Part 3: Simultaneous composition for pasting diagrams in a multitopic set

Now, let's make the context even more special. Let $\boldsymbol{R}$ be an arbitrary ms; we will use the notation $R_n \,\bar{\bar{\mathrm{def}}}\, \mathrm{C}_n(\boldsymbol{R})$, $S_n \,\bar{\bar{\mathrm{def}}}\, \mathrm{P}_n(\boldsymbol{R})$.

(The reason we change the notation from $\boldsymbol{M}$ to $\boldsymbol{R}$ is that later we will also have an $\boldsymbol{M}$, in fact, one that is related to some $\boldsymbol{X}$ as in section 2.)

Let $n\geq 1$, $\boldsymbol{C}=\mathbf{C}_n(\boldsymbol{R})$, $\boldsymbol{D}=\mathbf{D}_{n-1}(\boldsymbol{R})$. Recall the strict morphism $\mathrm{d}:\boldsymbol{C}\to\boldsymbol{D}$ which is the identity on the upper-level objects (which are the elements of R_n). In other words, we have that, with $\mathrm{s}=\mathrm{s}_{\boldsymbol{C}}$, $\langle\text{-}\rangle=\mathrm{s}_{\boldsymbol{D}}$, the items in

$$\mathrm{s}(\alpha)\backslash i \xrightarrow{\psi} \mathrm{s}(\alpha\circ_i\beta) \xleftarrow{\varphi} \mathrm{s}(\beta) \qquad \varphi=\varphi[\alpha,\beta,i]_{\boldsymbol{C}},\ \psi=\psi[\alpha,\beta,i]_{\boldsymbol{C}},$$

are identical to the ones in

$$\langle \mathrm{d}\alpha\rangle\backslash i \xrightarrow{\psi} \langle \mathrm{d}\alpha\square_i \mathrm{d}\beta\rangle \xleftarrow{\varphi} \langle \mathrm{d}\beta\rangle \qquad \varphi=\varphi[\mathrm{d}\alpha,\mathrm{d}\beta,i]_{\boldsymbol{D}},\ \psi=\psi[\mathrm{d}\alpha,\mathrm{d}\beta,i]_{\boldsymbol{D}}.$$

This fact has an immediate generalization to poly-composition. For a polyarrow Φ in $\boldsymbol{C}$, we define the polyarrow $\mathrm{d}\Phi$ in $\boldsymbol{D}$ pointwise: $|\mathrm{d}\Phi| = |\Phi|$, $(\mathrm{d}\Phi)(i)=\mathrm{d}(\Phi i)$ ($i\in|\Phi|$). Then one immediately verifies that the equality $\mathrm{s}\alpha=\langle\mathrm{d}\alpha\rangle$ generalizes to $\mathrm{s}\Phi = \langle\mathrm{d}\Phi\rangle$. Supposing that the polycomposite $\Phi\circ_\kappa\Psi$ is well-defined in $\boldsymbol{C}$, we have that

$$\mathrm{s}(\Phi)\backslash\mathrm{Im}\kappa \xrightarrow{\psi} \mathrm{s}(\Phi\circ_\kappa\Psi) \xleftarrow{\varphi} \mathrm{s}(\Psi) \qquad \varphi=\varphi[\Phi,\Psi,\kappa]_{\boldsymbol{C}},\ \psi=\psi[\Phi,\Psi,\kappa]_{\boldsymbol{C}},$$

are the same as

$$\langle\mathrm{d}\Phi\rangle\backslash\mathrm{Im}\kappa \xrightarrow{\psi} \langle\mathrm{d}\Phi\square_\kappa\mathrm{d}\Psi\rangle \xleftarrow{\varphi} \langle\mathrm{d}\Psi\rangle \qquad \varphi=\varphi[\mathrm{d}\Phi,\mathrm{d}\Psi,\kappa]_{\boldsymbol{D}},\ \psi=\psi[\mathrm{d}\Phi,\mathrm{d}\Psi,\kappa]_{\boldsymbol{D}}.$$

The upshot for us is that in case $\mathrm{d}\Phi'=\mathrm{d}\Phi$, $\mathrm{d}\Psi'=\mathrm{d}\Psi$, all the respective items in the following two diagrams are the same:

$$\mathrm{s}(\Phi)\backslash\kappa \xrightarrow{\psi} \mathrm{s}(\Phi\circ_\kappa\Psi) \xleftarrow{\varphi} \mathrm{s}(\Psi) \qquad \varphi=\varphi[\Phi,\Psi,\kappa]_{\boldsymbol{C}},\ \psi=\psi[\Phi,\Psi,\kappa]_{\boldsymbol{C}},$$

$s(\Phi')\backslash\kappa \xrightarrow{\psi'} s(\Phi' \circ_\kappa \Psi') \xleftarrow{\varphi'} s(\Psi')$, $\varphi'=\varphi[\Phi',\Psi',\kappa]_{\boldsymbol{C}}$, $\psi'=\psi[\Phi',\Psi',\kappa]_{\boldsymbol{C}}$.

The $\ulcorner\lrcorner$–construction

What follows is a rather technical development; it may be better read when later parts make references to it. The reason why it is here is the logic of the thing: this is exactly the context in which these matters should be dealt with.

Recall that for a positive $\alpha\in S_n$, $n\geq 1$, we had $\alpha=P\alpha\circ\hat{\alpha}$, $|\hat{\alpha}| = |\langle dP\alpha\rangle|$. Let $\langle\alpha\rangle^-=\langle\alpha\rangle\restriction(|\langle\alpha\rangle| - \{p_\alpha\})$ for $n\geq 2$ and α positive; and $\langle\alpha\rangle^-=\langle\alpha\rangle$ otherwise.

Let $\vec{\alpha}_{def}\hat{\alpha}\restriction(|\langle dP\alpha\rangle^-|)$ provided $n\geq 2$ and $dP\alpha$ is positive, and $\vec{\alpha}=\hat{\alpha}$ in every other case of a positive α .

We define the set $Z_n\subset S_n$ recursively in n . For $n=0$ and $n=1$, $Z_n=S_n$. Let $n\geq 2$. Then $\alpha\in S_n$ is in Z_n iff α is positive, $P\alpha\in Z_{n-1}$, and $\alpha=P\alpha\circ\vec{\alpha}$. The last condition is equivalent to saying that $\hat{\alpha}(p_{dP\alpha})$ is an identity n-pd. Roughly, α is in Z if $dP\alpha$ is in Z , and in α , "there is nothing composed into $p_{dP\alpha}$ ".

Let $\alpha\in Z_n$, $n\geq 2$. In $\boldsymbol{C}$, we have the amalgamating maps

$$s(P\alpha)\backslash Im(\iota) \xrightarrow{\psi} s(\alpha=P\alpha\circ\vec{\alpha}) \xleftarrow{\varphi} s(\vec{\alpha}) ,$$

i.e.,

$$\langle dP\alpha\rangle\backslash Im(\iota) \xrightarrow{\psi} \langle d\alpha\rangle \xleftarrow{\varphi} \langle d\vec{\alpha}\rangle ,$$

where ι is the inclusion of $|\vec{\alpha}|$ into $|\langle dP\alpha\rangle|$. Let's write $\gamma=dP\alpha$. Of course, $|\langle\gamma\rangle\backslash Im(\iota)|$ is a singleton, $\{p_\gamma\}$; and ψ maps p_γ to $p_{d\alpha}$ ((11)). It follows that φ induces an isomorphism

$$\langle d\alpha\rangle^{-} \xleftarrow[\cong]{\varphi} \langle d\vec{\alpha}\rangle \quad . \tag{16}$$

(17) Suppose $\alpha\in Z_n$, Φ is a polyarrow in $\boldsymbol{R}$, and $\alpha\circ_\kappa\Phi$ is well-defined. Then $\alpha\circ_\kappa\Phi\in Z_n$. Moreover, when $n\geq 2$, then

$$P(\alpha\circ_\kappa\Phi)=P\alpha, \qquad \overrightarrow{\alpha\circ_\kappa\Phi} = \vec{\alpha}\circ_{\varphi^{-1}\kappa}\Phi$$

with φ from (16).

Proof By (14), $\alpha\circ_\kappa\Phi = P\alpha\circ(\vec{\alpha}\circ_{\varphi^{-1}\kappa}\Phi)$. From this, and by looking at the definition of Z, all assertions are clear.

Assume $\alpha\in Z_n$. We have an isomorphism

$$\langle\vec{\alpha}\rangle \xrightarrow[\rho]{\cong} \langle\alpha\rangle^{-} . \tag{18}$$

Namely, when $n\leq 2$, ρ is the identity. When $n\geq 2$, we have

$$\langle P\alpha\rangle \xrightarrow{\sigma} \langle\alpha=P\alpha\circ\vec{\alpha}\rangle \xleftarrow{\rho} \langle\vec{\alpha}\rangle$$

coming from the HAA-structure of $\mathbf{C}_n(\boldsymbol{R})$. Since, by (11), σ takes the one place in $|\langle P\alpha\rangle|$ to p_α, ρ reduces to the isomorphism called ρ above.

(19) If $\alpha\in Z_{\geq 1}$, Φ is a polyarrow in $\boldsymbol{R}$, $\alpha\square_\lambda\Phi$ is well-defined, and $p_\alpha\notin \mathrm{Im}(\lambda)|$, then $\alpha\square_\lambda\Phi\in Z_{\geq 1}$ as well. Moreover, when $\dim(\alpha)\geq 2$, we have, with ρ from (18),

$$\alpha\square_\lambda\Phi = (P\alpha)\circ(\vec{\alpha}\square_{\rho^{-1}\lambda}\Phi) \tag{19.1}$$

i.e.,

$$\mathrm{P}(\alpha\square_\lambda\Phi) = \mathrm{P}\alpha \quad \text{and} \quad \overrightarrow{\alpha\square_\lambda\Phi} = \vec{\alpha}\square_{\rho^{-1}\lambda}\Phi \tag{19.2}$$

Proof We may assume that $n=\dim(\alpha)\geq 2$. (19.1) follows from (15).

The fact that $\alpha\square_\lambda\Phi \in Z$ is clear.

Now, we can show

(20) For $\alpha\in Z_n$, $n>0$, we have $\mathrm{d}\alpha\in Z_{n-1}$.

Proof When $n\leq 2$, there is nothing to prove. Let $n\geq 3$. Assume $\alpha\in Z_n$. We have

$$\mathrm{d}\alpha = \mathrm{d}(\mathrm{P}\alpha\circ\vec{\alpha}) = \mathrm{dP}\alpha\square\mathrm{d}\vec{\alpha} \tag{21}$$

The assertion follows from (19) applied to $\mathrm{dP}\alpha$: $\mathrm{dP}\alpha\in Z_{n-1}$, and $|\mathrm{d}\vec{\alpha}| = |\vec{\alpha}| \subset |\langle \mathrm{dP}\alpha\rangle^-$, that is, $\mathrm{p}_{\mathrm{dP}\alpha} \not\in |\mathrm{d}\vec{\alpha}|$.

By (21) and (19.2), we have

$$\mathrm{Pd}\alpha = \mathrm{PdP}\alpha \qquad (\alpha\in Z_{\geq 3}). \tag{22}$$

For the rest of this section, we assume that we are given, for each n, an assignment

$$\gamma\mapsto\gamma^{@}:S_n\to R_{n+1}$$

such that $\mathrm{d}\gamma^{@}=\gamma$. (Of course, there may be no such assignment for an arbitrary $\boldsymbol{R}$: there may be pd's that are not domains of any cell. When we apply the present work, said

assignment will be one given in a natural way.)

We define, recursively in n, an assignment $\alpha \mapsto \ulcorner\alpha\urcorner : Z_n \to Z_n$, together with an isomorphism $\theta_\alpha : \langle\alpha\rangle^- \xrightarrow{\cong} \langle \ulcorner\alpha\urcorner \rangle^-$ in $(Z_n)^\#$.

For $n=0$ and $n=1$, $\ulcorner\alpha\urcorner = \alpha$, and θ_α is the identity.

Let $n \geq 2$, $\alpha \in Z_n$, $\alpha = \mathrm{P}\alpha \circ \vec{\alpha}$. $\mathrm{dP}\alpha \in Z_{n-1}$. We have $\ulcorner \mathrm{dP}\alpha \urcorner \in Z_{n-1}$, and

$$\theta_{\mathrm{dP}\alpha} : \langle \mathrm{dP}\alpha \rangle^- \xrightarrow{\cong} \langle \ulcorner \mathrm{dP}\alpha \urcorner \rangle^- .$$

Put

$$\ulcorner\alpha\urcorner \stackrel{\mathrm{def}}{=} (\ulcorner \mathrm{dP}\alpha \urcorner)^@ \circ \theta_{\mathrm{dP}\alpha} \vec{\alpha} . \tag{23}$$

This is well-defined: $\mathrm{t} \circ \vec{\alpha} = \langle \mathrm{dP}\alpha \rangle^-$, and thus

$$\theta_{\mathrm{dP}\alpha} : \mathrm{t} \circ \vec{\alpha} \xrightarrow{\cong} \langle \ulcorner \mathrm{dP}\alpha \urcorner \rangle^- = \langle \mathrm{d}((\ulcorner \mathrm{dP}\alpha \urcorner)^@) \rangle^- .$$

When we calculate $\ulcorner \mathrm{P}\alpha \urcorner$, we find $\ulcorner \mathrm{P}\alpha \urcorner = (\ulcorner \mathrm{dP}\alpha \urcorner)^@$. This allows a slight simplification in the notation:

$$\ulcorner\alpha\urcorner = \ulcorner \mathrm{P}\alpha \urcorner \circ \theta_{\mathrm{dP}\alpha} \vec{\alpha} . \tag{24}$$

Of course, also, $\mathrm{P} \ulcorner\alpha\urcorner = \ulcorner \mathrm{P}\alpha \urcorner$, and $\overrightarrow{\ulcorner\alpha\urcorner} \cong \ulcorner\alpha\urcorner$.

We have the canonical coproducts

$$\langle \mathrm{P}\alpha \rangle \xrightarrow{\sigma} \langle \alpha \rangle \xleftarrow{\rho} \langle \vec{\alpha} \rangle \qquad \rho = \rho[\mathrm{P}\alpha, \vec{\alpha}, \iota]$$

$$\langle \ulcorner \mathrm{P}\alpha \urcorner \rangle \xrightarrow{\sigma'} \langle \ulcorner\alpha\urcorner \rangle \xleftarrow{\rho'} \langle \vec{\alpha} \rangle , \qquad \rho' = \rho[\ulcorner \mathrm{P}\alpha \urcorner, \vec{\alpha}, \theta_{\mathrm{dP}\alpha}] ;$$

and the image of $|\sigma|$ is $\{p_{P\alpha}\}$, that of σ' is $\{p_{P\ulcorner\alpha\urcorner}\}$. Therefore,

$$\langle\alpha\rangle^- \xleftarrow[\cong]{\rho} \langle\vec{\alpha}\rangle$$
$$\langle\ulcorner\alpha\urcorner\rangle^- \xleftarrow[\cong]{\rho'} \langle\vec{\alpha}\rangle \ .$$

We define $\theta_\alpha = \rho' \circ \rho^{-1} : \langle\alpha\rangle^- \xrightarrow{\cong} \langle\ulcorner\alpha\urcorner\rangle^-$.

This completes the recursive definition of the entities $\ulcorner\alpha\urcorner$, θ_α.

We have the isomorphisms

$$\varphi : \langle d\vec{\alpha}\rangle \xrightarrow{\cong} \langle d\alpha\rangle^- ,$$
$$\varphi' : \langle d\vec{\alpha}\rangle \xrightarrow{\cong} \langle d\ulcorner\alpha\urcorner\rangle^- \ .$$

The first was given in (16); the second is the φ-map for $\ulcorner\alpha\urcorner$ as the composite in (24).

We use the temporary notations $\alpha \circ \bar{}_\kappa \Phi$, $\alpha \square \bar{}_\kappa \Phi$ in the following sense. $\alpha \circ \bar{}_\kappa \Phi$ is the same as $\alpha \circ_\kappa \Phi$, but it is considered defined only when $\mathrm{Im}\kappa \subset |\langle d\alpha\rangle^-|$, i.e., $\kappa : |\Phi| \to |\langle d\alpha\rangle^-|$. Similarly, for $\alpha \square \bar{}_\kappa \Phi$ to be well-defined, we need that $\kappa : |\Phi| \to |\langle\alpha\rangle^-|$.

(25) Lemma For $\alpha \in Z_n$, $\Phi \in \overrightarrow{S}_n$, if $\alpha \circ \bar{}_\kappa \Phi$ is well-defined, we have

$$\ulcorner\alpha \circ \bar{}_\kappa \Phi\urcorner = \ulcorner\alpha\urcorner \circ {}^{\varphi' \varphi^{-1} \kappa}\Phi$$

Proof By definition,

$$\ulcorner\alpha \circ_\kappa \Phi\urcorner = \ulcorner P(\alpha \circ_\kappa \Phi)\urcorner \circ \theta_{dP(\alpha \circ_\kappa \Phi)} \overrightarrow{\alpha \circ_\kappa \Phi} = \ulcorner P\alpha\urcorner \circ \theta_{dP\alpha} \overrightarrow{\alpha \circ_\kappa \Phi} \ .$$

But, by (17), $\overrightarrow{\alpha\circ_\kappa\Phi} = \vec{\alpha}\circ_{\varphi^{-1}\kappa}\Phi$. Thus, writing $\theta=\theta_{\mathrm{dP}\alpha}$, by the GAL, we have

$$\ulcorner\alpha\circ_{\bar\kappa}\Phi\urcorner = \ulcorner\mathrm{P}\alpha\urcorner\circ_\theta(\vec{\alpha}\circ_{\varphi^{-1}\kappa}\Phi) = (\ulcorner\mathrm{P}\alpha\urcorner\circ_\theta\vec{\alpha})\circ_{\varphi'\varphi^{-1}\kappa}\Phi = \ulcorner\alpha\urcorner\circ_{\varphi'\varphi^{-1}\kappa}\Phi .$$

(25') **Lemma** For $\alpha\in Z_n$, $\Phi\in\overrightarrow{Z_n}$, if $\alpha\square_{\bar\kappa}\Phi$ is well-defined, we have

$$\ulcorner\alpha\square_{\bar\kappa}\Phi\urcorner = \ulcorner\alpha\urcorner\square_{\theta_\alpha\kappa}\Phi .$$

Proof Again, let us write $\theta=\theta_{\mathrm{dP}\alpha}$. Then

$$\begin{aligned}\ulcorner\alpha\square_\kappa\Phi\urcorner &= \ulcorner\mathrm{P}(\alpha\square_\kappa\Phi)\urcorner\circ_{\theta_{\mathrm{dP}(\alpha\square_\kappa\Phi)}}\overrightarrow{\alpha\square_\kappa\Phi} = \\ &= \ulcorner\mathrm{P}\alpha\urcorner\circ_\theta(\vec{\alpha}\square_{\rho^{-1}\kappa}\Phi) \\ &\overset{*}{=} (\ulcorner\mathrm{P}\alpha\urcorner\circ_\theta\vec{\alpha})\square_{\theta_\alpha\kappa}\Phi \\ &= \ulcorner\alpha\urcorner\square_{\theta_\alpha\kappa}\Phi\ ;\end{aligned}$$

the equality $*$ is an application of (5) ("backwards"), under the hypothesis that $\alpha\square_{\bar\kappa}\Phi$ is well-defined.

(26) **Lemma** $\ulcorner\mathrm{d}\alpha\urcorner = \mathrm{d}\ulcorner\alpha\urcorner$ $(\alpha\in Z_n ,\ n\geq 1)$

Proof We assume $n\geq 3$, and leave the cases $n=1$, $n=2$ to the reader.

Since $\alpha=\mathrm{P}\alpha\circ\vec{\alpha}$, we have $\mathrm{d}\alpha=\mathrm{dP}\alpha\square\mathrm{d}\vec{\alpha}$. Let us write $\gamma=\mathrm{dP}\alpha$ and $\Phi=\mathrm{d}\vec{\alpha}$. From $\mathrm{d}\alpha=\gamma\square\Phi$ and $\gamma=\mathrm{P}\gamma\circ\vec{\gamma}$ we get, by (15), that $\mathrm{d}\alpha=\mathrm{P}\gamma\circ(\vec{\gamma}\square_{\rho^{-1}}\Phi)$ with $\rho=\rho[\mathrm{P}\gamma,\vec{\gamma},\iota]$. Thus, $\mathrm{Pd}\alpha = \mathrm{P}\gamma$, and $\vec{\mathrm{d}}\vec{\alpha}=(\vec{\gamma}\square_{\rho^{-1}}\Phi)$; and we have, with $\theta=\theta_{\mathrm{dP}\gamma}$:

$$\ulcorner d\alpha\urcorner \;=\; \ulcorner P\gamma\urcorner \circ_{\theta}(\vec{\gamma}\square_{\rho^{-1}}\Phi)\ .$$

On the other hand, $\ulcorner\alpha\urcorner = (\ulcorner\gamma\urcorner)^{@}\circ_{\theta_\gamma}\vec{\alpha}$. Thus,

$$d\ulcorner\alpha\urcorner = d(\ulcorner\gamma\urcorner)^{@}\square_{\theta_\gamma} d\vec{\alpha} = \ulcorner\gamma\urcorner\square_{\theta_\gamma} d\vec{\alpha}\ .$$

Also $\ulcorner\gamma\urcorner = \ulcorner P\gamma\urcorner \circ_{\theta}\vec{\gamma}$ (with $\theta=\theta_{dP\gamma}$ as before). Thus, by the same (15), with $\rho'=\rho[\ulcorner P\gamma\urcorner, \vec{\gamma}, \theta=\theta_{dP\gamma}]$,

$$d\ulcorner\alpha\urcorner \;=\; \ulcorner P\gamma\urcorner \circ_{\theta}(\vec{\gamma}\square_{(\rho')^{-1}\theta_\gamma}\Phi)\ .$$

However, $\theta_\gamma=\rho'\circ\rho^{-1}$, with exactly the right ρ and ρ', and thus $(\rho')^{-1}\theta_\gamma = \rho^{-1}$. Now we see that we obtained the same formula for $\ulcorner d\alpha\urcorner$ and $d\ulcorner\alpha\urcorner$.

Let $\alpha\in Z_n$. We have the isomorphisms

$$\varphi: \langle d\vec{\alpha}\rangle \xrightarrow{\cong} \langle d\alpha\rangle^{-}\ ,\ \varphi': \langle d\vec{\alpha}\rangle \xrightarrow{\cong} \langle d\ulcorner\alpha\urcorner\rangle^{-}\ .$$

On the other hand,

$$\theta=\theta_{d\alpha}: \langle d\alpha\rangle^{-} \xrightarrow{\cong} \langle \ulcorner d\alpha\urcorner\rangle^{-}\ .$$

By (26), $\ulcorner d\alpha\urcorner = \ulcorner d\ulcorner\alpha\urcorner$. Thus, $\varphi'\circ\varphi^{-1}$ and θ are parallel.

(27) Lemma $\varphi'\circ\varphi^{-1} = \theta$.

Proof Note the obvious fact that if the map $\langle d\alpha\rangle^{-}: |\langle d\alpha\rangle^{-}| \longrightarrow R_n$ is one-to-one,

then, automatically, any two parallel maps into $\langle \mathrm{d}\alpha\rangle^-$, $s \overset{\mu}{\underset{\mu'}{\rightrightarrows}} \langle \mathrm{d}\alpha\rangle^-$, in $(R_n)^{\#}$, must be the same. Thus, in this case, the lemma's equality holds (the remark being applied the inverses of the isomorphisms involved). The proof of the lemma in the general case will use an invariance argument reducing the statement to said special case.

For the rest of the proof of (27), we rename our ms $\boldsymbol{R}$ as $\bar{\boldsymbol{R}}$ so that we can talk about a map to it from a new ms in the form $\boldsymbol{R}\to\bar{\boldsymbol{R}}$, and denote its effect by $a\mapsto\bar{a}$, on cells as well as pd's. What was S_n before is now $\bar{S}_n$, *etc*. -- and S_n is the entity that is related to the *new* $\boldsymbol{R}$, *etc*.

(28) Sublemma Suppose we are given the ms $\bar{\boldsymbol{R}}$, together with the assignment $\bar{\gamma}\mapsto\bar{\gamma}^{@}$: $\bar{S}_{\geq 0}\longrightarrow\bar{R}_{\geq 1}$ such that $\mathrm{d}\bar{\gamma}^{@}=\bar{\gamma}$. Then, there are: an ms $\boldsymbol{R}$, an assignment $\gamma\mapsto\gamma^{@}$: $S_{\geq 0}\longrightarrow R_{\geq 1}$ and a morphism $\overline{(\text{-})}:\boldsymbol{R}\longrightarrow\bar{\boldsymbol{R}}$ with the following properties:

(i) $(\bar{\gamma})^{@}=\overline{\gamma^{@}}$ for all $\gamma\in S_{\geq 0}$;

(ii) for every $\bar{\alpha}\in\bar{S}_{\geq 0}$, there is $\alpha\in S_{\geq 0}$ such that $\mathrm{d}\alpha$ is separated, and $\bar{\alpha}$ is mapped by $\overline{(\text{-})}$ to $\bar{\alpha}$.

Proof We use 1.(56). Start by adjoining to $\bar{\boldsymbol{R}}$ infinitely many new $(n\text{-}1)$-cells $b^{[i]}$, one for each $b\in R_{n-1}$ and $i\in\mathbb{N}$, so that $b^{[i]}$ is parallel to b. (It is understood that $b^{[i]}$ is not in $\bar{R}_{n-1}$, and $b^{[i]}=(b')^{[i']}$ only if $b=b'$ and $i=i'$.) Denote the resulting ms by $\hat{\boldsymbol{R}}$. We have a map $\overline{\overline{(\text{-})}}:\hat{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ which is the identity on $\bar{\boldsymbol{R}}\subset\hat{\boldsymbol{R}}$ and maps $b^{[i]}$ to b. Note that $\overline{(\text{-})}:\hat{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ is a cover at each $k\leq n\text{-}2$ (as an identity map is), and an ∞-cover at $n\text{-}1$.

Next, extend $\hat{\boldsymbol{R}}$ to $\tilde{\boldsymbol{R}}$ (meaning that $\hat{\boldsymbol{R}}\subset\tilde{\boldsymbol{R}}$), and $\overline{\overline{(\text{-})}}:\hat{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ to $\overline{(\text{-})}:\tilde{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ so that

$\overline{(-)}:\tilde{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ is a cover at n. This is done by adding appropriate n-cells to $\hat{\boldsymbol{R}}$ to fulfill the cover-condition at n.

Note that the map $\overline{(-)}:\hat{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ satisfies the hypotheses of 1.(56).

Finally, we extend $\tilde{\boldsymbol{R}}$ to $\boldsymbol{R}$, and $\overline{(-)}:\tilde{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ to $\overline{(-)}:\boldsymbol{R}\to\bar{\boldsymbol{R}}$ as follows.

Recursively in $k\in\mathbb{N}$, we define the ms $\boldsymbol{R}^{(k)}$, $\boldsymbol{R}^{(k+1)}$ extending $\boldsymbol{R}^{(k)}$. We put $\boldsymbol{R}^{(0)}=\tilde{\boldsymbol{R}}$. Having obtained $\boldsymbol{R}^{(k)}$, we construct $\boldsymbol{R}^{(k+1)}$ thus. Simultaneously for each $\gamma\in S_n^{(k)}$, we adjoin to $\boldsymbol{R}^{(k)}$ a new n-cell $\tilde{\gamma}$ parallel to γ, and then, adjoin a new $(n+1)$-cell $\gamma^{@}$ subject to $\mathrm{d}\gamma^{@}=\gamma$, $\mathrm{c}\gamma^{@}=\tilde{\gamma}$; the resulting ms is $\boldsymbol{R}^{(k+1)}$.

We recursively define $\overline{(-)}:\boldsymbol{R}^{(k)}\to\bar{\boldsymbol{R}}$. We start with the given $\overline{\overline{(-)}}:\tilde{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$ for $k=0$.

Assume having the map $\overline{(-)}:\boldsymbol{R}^{(k)}\to\bar{\boldsymbol{R}}$; extend it to $\overline{\overline{(-)}}:\boldsymbol{R}^{(k+1)}\to\bar{\boldsymbol{R}}$ as follows. Take any $\gamma\in S_n^{(k)}$; we have $(\bar{\gamma})^{@}=\bar{\gamma}^{@}$ given in $\bar{\boldsymbol{R}}$. We want to map $\tilde{\gamma}$ to $\mathrm{c}(\bar{\gamma}^{@})$. This is all right, since $\mathrm{c}(\bar{\gamma}^{@})$ is parallel to $\bar{\gamma}$, and $\tilde{\gamma}$ is parallel to γ, thus $\overline{(-)}:\boldsymbol{R}^{(k)}\to\bar{\boldsymbol{R}}$ maps $\mathrm{d}\tilde{\gamma}=\mathrm{d}\gamma$ to $\mathrm{dc}(\bar{\gamma}^{@})=\mathrm{d}\bar{\gamma}$, and similarly for c in place of d. Having mapped $\tilde{\gamma}$ to $\mathrm{c}(\bar{\gamma}^{@})$, we can map $\gamma^{@}$ to $\bar{\gamma}^{@}$, because $\overline{(-)}:\boldsymbol{R}^{(k)}\to\bar{\boldsymbol{R}}$ maps $\mathrm{d}(\gamma^{@})=\gamma$ to $\mathrm{d}(\bar{\gamma}^{@})=\bar{\gamma}$ and we have already mapped $\mathrm{c}(\gamma^{@})=\tilde{\gamma}$ to $\mathrm{c}(\bar{\gamma}^{@})$.

$\boldsymbol{R}$ is defined to be the union of all the $\boldsymbol{R}^{(k)}$; and $\overline{(-)}:\boldsymbol{R}\to\bar{\boldsymbol{R}}$ is the union of all the $\overline{(-)}:\boldsymbol{R}^{(k)}\to\bar{\boldsymbol{R}}$.

We have defined $\overline{(-)}:\boldsymbol{R}\to\bar{\boldsymbol{R}}$ as an extension of $\overline{(-)}:\hat{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$. Clause (i) is made true by the construction; clause is true by 1.(56) applied to $\overline{(-)}:\hat{\boldsymbol{R}}\to\bar{\boldsymbol{R}}$.

Continuing the proof of (27). In addition to the data given for (27), let us take the data as supplied by (28). Let us write $\tau_\beta : \overline{\langle \beta \rangle} \xrightarrow{\cong} \langle \bar{\beta} \rangle$ ($\beta \in S_n$) for the transition isomorphism associated with the part $\overline{\overline{(\,-\,)}} : \boldsymbol{D}_n \longrightarrow \bar{\boldsymbol{D}}_n$ of the morphism $\overline{\overline{(\,-\,)}} : \boldsymbol{R} \to \bar{\boldsymbol{R}}$. Then $\tau_{\mathrm{d}\beta} : \overline{\langle \mathrm{d}\beta \rangle} \xrightarrow{\cong} \langle \mathrm{d}\bar{\beta} \rangle$ is the transition isomorphism associated with $\overline{(\,-\,)} : \boldsymbol{C}_n \longrightarrow \bar{\boldsymbol{C}}_n$ ($\beta \in S_n$).

Application of $\overline{(\,-\,)} : \boldsymbol{C}_n \longrightarrow \bar{\boldsymbol{C}}_n$ to a polyarrow Φ in $\boldsymbol{C}_n$ is defined naturally pointwise: $|\bar{\Phi}| = |\Phi|$, $\bar{\Phi}(p) = \overline{\Phi(p)}$ ($p \in |\Phi|$). We have $\tau_\Phi : \overline{\langle \Phi \rangle} \xrightarrow{\equiv} \langle \bar{\Phi} \rangle$; it is defined "componentwise".

Applying $\overline{\overline{(\,-\,)}} : \boldsymbol{C}_n \longrightarrow \bar{\boldsymbol{C}}_n$ to $\beta \circ_p \gamma$, we get $\overline{\beta \circ_p \gamma} = \bar{\beta} \circ_{\bar{p}} \bar{\gamma}$, where $\bar{p} = \tau_{\mathrm{d}\beta}(p)$. It follows that, more generally, for polycomposition in $\boldsymbol{C}_n$, we have

$$\overline{\overline{\Phi \circ_\kappa \Psi}} = \bar{\Phi} \circ_{\tau_{\mathrm{d}\Phi} \circ \kappa} \bar{\Psi} \quad . \tag{29}$$

In particular, for any $\beta \in Z_n$, $\beta = \mathrm{P}\beta \circ \vec{\beta}$, we have

$$\bar{\beta} = \mathrm{P}\bar{\beta} \circ_{\tau_{\mathrm{dP}\beta}} \bar{\vec{\beta}} = \mathrm{P}\bar{\beta} \circ ((\tau_{\mathrm{dP}\beta})^{-1} \circ \bar{\vec{\beta}})$$

by "renaming". Therefore,

$$\vec{\bar{\beta}} = (\tau_{\mathrm{dP}\beta})^{-1} \circ \bar{\vec{\beta}} \,. \tag{30}$$

We **claim** that for all $\beta \in Z_{\geq 1}$, $\overline{\ulcorner \beta \urcorner} = \ulcorner \bar{\beta} \urcorner$; and that

$$\begin{array}{ccc} |\langle \beta \rangle| & \xrightarrow{\theta_\beta} & |\langle \ulcorner \beta \urcorner \rangle| \\ {\scriptstyle \tau_\beta}\downarrow & \circ & \downarrow{\scriptstyle \tau_{\ulcorner \beta \urcorner}} \\ |\langle \bar{\beta} \rangle| & \xrightarrow[\theta_{\bar{\beta}}]{} & |\langle \ulcorner \bar{\beta} \urcorner \rangle| \end{array} \tag{31}$$

commutes. The proof is an induction on the dimension n of β.

For $n \leq 1$, the assertions are clearly true. Thus, for a given β of dimension ≥ 2, we may assume

$$\overline{\ulcorner \mathrm{dP}\beta \urcorner} = \ulcorner \overline{(\mathrm{dP}\beta)} \urcorner = \ulcorner (\mathrm{dP}\bar{\beta}) \urcorner = \mathrm{dP} \ulcorner \bar{\beta} \urcorner \ , \tag{32}$$

and (31) with $\mathrm{dP}\beta$ replacing β, that is,

$$\begin{array}{ccc} |\langle \mathrm{dP}\beta \rangle| & \xrightarrow{\theta_{\mathrm{dP}\beta}} & |\langle \mathrm{dP} \ulcorner \beta \urcorner \rangle| \\ {\scriptstyle \tau_{\mathrm{dP}\beta}}\downarrow & \circ & \downarrow{\scriptstyle \tau_{\mathrm{dP} \ulcorner \beta \urcorner}} \\ |\langle \mathrm{dP}\bar{\beta} \rangle| & \xrightarrow[\theta_{\mathrm{dP}\bar{\beta}}]{} & |\langle \mathrm{dP} \ulcorner \bar{\beta} \urcorner \rangle| \end{array} \tag{33}$$

Let us compare

$$\ulcorner \beta \urcorner \ = \ (\mathrm{dP} \ulcorner \beta \urcorner)^{@} \circ \theta_{\mathrm{dP}\beta}{}^{\vec{\beta}} \tag{34}$$

and

$$\ulcorner \bar{\beta} \urcorner \ = \ (\mathrm{dP} \ulcorner \bar{\beta} \urcorner)^{@} \circ \theta_{\mathrm{dP}\bar{\beta}}{}^{\vec{\bar{\beta}}} \ . \tag{35}$$

When we apply (29) to the right-hand side of (34), and take into account (30), (32), (33) and (35), and use "renaming", we obtain the desired equality $\overline{\ulcorner \beta \urcorner} = \ulcorner \bar{\beta} \urcorner$.

We leave the proof of (31) to the reader; it needs the fact that the τ_β are components of an HAA-morphism $(\boldsymbol{D}_n, \langle - \rangle, \rho, \sigma) \longrightarrow (\bar{\boldsymbol{D}}_n, \langle - \rangle, \rho, \sigma)$.

Having proved the **claim**, let us use the notation for (27), for both α and $\bar{\alpha}$, and consider the diagram

$$\begin{array}{ccc} \langle \mathrm{d}\alpha \rangle^- & \underset{\varphi' \circ \varphi^{-1}}{\overset{\theta}{\rightrightarrows}} & \langle \ulcorner \mathrm{d}\alpha \urcorner \rangle^- \\ {\scriptstyle \tau_{\mathrm{d}\alpha}}\downarrow & & \downarrow{\scriptstyle \tau_{\ulcorner \mathrm{d}\alpha \urcorner}} \\ \langle \mathrm{d}\bar{\alpha} \rangle^- & \underset{\varphi' \circ \varphi^{-1}}{\overset{\theta}{\rightrightarrows}} & \langle \ulcorner \mathrm{d}\bar{\alpha} \urcorner \rangle^- \end{array}$$

The square with the two θ's in it commutes by (31). The square with the two $\varphi' \circ \varphi^{-1}$'s in it commutes because the τ's are transition isomorphisms (at α and $\ulcorner \alpha \urcorner$) for the morphism $\overline{(-)} : \boldsymbol{C}_n \longrightarrow \bar{\boldsymbol{C}}_n$. As we said, the two uppermost horizontals are equal. It follows that the two lower ones are equal too.

We leave the proof of the following facts to the reader.

(35') Suppose $\delta = \delta_1 \circ_k \bar{\delta}_2$ is well-defined. Then $\ulcorner \delta \urcorner = \ulcorner \delta_1 \urcorner \circ_{\theta k} \bar{\delta}_2$ ($\theta = \theta_{\delta_1}$), and the diagram

$$\begin{array}{ccccc} \langle \delta_1 \rangle^- & \overset{\sigma}{\longrightarrow} & \langle \delta \rangle^- & \overset{\rho}{\longleftarrow} & \langle \delta_2 \rangle \\ {\scriptstyle \theta}\downarrow & \circ & {\scriptstyle \theta}\downarrow & \circ & \| \\ \langle \ulcorner \delta_1 \urcorner \rangle^- & \underset{\sigma}{\longrightarrow} & \langle \ulcorner \delta \urcorner \rangle^- & \underset{\rho}{\longleftarrow} & \langle \delta_2 \rangle \end{array}$$

commutes.