$, $\mathtt{c}\alpha = \mathtt{c}a$), there is *exactly one* indeterminate $u \in R_{n+1}^+ - R_{n+1}$ such that $\mathtt{d}u = \alpha$, $\mathtt{c}u = a$.

It is obvious that such $\boldsymbol{R}^+$ exists; the construction of $\boldsymbol{R}^+$ is by successive adjunction of indeterminates.

Intuitively, what $\boldsymbol{R}^+$ allows us to do is speak about pd's that are only partially filled with elements of $\boldsymbol{R}$; each of the non-filled positions is understood to be filled with the unique indeterminate fitting into that position (which position itself may be defined by using indeterminates of lower dimensions).

As usual, we treat the canonical embedding $\mathtt{i}: R_n^+ \longrightarrow S_n^+$ as an inclusion; therefore, an n-cell v of $\boldsymbol{R}^+$ is automatically also an n-pd of $\boldsymbol{R}^+$.

Assume $n \geq 2$ and let $\gamma \in S_{n-1}^+$. We construct $\gamma^+ \in R_n^+ \subset S_n^+$ as follows. Let u be *the* (!) $(n-1)$-indeterminate with $u \| \gamma$, and $v \in R_n^+$ the n-indeterminate such that $\mathtt{d}v = \gamma$, $\mathtt{c}v = u$. We let $\gamma^+ \underline{\underline{\mathtt{def}}}\, v \in R_n^+$. Note that $\mathtt{d}\gamma^+ = \gamma$.

Although we need it only later, here we make the additional specification that for $\gamma \in R_0$, $\gamma^+ = 1_\gamma \in R_1$, the identity 1-pd.

We are going to use the notation and the constructions of §3, Part 3, in the multitopic set $\boldsymbol{R}^+$; we write Z_n^+ for $\mathtt{Z}_n(\boldsymbol{R}^+)$.

(**Definition of " n-cell" in the free ω-category Y on R**)

A pd α of R^+ is called *good* according to the following inductive definition. We write Y_n for the set of good n-pd's.

The *good* 1-dimensional and 2-dimensional pd's are those that are already in R (contain no indeterminates).

Let $n \geq 2$. A good n-dim pd α is obtained as follows.

Take any $\gamma \in Y_{n-1}$. We take $\gamma^+ \in S_n^+$ as above. Next, we let Φ be any n-polyarrow in S_n (no indeterminates in Φ) such that $|\Phi| \subset |\langle \gamma \rangle^-|$, and such that the inclusion $\iota : |\Phi| \to |\langle \gamma \rangle| = |\langle \mathrm{d}\gamma^+ \rangle|$ is a correct placing of Φ into γ^+.

The sought-after α is $\gamma^+ \circ \Phi$.

In other words, for $n \leq 1$, $Y_n = S_n$; and for $n \geq 2$, Y_n is the set of all $\alpha \in S_n^+$ such that, for some $v \in R_n^+$ and $\Phi \in \overrightarrow{S_n}$,

(1)(i) $\alpha = v \circ \Phi$;
(1)(ii) v is an indeterminate, $\mathrm{c}v$ is an indeterminate;
(1)(iii) $\mathrm{d}v \in Y_{n-1}$;
(1)(iv) Φ is a polyarrow in R.

Note that these conditions imply that

(1)(v) $|\Phi| \subset |\langle \mathrm{d}v \rangle^-|$.

Indeed, $|\Phi| \subset |\langle \mathrm{d}v \rangle|$ by the convention associated with the expression $v \circ \Phi$. Thus, when $n=2$, (1)(v) is automatic. When $n \geq 3$, $\mathrm{p}_{\mathrm{d}v} \notin |\Phi|$, since otherwise for $\beta = \Phi(\mathrm{p}_{\mathrm{d}v})$,

$c\beta=\langle dv\rangle(p_{dv})=Pdv\in S^+_{n-1}-S_{n-1}$ by (1)(ii) applied to $dv\in Y_{n-1}$ (see (1)(iii)) , thus $\beta\notin S_n$, contradiction.

Under (1)(i)-(iv) , we have $v=P\alpha$ and $\gamma=dv=dP\alpha$. We also write $\hat{c}\alpha$ for $dP\alpha$ (with good reason; see below). Further, $\Phi = \vec{\alpha}\restriction|\Phi|$. Note that we may always assume that $\Phi=\vec{\alpha}$, by changing the given Φ to Φ' for which $|\Phi'| = |\langle dP\alpha\rangle^-|$, $\Phi'\restriction|\Phi| = \Phi$, and $\Phi'(p) = 1_{\langle dP\alpha\rangle}$ for $p\in|\langle dP\alpha\rangle^-| - |\Phi'|$.

Reformulated, these conditions become: when $n\geq 2$, $\alpha\in S^+_n$ is good iff

(2)(i) α is positive and $\alpha = P\alpha\circ\vec{\alpha}$;

(2)(ii) $cP\alpha$ is an indeterminate (and as a consequence, $P\alpha$ is an indeterminate);

(2)(iii) $dP\alpha\in Y_{n-1}$;

(2)(iv) $\vec{\alpha}\in \overrightarrow{S_n}$.

Note that $Y_n\subset Z^+_n$.

By 3.(17)-(21), we easily see the following.

(3) Suppose $\alpha\in Y_n$, Φ is a polyarrow in $\boldsymbol{R}$, and $\alpha\circ_\kappa\Phi$ is well-defined. Then $\alpha\circ_\kappa\Phi\in Y_n$.

(4) If α is good, Φ is a polyarrow in $\boldsymbol{R}$, and $\alpha\square_\lambda\Phi$ is well-defined, then $\alpha\square_\lambda\Phi$ is good as well.

(5) For $\alpha\in Y_n$, $n>0$, we have $d\alpha\in Y_{n-1}$.

We define an ω-category $\boldsymbol{Y}$ (this will be $\langle \boldsymbol{R} \rangle$). The n-cells of $\boldsymbol{Y}$ are the good n-pd's of $\boldsymbol{R}^+$; this is why we wrote Y_n for $S_n^+(\mathtt{good})$.

Let us write $\hat{\mathtt{d}}$, $\hat{\mathtt{c}}$ for the operators $\mathtt{d}_{(\boldsymbol{Y})}$, $\mathtt{c}_{(\boldsymbol{Y})}$, resp.

For $\alpha \in Y_n$, $n>0$, $\hat{\mathtt{d}}\alpha \stackrel{\mathtt{def}}{=} \mathtt{d}\alpha$ (with " $\mathtt{d}$ " in the sense of the ms $\boldsymbol{R}$). When $\mathtt{n=1}$, $\hat{\mathtt{c}}\alpha \stackrel{\mathtt{def}}{=} \mathtt{c}\alpha$. For $\mathtt{n}\geq 2$, $\hat{\mathtt{c}}\alpha \stackrel{\mathtt{def}}{=} \mathtt{dP}\alpha$ (as we anticipated above). $\hat{\mathtt{d}}\alpha \in Y_{n-1}$ by (5); $\hat{\mathtt{c}}\alpha \in Y_{n-1}$ by (1)(iii).

We have (as required):

For $\alpha \in Y_{\geq 2}$, $\hat{\mathtt{d}}\hat{\mathtt{d}}\alpha = \hat{\mathtt{d}}\hat{\mathtt{c}}\alpha$, $\hat{\mathtt{c}}\hat{\mathtt{d}}\alpha = \hat{\mathtt{c}}\hat{\mathtt{c}}\alpha$:

Namely, $\alpha = \mathtt{P}\alpha \circ \vec{\alpha}$ gives $\mathtt{d}\alpha = \mathtt{dP}\alpha \square \mathtt{d}\vec{\alpha}$ from which $\mathtt{dd}\alpha = \mathtt{ddP}\alpha$; now, the first claimed equality is clear. For the second, when $n=2$,

$\hat{\mathtt{c}}\hat{\mathtt{d}}\alpha = \mathtt{c}(\mathtt{dP}\alpha \square \mathtt{d}\vec{\alpha}) = \mathtt{cdP}\alpha$; and

$\hat{\mathtt{c}}\hat{\mathtt{c}}\alpha = \hat{\mathtt{c}}\mathtt{dP}\alpha = \mathtt{cdP}\alpha = \mathtt{ccP}\alpha$,

which suffices. For $n \geq 3$,

$\hat{\mathtt{c}}\hat{\mathtt{d}}\alpha = \mathtt{dPd}\alpha$, and $\hat{\mathtt{c}}\hat{\mathtt{c}}\alpha = \hat{\mathtt{c}}\mathtt{dP}\alpha = \mathtt{dPdP}\alpha$;

thus, (9) settles it.

(**Definition of identity cells in $\boldsymbol{Y}$**)

We will write $\mathtt{id}_\alpha$ for the identity $(n+1)$-cell on the n-cell α in the ω-category $\boldsymbol{Y}$; and $\mathtt{1}_\alpha$ is the identity multiarrow on the object α in the multicategory $\boldsymbol{C}_{n+1}(\boldsymbol{R})$.

For $\alpha \in Y_0$, $\mathrm{id}_\alpha = 1_\alpha$.

For $\alpha \in Y_n$, $n \geq 1$, $\mathrm{id}_\alpha = \alpha^+$. Note that since $\alpha^+ \in R_{n+1}$, also $\alpha^+ \in S_{n+1}$. It is clear, by the definition in the form (1), that $\alpha^+ \in Y_{n+1}$.

It is clear that $\hat{d}(\mathrm{id}_\alpha) = \hat{c}(\mathrm{id}_\alpha) = \alpha$.

(**Definition of composition in *Y***)

By recursion on $n > 1$, we define, for each $0 \leq k < n$, the composition operation

$$\circ_k : Y_n \times_{Y_k} Y_n \longrightarrow Y_n$$
$$(\alpha, \beta) \longmapsto \alpha \circ_k \beta$$
$$\hat{c}(k)\beta = \hat{d}(k)\alpha$$

satisfying the laws:

(domain/codomain) $\hat{d}(\alpha \circ_{n-1} \beta) = \hat{d}\beta$, $\hat{c}(\alpha \circ_{n-1} \beta) = \hat{c}\beta$

provided: $\hat{c}\beta = \hat{d}\alpha$;

$\hat{d}(\alpha \circ_k \beta) = \hat{d}\alpha \circ_k \hat{d}\beta$, $\hat{c}(\alpha \circ_k \beta) = \hat{c}\alpha \circ_k \hat{c}\beta$

provided: $k \leq n-2$, $\hat{c}(k)\beta = \hat{d}(k)\alpha$;

(associativity) $(\alpha \circ_k \beta) \circ_k \gamma = \alpha \circ_k (\beta \circ_k \gamma)$

provided: $k < n$, $\hat{c}(k)\beta = \hat{d}(k)\alpha$, $\hat{c}(k)\gamma = \hat{d}(k)\beta$;

(middle exchange) $(\alpha \circ_k \beta) \circ_\ell (\gamma \circ_k \delta) = (\alpha \circ_\ell \gamma) \circ_k (\beta \circ_\ell \delta)$

provided: $\ell < k < n$, $\hat{c}(k)\beta = \hat{d}(k)\alpha$, $\hat{c}(k)\delta = \hat{d}(k)\gamma$,
$(\hat{c}(\ell)\gamma =)\hat{c}(\ell)\delta = \hat{d}(\ell)\alpha (= \hat{d}(\ell)\beta)$.

Simultaneously, we define, for the fixed n : for every $k<n$, and for each pair of α and β in Y_n , with $\hat{c}^{(k)}\beta=\hat{d}^{(k)}\alpha$, morphisms in $(R_n)^{\#}$:

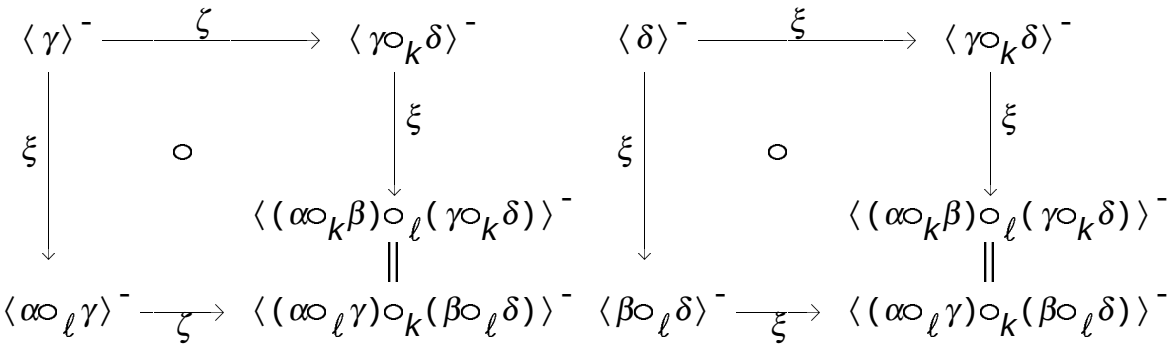
$$\langle\alpha\rangle^{-} \xrightarrow{\zeta} \langle\alpha\circ_k\beta\rangle^{-} \xleftarrow{\xi} \langle\beta\rangle^{-} \ ,$$

$\zeta=\zeta[\alpha, \beta, k]$, $\xi=\xi[\alpha, \beta, k]$, that form a coproduct in $(R_n)^{\#}$. Moreover, we also inductively establish the following coherence conditions for the (ζ, ξ)'s; the conditions correspond to associativity and middle exchange laws, and each is to hold under the conditions for the corresponding law:

$$\begin{array}{ccc}
\langle\alpha\rangle^{-} & \xrightarrow{\zeta} & \langle\alpha\circ_k(\beta\circ_k\gamma)\rangle^{-} \\
{\scriptstyle\zeta}\downarrow & \circ & \| \\
\langle\alpha\circ_k\beta\rangle^{-} & \xrightarrow[\zeta]{} & \langle(\alpha\circ_k\beta)\circ_k\gamma\rangle^{-}
\end{array}
\qquad
\begin{array}{ccc}
\langle\beta\rangle^{-} & \xrightarrow{\zeta} & \langle\beta\circ_k\gamma\rangle^{-} \\
{\scriptstyle\xi}\downarrow & \circ & \downarrow{\scriptstyle\xi} \\
 & & \langle\alpha\circ_k(\beta\circ_k\gamma)\rangle^{-} \\
 & & \| \\
\langle\alpha\circ_k\beta\rangle^{-} & \xrightarrow[\zeta]{} & \langle(\alpha\circ_k\beta)\circ_k\gamma\rangle^{-}
\end{array}$$

$$\begin{array}{ccc}
\langle\gamma\rangle^{-} & \xrightarrow{\xi} & \langle(\alpha\circ_k\beta)\circ_k\gamma\rangle^{-} \\
{\scriptstyle\xi}\downarrow & \circ & \| \\
\langle\beta\circ_k\gamma\rangle^{-} & \xrightarrow[\xi]{} & \langle\alpha\circ_k(\beta\circ_k\gamma)\rangle^{-}
\end{array}$$

$$\begin{array}{ccc}
\langle\alpha\rangle^{-} & \xrightarrow{\zeta} & \langle\alpha\circ_k\beta\rangle^{-} \\
{\scriptstyle\zeta}\downarrow & \circ & {\scriptstyle\zeta}\downarrow \\
 & & \langle(\alpha\circ_k\beta)\circ_\ell(\gamma\circ_k\delta)\rangle^{-} \\
 & & \| \\
\langle\alpha\circ_\ell\gamma\rangle^{-} & \xrightarrow[\zeta]{} & \langle(\alpha\circ_\ell\gamma)\circ_k(\beta\circ_\ell\delta)\rangle^{-}
\end{array}
\qquad
\begin{array}{ccc}
\langle\beta\rangle^{-} & \xrightarrow{\xi} & \langle\alpha\circ_k\beta\rangle^{-} \\
{\scriptstyle\zeta}\downarrow & \circ & {\scriptstyle\zeta}\downarrow \\
 & & \langle(\alpha\circ_k\beta)\circ_\ell(\gamma\circ_k\delta)\rangle^{-} \\
 & & \| \\
\langle\beta\circ_\ell\delta\rangle^{-} & \xrightarrow[\xi]{} & \langle(\alpha\circ_\ell\gamma)\circ_k(\beta\circ_\ell\delta)\rangle^{-}
\end{array}$$



(**The case** $k=n-1$)

There is one class of cases when the definition is explicit: $k=n-1$. That is, for $n\geq 1$, $\alpha,\ \beta\in Y_n$, $\hat{c}\beta=\hat{d}\alpha$, we define $\alpha\circ_{n-1}\beta$ and

$$\langle\alpha\rangle^- \xrightarrow{\zeta} \langle\alpha\circ_{n-1}\beta\rangle^- \xleftarrow{\xi} \langle\beta\rangle^- \ . \tag{6}$$

We put

$$\alpha\circ_{n-1}\beta \ \overset{\text{def}}{=}\ \alpha\circ\vec{\beta} \tag{7}$$

(here, $\circ=\circ^{(\boldsymbol{C}_n)}$; we are doing poly-composition in $\boldsymbol{C}_n=\mathbf{C}_n(\boldsymbol{R}^+)$). The fact that $\alpha\circ_{n-1}\beta\in Y_n$ is ensured by (4).

We derive (6) from the commutative diagram

$$\begin{array}{ccccc}
\langle\alpha\rangle^- & \xrightarrow{\zeta} & \langle\alpha\circ_{n-1}\beta\rangle^- & \xleftarrow{\xi} & \langle\beta\rangle^- \\
\| & \circ & \| & \circ & \rho\uparrow\cong \\
\langle\alpha\rangle^- & \xrightarrow{\sigma} & \langle\alpha\circ\vec{\beta}\rangle^- & \xleftarrow{\rho} & \langle\vec{\beta}\rangle
\end{array}$$

Here, the rightmost vertical is an instance of 3.(18); the lower horizontal is (essentially) the

canonical (σ, ρ)-pair associated with the HAA structure on $\boldsymbol{C}_n$.

(We note that, in this case, we can define $\alpha\circ_{n-1}\beta$ equivalently by $\alpha\circ_{n-1}\beta = \beta\square p_\beta\alpha$, without simultaneous composition, but using, instead, the $\square$-composition. However, in the later cases we cannot make a similar replacement.)

For the minimal value: $n=1$, $k=0$, as a special case, we have: $\alpha, \beta\in S_1$, $c\beta = d\alpha$. $\alpha\circ_0\beta = \alpha\circ_*\beta$, where $\{*\}= |s_{\boldsymbol{C}_1}(\alpha)|$. The ζ and the ξ are the same as σ and ρ ; see §1.

Here is an argument that, for $n\geq 2$, in (7), $\alpha\circ\vec{\beta}$ is well-defined: Since $\beta=P\beta\circ\vec{\beta}$, we have $t\circ\vec{\beta} = sP\beta$. But, $sP\beta = \langle dP\beta\rangle = \langle P\beta\rangle$, and $s\alpha=\langle d\alpha\rangle$. $\hat{c}\beta=\hat{d}\alpha$ means that $dP\beta=d\alpha$; thus, $t\circ\vec{\beta} = s\alpha$, and $\alpha\circ\vec{\beta}$ is well-defined.

(**The case** $k\leq n-2$)

Next, assume $k\leq n-2$.

Inductively, from the case $n-1$, $k\leq(n-1)-1$, we have $\hat{c}\alpha\circ_k\hat{c}\beta\in Y_{n-1}$, and the coproduct

$$\langle\hat{c}\alpha\rangle^- \xrightarrow{\ \zeta\ } \langle\hat{c}\alpha\circ_k\hat{c}\beta\rangle^- \xleftarrow{\ \xi\ } \langle\hat{c}\beta\rangle^- \qquad (8)$$

in $(R^+_{n-1})^\#$. Note that $|\vec{\alpha}| = |\langle\hat{c}\alpha\rangle^-|$, $|\vec{\beta}| = |\langle\hat{c}\beta\rangle^-|$; therefore, we have a unique tuple $\vec{\alpha}\wedge\vec{\beta}\in(S_n)^*$ with a coproduct

$$\vec{\alpha} \xrightarrow{\ \tilde{\zeta}\ } \vec{\alpha}\wedge\vec{\beta} \xleftarrow{\ \tilde{\xi}\ } \vec{\beta} \qquad (8')$$

in $(S_n)^\#$ such that, as maps of sets, $\tilde{\zeta}$ and $\tilde{\xi}$ coincide with ζ and ξ , respctively. For $t=t_{(\boldsymbol{C}_{n-1})}$, we have $t\circ\vec{\alpha}=\langle\hat{c}\alpha\rangle^-$, $t\circ\vec{\beta}=\langle\hat{c}\beta\rangle^-$; it follows that

$$t \circ (\vec{\alpha} \wedge \vec{\beta}) = \langle \hat{c}\alpha \circ_k \hat{c}\beta \rangle^- . \qquad (9)$$

As in 3.(2') ("trivial distributivity"), (8') induces

$$\langle \vec{\alpha} \rangle \xrightarrow{\bar{\zeta}} \langle \vec{\alpha} \wedge \vec{\beta} \rangle \xleftarrow{\bar{\xi}} \langle \vec{\beta} \rangle \qquad (10)$$

We define

$$\alpha \circ_k \beta \ \overline{\text{def}} \ (\hat{c}\alpha \circ_k \hat{c}\beta)^+ \circ \vec{\alpha} \wedge \vec{\beta} \ ; \qquad (10')$$

of course, (9) shows that $\alpha \circ_k \beta$ is well-defined. $\alpha \circ_k \beta \in Y_n$ by (3). In particular,

$$\overrightarrow{\alpha \circ_k \beta} = \vec{\alpha} \wedge \vec{\beta} \qquad (11)$$

(Note that whereas $\vec{\alpha} \wedge \vec{\beta}$ is indeed a kind of concatenation of $\vec{\alpha}$ and $\vec{\beta}$, it is not quite determined by $\vec{\alpha}$ and $\vec{\beta}$ alone; thus, $\vec{\alpha} \wedge \vec{\beta}$ is an incomplete notation.)

To define the (ζ, ξ)-pair, appearing in the upper row in the diagram that follows, we construct the commutative diagram:

$$\begin{array}{ccccc} \langle \alpha \rangle^- & \xrightarrow{\zeta} & \langle \alpha \circ_k \beta \rangle^- & \xleftarrow{\xi} & \langle \beta \rangle^- \\ \rho \uparrow \cong & \circ & \rho \uparrow \cong & \circ & \rho \uparrow \cong \\ \langle \vec{\alpha} \rangle & \xrightarrow[\bar{\zeta}]{} & \langle \vec{\alpha} \wedge \vec{\beta} \rangle & \xleftarrow[\bar{\xi}]{} & \langle \vec{\beta} \rangle \end{array} \qquad (12)$$

Here, the three vertical ρ's are instances of 3.(18); the lower horizontal row comes from (10); the upper row is uniquely determined by this.

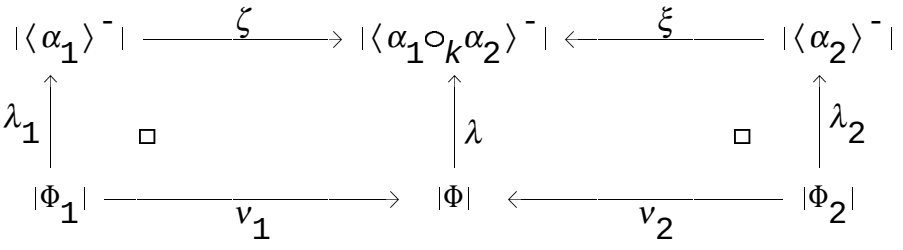
(**Preparing for the verification of the ω-category laws**)

(13) Lemma Suppose $\alpha_1, \alpha_2 \in Y_n$, $k<n$, $\alpha_1 \circ_k \alpha_2 \in Y_n$ is well-defined. Suppose

that $\Phi \in \overrightarrow{S_n}$, and that $(\alpha_1 \circ_k \alpha_2) \square_\lambda \Phi$ is well-defined. Then

$$(\alpha_1 \circ_k \alpha_2) \square_\lambda \Phi = (\alpha_1 \square_{\lambda_1} \Phi_1) \circ_k (\alpha_2 \square_{\lambda_2} \Phi_2)$$

for $\Phi_1, \Phi_2, \lambda_1, \lambda_2$ determined by any diagram of the form



(14)

Proof First, let us note that the right-hand side in the claimed equality is well-defined. The reason is that

$$\hat{d}(\alpha_1 \square_{\lambda_1} \Phi_1) = d(\alpha_1 \square_{\lambda_1} \Phi_1) = d\alpha_1 = \hat{d}\alpha_1$$

and

$$\hat{c}(\alpha_2 \square_{\lambda_2} \Phi_2) = dP(\alpha_2 \square_{\lambda_2} \Phi_2) = dP\alpha_2 = \hat{c}\alpha_2$$

by 1.(45.1); thus, $\hat{d}^{(k)}\alpha_1 = \hat{c}^{(k)}\alpha_2$ implies that

$$\hat{d}^{(k)}(\alpha_1 \square_{\lambda_1} \Phi_1) = \hat{c}^{(k)}(\alpha_2 \square_{\lambda_2} \Phi_2) \ .$$

To establish the desired equality, first let $k=n-1$. Then, by the distributive law,

$$(\alpha_1 \circ_{n-1} \alpha_2) \square_\lambda \Phi = (\alpha_1 \circ \vec{\alpha}_2) \square_\lambda \Phi = (\alpha_1 \square_\lambda \Phi_1) \circ (\vec{\alpha}_2 \square_{\rho^{-1}\lambda_2} \Phi_2) \ , \quad (15)$$

from the diagram