Part 4. Simultaneous composition in the multicategories associated with $\boldsymbol{M}=[\boldsymbol{X}]$.

As in §2, we have $\boldsymbol{X}$, $\boldsymbol{M}=[\boldsymbol{X}]$, etc. We use the notation of §2.

We develop a multi-variable version of the formula 2.(32).

Let $\alpha\in Q_n$. We use the notation Δ , Γ for $\mathsf{d}^{(n)}=\hat{\mathsf{d}}$, resp. $\mathsf{c}^{(n)}=\hat{\mathsf{c}}$ as in 2.(32).

We used the notation $\alpha\square_p u$ with u an $\boldsymbol{X}$-cell "fitting into" α at the place p . We can extend this to a "poly-notation". Let $U\colon |U| \longrightarrow X_{\geq n}$ be a finite tuple of $\geq n$-cells of $\boldsymbol{X}$, and let $\kappa\colon |U| \longrightarrow |\langle\alpha\rangle|$ be a 1-1 mapping such that, for all $q\in |U|$, $U(q)$ fits into α at $\kappa(q)\in |\langle\alpha\rangle|$. Then $\alpha\square_\kappa U$ is defined to be $\alpha\square_\kappa\Phi$, where $|\Phi| = |U|$, and $\Phi(q)=\mathsf{b}[Uq,\ \alpha,\ \kappa q]$. Using $[\![\Phi]\!]$ in the pointwise sense, we have, every time when $\alpha\square_\kappa\Phi$ is well-defined, that $\alpha\square_\kappa\Phi = \alpha\square_\kappa[\![\Phi]\!]$.

Let us fix n , $\alpha\in Q_n$. Let $m=\#|\langle\alpha\rangle|$. We write $[m)=\{i\in\mathbb{N}\colon 0\leq i<m\}$. We are going to consider total orders (!) on the set $|\langle\alpha\rangle|$, in the form of a bijection $\pi\colon [m) \xrightarrow{\cong} |\langle\alpha\rangle|$. Temporarily, fix such a π . Let us write $u_i \operatorname{d\bar{e}f} [\![\langle\alpha\rangle(\pi i)]\!]$. (When we need to show dependence on α and/or π , we write $u_i^{(\alpha)}$, or $u_i^{(\alpha,\ \pi)}$, and similarly for other notation.) Further, let us define, for $i\in[m)$, $j\in[m+1)$, the tuples U_i , V_j of cells of $\boldsymbol{X}$ by $|U_i| = |V_j| = |\langle\alpha\rangle|$, and

$$U_i(\pi k) = \begin{array}{ll} \Delta u_k & \text{for } k>i \\ u_i & \text{for } k=i \\ \Gamma u_\ell & \text{for } \ell<i \end{array} \qquad\qquad V_j(\pi k) = \begin{array}{ll} \Delta u_k & \text{for } k\geq j \\ \Gamma u_\ell & \text{for } \ell<j \end{array} .$$

We have that, for i and j as before, $\alpha_i = \alpha\square U_i$, $\beta_j = \alpha\square V_j$ are well-defined.

Let $A_i=[\![\alpha_i]\!]$, $A_j=[\![\beta_j]\!]$. We have $\Delta A_i=B_i$, $\Gamma A_i=B_{i+1}$, similarly to (by a repeated use of) 2.(22) and 2.(23).

Therefore, the $\circ_n$-composite of the following chain of $X_{\geq n}$-cells:

$$B_m \xleftarrow{\quad A_{m-1}\quad} B_{m-1} \xleftarrow{\quad A_{m-2}\quad} \ldots B_1 \xleftarrow{\quad A_0\quad} B_0$$

is meaningful; let us denote it by A_π.

We **claim** that $A_\pi = [\![\alpha]\!]$, independently of π.

To prove this, we first show that the value A_π is independent of π. It suffices to show that $A_\pi = A_{\pi'}$ whenever π' is obtained from π by a transposition of adjacent items: there is $i<m\text{-}1$ such that $\pi'(k)=\pi(k)$ for $k\in[m)\text{-}\{i, i+1\}$, and $\pi'(i)=\pi(i+1)$, $\pi'(i+1)=\pi(i)$.

The reason is the equality 2.(32), more precisely, the equality of the first two terms in 2.(32). The equality of the two partial composites:

$$B_i \xrightarrow{A_i} B_{i+1} \xrightarrow{A_{i+1}} B_{i+2} \qquad B'_i \xrightarrow{A'_i} B'_{i+1} \xrightarrow{A'_{i+1}} B'_{i+2} ,$$

with $B_i=B_i{}^{(\pi)}$, $B'_i=B_i{}^{(\pi')}$, etc, is an instance *loc. cit.* as inspection will show. The rest of the chains defining A_π and $A_{\pi'}$ are identical; thus, $A_\pi=A_{\pi'}$.

Finally, we prove, by induction on $\#|\langle\alpha\rangle|$, that $A_\pi^{(\alpha)}$ is equal to $[\![\alpha]\!]$, for at least one π.

When $\#|\langle\alpha\rangle|=0$ or 1, there is nothing to prove. Assume that $\#|\langle\alpha\rangle|>1$.

We can write $\alpha=\hat{\alpha}\circ_h\check{\alpha}$, with $\#|\langle\check{\alpha}\rangle|=1$, $\#|\langle\hat{\alpha}\rangle| = \#|\langle\alpha\rangle|-1$. (The choice of $\check{\alpha}$ with $\#|\langle\check{\alpha}\rangle|=1$, rather than just $\#|\langle\check{\alpha}\rangle|<\#|\langle\alpha\rangle|$, makes the argument slightly simpler.) We have the coproduct

$$\langle\hat{\alpha}\rangle \xrightarrow{\sigma} \langle\alpha\rangle \xleftarrow{\rho} \langle\check{\alpha}\rangle .$$

Let $m=\#|\langle\alpha\rangle|$. Let us write $|\langle\check{\alpha}\rangle|=\{q_0\}$. Take any $\hat{\pi}:[m\text{-}1)\xrightarrow{\cong}|\langle\delta\rangle|$, and define $\pi:[m)\xrightarrow{\cong}|\langle\alpha\rangle|$ so that $\pi(0)=\rho(q_0)$, and $\sigma\circ\hat{\pi}=\pi\restriction([m)\text{-}\{0\})$.

Referring to the notation above, let us refer to the items $\alpha_i^{(\alpha)}$, $\beta_j^{(\alpha)}$, $A_i^{(\alpha)}$, *etc.* as α_i, β_j, A_i ($0 \le i \le m-1$, $0 \le j \le m$), *etc.*; to $\alpha_i^{(\hat{\alpha})}$, $\beta_j^{(\hat{\alpha})}$, $A_i^{(\hat{\alpha})}$, *etc.* as $\hat{\alpha}_i$, $\hat{\beta}_j$, $\hat{A}_i$ ($0 \le i \le m-2$, $0 \le j \le m-1$), *etc.*; and to $\alpha_0^{(\tilde{\alpha})}$, $\beta_0^{(\tilde{\alpha})}$, $\beta_1^{(\tilde{\alpha})}$, $A_0^{(\tilde{\alpha})}$, *etc.* as $\tilde{\alpha}_0$, $\tilde{\beta}_0$, $\tilde{\beta}_1$, $\tilde{A}_0$, *etc.* Note that

$$\tilde{\alpha}_0 = \tilde{\alpha}, \quad \tilde{B}_0 = \mathrm{d}\tilde{A}_0, \quad \tilde{B}_1 = \mathrm{d}\tilde{A}_0 .$$

From $\alpha = \hat{\alpha} \circ_h \tilde{\alpha}$, we obtain that

$$\beta_0 = \hat{\beta}_0 \circ_h \tilde{\beta}_0, \qquad \alpha_0 = \hat{\beta}_0 \circ_h \tilde{\alpha}_0, \qquad \beta_1 = \hat{\beta}_0 \circ_h \tilde{\beta}_1,$$
$$\beta_j = \hat{\beta}_{j-1} \circ_h \tilde{\beta}_1 \ (1 \le j \le m), \qquad \alpha_i = \hat{\alpha}_{i-1} \circ_h \tilde{\beta}_1 \ (1 \le i \le m-1).$$

For all $i \in [m-1)$, $j \in [m)$, $\mathrm{d}\hat{\alpha}_i = \mathrm{d}\hat{\beta}_j = \mathrm{d}\hat{\alpha}$ $\bar{\mathrm{def}}$ γ. Thus, with

$$x_0 \ \bar{\mathrm{def}} \ [\![\gamma \square_h \tilde{A}_0]\!] , \qquad y_k \ \bar{\mathrm{def}} \ [\![\gamma \square_h \hat{B}_k]\!] \ (k \in \{0, 1\}),$$

we have

$$A_0 = [\![\alpha_0]\!] = [\![\hat{\beta}_0 \circ_h \tilde{\alpha}_0]\!] = \hat{B}_0 \circ_{n-1} x_0$$

$$A_i = [\![\alpha_i]\!] = [\![\hat{\alpha}_{i-1} \circ_h \tilde{\beta}_1]\!] = \hat{A}_{i-1} \circ_{n-1} y_1 \ (1 \le i \le m-1) .$$

The composite $A_{1|(m-1)}$ $\bar{\mathrm{def}}$ $A_{m-1} \circ_n A_{m-2} \circ_n \cdots \circ_n A_1$ thus equals

$$A_{1|(m-1)} = (\hat{A}_{m-2} \circ_n \hat{A}_{m-3} \circ_n \cdots \circ_n \hat{A}_0) \circ_{n-1} y_1 = [\![\hat{\alpha}]\!] \circ_{n-1} y_1$$

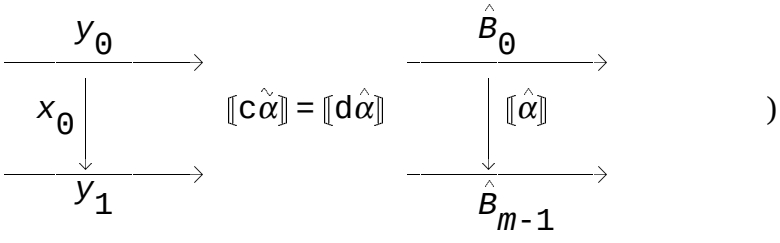
by the induction hypothesis applied to $\hat{\alpha}$. Hence,

$$A_\pi = A_{1|(m-1)} \circ_n A_0 = ([\![\hat{\alpha}]\!] \circ_{n-1} y_1) \circ_n (\hat{B}_0 \circ_{n-1} x_0) ,$$

which equals

$$A_\pi \overset{(*)}{=} [\![\hat{\alpha}]\!] \circ_{n-1} x_0 = [\![\hat{\alpha}]\!] \circ_{n-1} [\![\mathrm{d}\hat{\alpha} \square_h \mathrm{B}\tilde{\alpha}]\!] = [\![\hat{\alpha} \circ_h \tilde{\alpha}]\!] = [\![\alpha]\!]$$

as desired (for the equality (*), consider the following picture:



)

We draw the following concise conclusion from the above:

(36) Lemma Suppose $\gamma \in P_n$, $\Psi \in \overrightarrow{Q}_n$, and that $\gamma \square_\lambda \Psi$ is well-defined. Assume that $\Psi = \Psi_1 \dot{\cup} \Psi_2$. Then

$$[\![\gamma \square_\lambda \Psi]\!] = [\![\gamma \square_\lambda (\Gamma\Psi_1 \dot{\cup} \Psi_2)]\!] \circ_n [\![\gamma \square_\lambda (\Psi_1 \dot{\cup} \Delta\Psi_2)]\!]$$

Indeed, this results from the above as follows. We let $\alpha = \gamma \square_\lambda [\![\Psi]\!]$, $\hat{\alpha} = \gamma \square_\lambda [\![\Gamma\Psi_1 \dot{\cup} \Psi_2]\!]$, $\tilde{\alpha} = \gamma \square_\lambda [\![\Psi_1 \dot{\cup} \Delta\Psi_2]\!]$. Of course, $[\![\alpha]\!] = [\![\gamma \square_\lambda \Psi]\!]$, *etc.* With $m = \#|\langle \alpha \rangle|$, we define $\pi : [m) \xrightarrow{\cong} |\langle \alpha \rangle|$ so that $\pi^{-1}(\mathrm{Im}(\lambda \upharpoonright |\Psi_1|))$ is an initial segment $[\ell)$ of $[m)$.

Let $A_i = A_i^{(\alpha, \pi)}$, $\tilde{A}_i = A_i^{(\tilde{\alpha}, \pi)}$, $\hat{A}_i = A_i^{(\hat{\alpha}, \pi)}$, and similarly for B's.

Observe that, since $\gamma \in P_n$, we have that for $i \in |\langle \gamma \rangle| - \mathrm{Im}(\lambda)$, $\Delta(\pi i) = \Gamma(\pi i) = \pi i$; in particular, for all such i, we have $B_i = B_{i+1}$ and $A_i = 1_{B_i}$.

Now it is fairly clear that $\tilde{A}_i = A_i$ for $i < \ell$, and $\tilde{A}_i = 1_{B_\ell}$ for $\ell \leq i < m$; and, on the other

hand, $\hat{A}_i = 1_{B_0}$ for $i<\ell$, and $\hat{A}_i = A_i$ for $\ell \le i < m$. Therefore,

$$[\![\alpha]\!] = A_{m-1} \circ_n \cdots \circ_n A_\ell \circ_n A_{\ell-1} \circ_n \cdots \circ_n A_0 = [\![\hat{\alpha}]\!] \circ_n [\![\tilde{\alpha}]\!]$$

as desired.

Recall the definition of $Bb \in N_{n-1}$ for $b \in N_n$ from 2.[5]. Let us extend the effect of B, first to $B\beta \underset{def}{=} B(\|\beta\|) \in N_{n-1}$ for $\beta \in Q_n$, and then to $B\Phi \in \overrightarrow{N_{n-1}}$ for $\Phi \in \overrightarrow{Q_n}$ pointwise: $|B\Phi| = |\Phi|$, $B\Phi(p) = B(\Phi p)$.

Finally, we establish the following:

(37) For $a \in N_n$, $\Lambda \in \overrightarrow{N_n}$, assuming that $a \Diamond_\mu \Lambda$ is well-defined, we have

$$[\![a \Diamond_\mu \Lambda]\!] = [\![a]\!] \circ_{n-1} [\![da \Box_\mu B\Lambda]\!] \quad .$$

Of course, when $\#|\Lambda| = 1$, this is 2.[8], the definition of $\Diamond$. For $\#|\Lambda| = 0$, the equality is trivially true. Arguing inductively, and using the formalism of the GAL (Part 1 of this section), we let $\Lambda = \Psi \dot\cup \Sigma$, with $\#|\Psi| < \#|\Lambda|$, $\#|\Sigma| < \#|\Lambda|$. We have, with the equalities marked 1 and 2 being true by the induction hypothesis, that

$$\begin{aligned}
[\![a \Diamond_\mu \Lambda]\!] &= [\![(a \Diamond_\kappa \Psi) \Diamond_{\bar\lambda} \Sigma]\!] \overset{1}{=} [\![a \Diamond_\kappa \Psi]\!] \circ_{n-1} [\![d(a \Diamond_\kappa \Psi) \Box_{\bar\lambda} B\Sigma)]\!] \\
&\overset{2}{=} ([\![a]\!] \circ_{n-1} [\![da \Box_\kappa B\Psi]\!]) \circ_{n-1} [\![(da \Box_\kappa d\Psi) \Box_{\bar\lambda} B\Sigma)]\!] = \\
&= [\![a]\!] \circ_{n-1} ([\![da \Box_\kappa B\Psi]\!] \circ_{n-1} [\![da \Box_\mu (d\Psi \dot\cup B\Sigma)]\!]) \quad .
\end{aligned}$$

It remains to show that

$$[\![da \Box_\mu B\Lambda]\!] \overset{?}{=} [\![da \Box_\kappa B\Psi]\!] \circ_{n-1} [\![da \Box_\mu (d\Psi \dot\cup B\Sigma)]\!] \quad .$$

This is an instance of the Lemma above, with $n-1$ in place of n, $\gamma = da$, $\Psi = B\Lambda$,

$\Psi_1 = \mathtt{B}\Sigma$, $\Psi_2 = \mathtt{d}\Psi$. It is important to observe that, because $a \Diamond_{\mu(q)} \Psi_1(q)$ is well-defined for each $q \in |\Psi_1|$, we have $\mathtt{d}a \Box_{\mu(q)} ((\Gamma\Psi_1)(q)) = \mathtt{d}a$, thus $[\![\mathtt{d}a \Box_\mu (\Gamma\mathtt{B}\Sigma \dot\cup \mathtt{d}\Psi)]\!] = [\![\mathtt{d}a \Box_\kappa \mathtt{B}\Psi]\!]$.

Let us rewrite the identity of (37) in a form that eliminates the use of the symbol $\Diamond$.

Recall the stict, identity-on-objects multicategory morphism $\|\text{-}\| : \boldsymbol{F}_n \to \boldsymbol{E}_n$, which gives rise to the definition of $[\![\gamma]\!]$, for $\gamma \in Q_n$, as $[\![\gamma]\!] = [\![\|\gamma\|]\!]$, using the map $[\![\text{-}]\!] : N_n \to X_{\geq n}$. When $\|\text{-}\|$ is applied to a polyarrow of $\boldsymbol{F}_n$, the meaning is, by definition, pointwise.

When $\mathtt{B}$ is applied to a polyarrow, it is meant to act pointwise.

(38) For $\alpha \in Q_n$, $\Lambda \in \overrightarrow{Q_n}$, assuming that $\alpha \circ_\mu \Lambda$ is well-defined, we have

$$[\![\alpha \circ_\mu \Lambda]\!] = [\![\alpha]\!] \circ_{n-1} [\![\mathtt{d}\alpha \Box_\mu \mathtt{B}\|\Lambda\|]\!] \quad .$$

Since $\|\text{-}\|$ is a multicategory morphism, we have $\|\alpha \circ_\mu \Lambda\| = \|\alpha\| \Diamond_\mu \|\Lambda\|$. Thus,

$$[\![\|\alpha \circ_\mu \Lambda\|]\!] = [\![\|\alpha\| \Diamond_\mu \|\Lambda\|]\!] = [\![\|\alpha\|]\!] \circ_{n-1} [\![\mathtt{d}\|\alpha\| \Box_\mu \mathtt{B}\|\Lambda\|]\!]_{(n-1)}$$

by (37). (Here, for added clarity, we distinguished the two maps $[\![\text{-}]\!] : Q_n \to X_{\geq n}$, $[\![\text{-}]\!] : Q_{n-1} \to X_{\geq n-1}$, by marking the second with $(n\text{-}1)$.) This shows (38).