§3. Simultaneous composition

Part 1: Simultaneous composition in a general multicategory

$\boldsymbol{C}$: fixed multicategory; not necessarily 1-level.

The symbols α , β , γ , ... are used to denote *multiarrows*, that is, elements of A ($=\mathsf{A}(\boldsymbol{C})$).

We call a tuple of multiarrows a *polyarrow*. A polyarrow Ψ is given by a finite set $|\Psi|$ and a function $|\Psi| \longrightarrow A$. In other words, Ψ is an object of the category $A^{\#}$.

We use Φ , Ψ , Γ , ... to denote polyarrows.

The *source of* Ψ , $\mathsf{s}\Psi$, is the coproduct $\mathsf{s}\Psi = \bigsqcup_{j \in |\Psi|} \mathsf{s}(\Psi j)$ (in the category $O^{\#}$), but with the particular domain

$$|\mathsf{s}\Psi| = \bigsqcup_{j \in |\Psi|} |\mathsf{s}(\Psi j)| = \{(j, p) : j \in |\Psi| , p \in |\mathsf{s}(\Psi v)| \} .$$

That is, $\mathsf{s}\Psi : |\mathsf{s}\Psi| \longrightarrow O$, and $(\mathsf{s}\Psi)(j, p) = (\mathsf{s}(\Psi j))(p)$.

The *target of* Ψ , $\mathsf{t}\Psi$, is the tuple $\mathsf{t} \circ \Psi : |\Psi| \longrightarrow \dot{O}$, the composite

$$|\Psi| \xrightarrow{\ \Psi\ } A \xrightarrow{\ \mathsf{t}\ } \dot{O} .$$

We introduce *polycomposition*: the composition of polyarrows. The result is again a polyarrow. The definition of polycomposition is a natural extension of "multicomposition": the given composition in $\boldsymbol{C}$.

(As a matter of fact, polycomposition was already discussed in [H/M/P], under the name of simultaneous composition; here, we expand that discussion, and introduce a streamlined notation.)

Let Φ and Ψ be given. We write $(s\Phi)^{\cdot}$ for the composite tuple $s\alpha\circ(-)^{\cdot}$:

$$|s\Phi| \xrightarrow{s\Phi} O \xrightarrow{(-)^{\cdot}} \dot{O} .$$

A (*correct*) *placement of* Ψ *in* Φ is an *injective* morphism $\kappa: t\Psi \longrightarrow (s\Phi)^{\cdot}$ in the category $\dot{O}^{\#}$. This means that κ is an *injective* map $\kappa: |\Psi| \longrightarrow |s\Phi|$; and
$t\circ\Psi = (s\alpha)^{\cdot}\circ\kappa$, i.e., with $(\kappa_1 j, \kappa_2 j)=\kappa j\in |s\Phi|$,
$\forall j\in |\Psi| . t(\Psi j)=(s(\Phi(\kappa_1 j)))(\kappa_2 j)^{\cdot}$.

For a correct placement κ of Ψ in Φ, the "poly-composite" polyarrow $\Phi\circ_{\kappa}\Psi$ is defined; $\Phi\circ_{\kappa}\Psi$ comes together with maps in the category $O^{\#}$:

$$s\Phi\backslash Im(\kappa) \xrightarrow{\psi} s(\Phi\circ_{\kappa}\Psi) \xleftarrow{\varphi} s\Psi , \qquad (1)$$

$\psi=\psi[\Phi, \Psi, \kappa]$, $\varphi=\varphi[\Phi, \Psi, \kappa]$, which form a coproduct diagram.

There are two aspects of the definition of these items. On the one hand, it is "pointwise in Φ"; that is, the general case is reduced to the one when $|\Phi|$ is a singleton in a direct, pointwise, manner. On the other hand, the definition proceeds recursively in $\#|\Psi|$, the case when $\#|\Psi|=1$ being, essentially, the case of ordinary composition.

When Φ is a singleton, with the one term α, we write $\alpha\circ_{\kappa}\Psi$ for $\Phi\circ_{\kappa}\Psi$; of course, there is slight change from one κ to the other.

The definition of $\Phi\circ_{\kappa}\Psi$ is reduced to that of $\alpha\circ_{\nu}\Psi$, for several appropriate α's and ν's, as follows. We note that, of course, $\alpha\circ_{\nu}\Psi$ is an (ordinary) multiarrow. Given Φ, Ψ and κ, a placement of Ψ in Φ, we write, as before $\kappa(j) = (\kappa_1 j, \kappa_2 j)$ ($j\in |\Psi|$). For $i\in |\Phi|$, $\Psi\restriction i$ is the restriction of Ψ as a mapping to the subset $|\Psi\restriction i| =\{j: \kappa_1 j=i\}$. Similarly, $\kappa\restriction i$ is the restriction of κ to the subset $|\Psi\restriction i|$. Then, the polyarrow $\Phi\circ_{\kappa}\Psi$ is defined as follows:

$$|\Phi\circ_\kappa\Psi| = |\Phi| ,$$
$$(\Phi\circ_\kappa\Psi)(i) = \Phi u\circ_{\kappa\restriction i}(\Psi\restriction i) .$$

The amalgamating maps

$$s\Phi\backslash \mathrm{Im}(\kappa) \xrightarrow{\psi} s(\Phi\circ_\kappa\Psi) \xleftarrow{\varphi} s\Psi$$

are deduced from the instances of (1) when Φ is replaced by the $\Phi(i)$, etc., in the obvious way.

The above "pointwise" character in the first component can be expressed in a "trivial distributive law" (later on, there will be a "non-trivial" distributive law as well) as follows. Suppose we have a coproduct

$$\Phi_1 \xrightarrow{\zeta_1} \Phi \xleftarrow{\zeta_2} \Phi_2 \qquad (2)$$

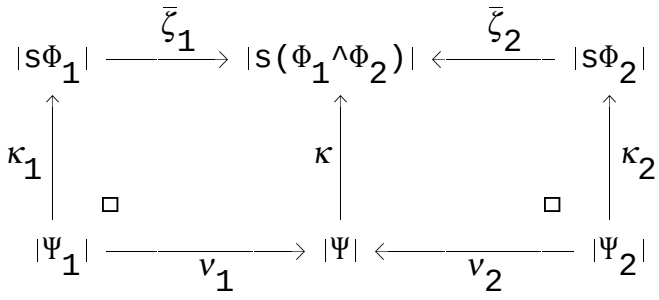
in the category $A^{\#}$; we write $\Phi = \Phi_1\hat{}\Phi_2$. (It would be a non-essential restriction to assume that ζ_1, ζ_2 are inclusions, that is, that $|\Phi|$ is the disjoint union of $|\Phi_1|$ and $|\Phi_2|$; but, the form given here is the one we will directly use.)

(2) induces the diagram

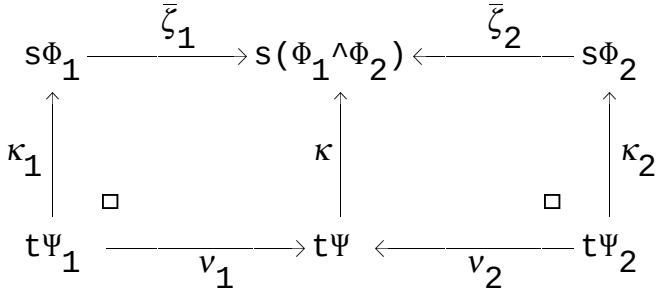
$$s\Phi_1 \xrightarrow{\bar{\zeta}_1} s(\Phi_1\hat{}\Phi_2) \xleftarrow{\bar{\zeta}_2} s\Phi_2 ; \qquad (2')$$

for $(u, p)\in |s\Phi_j|$, $u\in |s\Phi_j|$, $p\in |s(\Phi_j u)| = |s(\Phi(\zeta_j u))|$, we define $\bar{\zeta}_1(u, p) = (\zeta_1 u, p)$.

Suppose $(\Phi_1\hat{}\Phi_2)\circ_\kappa\Psi$ is well-defined. Form *any* diagram of the form



with two pullbacks as shown; for instance, ν_1, ν_2 may be, but do not have to be, chosen to be inclusions. For $\Psi_j=\Psi\circ\iota_j : |\Psi_j| \longrightarrow A$, $\Psi_j \in \mathrm{Ob}(A^{\#})$, we have the diagram in $A^{\#}$:



Thus, we can form $\Phi_1{}^{\circ}{}_{\kappa_1}\Psi_1$, $\Phi_2{}^{\circ}{}_{\kappa_2}\Psi_2$. Of course, $|\Phi_j{}^{\circ}{}_{\kappa_j}\Psi_j| = |\Phi_j|$. The polyarrow $(\Phi_1{}^{\circ}{}_{\kappa_1}\Psi_1)\hat{}(\Phi_2{}^{\circ}{}_{\kappa_2}\Psi_2)$ is uniquely determined by the conditions that

$$|(\Phi_1{}^{\circ}{}_{\kappa_1}\Psi_1)\hat{}(\Phi_2{}^{\circ}{}_{\kappa_2}\Psi_2)| = |\Phi_1\hat{}\Phi_2|$$

and

$$(\Phi_1{}^{\circ}{}_{\kappa_1}\Psi_1) \xrightarrow{\zeta_1} (\Phi_1{}^{\circ}{}_{\kappa_1}\Psi_1)\hat{}(\Phi_2{}^{\circ}{}_{\kappa_2}\Psi_2) \xleftarrow{\zeta_2} (\Phi_2{}^{\circ}{}_{\kappa_2}\Psi_2)$$

with ζ_1, ζ_2 the same functions as in (2). We have

$$(\Phi_1\hat{}\Phi_2)\circ\kappa^{\Psi} = (\Phi_1{}^{\circ}{}_{\kappa_1}\Psi_1)\hat{}(\Phi_2{}^{\circ}{}_{\kappa_2}\Psi_2) \ ;$$

the proof is easy.

The remaining parts of the definition of composition of polyarrows are, in essence, contained in the statement of a *generalized commutative law* (GCL) as follows.

Suppose $\Lambda = \Psi \dot\cup \Sigma$, whereby we mean that $|\Lambda| = |\Psi| \dot\cup |\Sigma|$, and each of Ψ, Σ is the restriction of Ψ to its respective domain. Suppose we have a correct placement μ of Λ in Φ; then $\kappa = \mu \restriction |\Psi|$ is a correct placement of Ψ in Φ; and similarly for $\lambda = \mu \restriction |\Sigma|$ and Σ. Moreover, we can define the placement $\bar\lambda$ of Σ in $\Phi \circ_\kappa \Psi$ as follows. Since μ is injective, λ induces a map

$$\lambda : t\Sigma \longrightarrow (s\Phi) \dot\setminus \mathrm{Im}(\kappa) ;$$

and ψ from (1) gives

$$\psi : (s\Phi) \dot\setminus \mathrm{Im}(\kappa) \longrightarrow (s(\Phi \circ_\kappa \Psi)) \dot\setminus .$$

Let's define $\bar\lambda = \psi \circ \lambda$. The law in question says that

$$\Phi \circ_{\kappa \dot\cup \lambda} (\Psi \dot\cup \Sigma) = (\Phi \circ_\kappa \Psi) \circ_{\bar\lambda} \Sigma ,$$

and

$$\begin{array}{ccccc}
s\Phi \setminus \mathrm{Im}(\kappa \dot\cup \lambda) & \xrightarrow{\ \psi\ } & s(\Phi \circ_{\kappa \dot\cup \lambda} (\Psi \dot\cup \Sigma)) & \xleftarrow{\ \varphi\ } & s(\Psi \dot\cup \Sigma) \\
\psi \restriction \downarrow & \circ & \| & \circ & \uparrow \mathrm{incl} \\
s(\Phi \circ_\kappa \Psi) \setminus \mathrm{Im}(\bar\lambda) & \xrightarrow[\ \psi\ ]{} & s((\Phi \circ_\kappa \Psi) \circ_{\bar\lambda} \Sigma) & \xleftarrow[\ \varphi\ ]{} & s(\Sigma)
\end{array}$$

One immediately deduces the more familiar form of the GCL:

$$(\Phi \circ_\kappa \Psi) \circ_{\bar\lambda} \Sigma = (\Phi \circ_\lambda \Sigma) \circ_{\bar\kappa} \Psi .$$

In fact, this holds every time $\Phi \circ_\kappa \Psi$, $\Phi \circ_\lambda \Sigma$ are well-defined, and $\bar\lambda$, $\bar\kappa$ are defined appropriately.

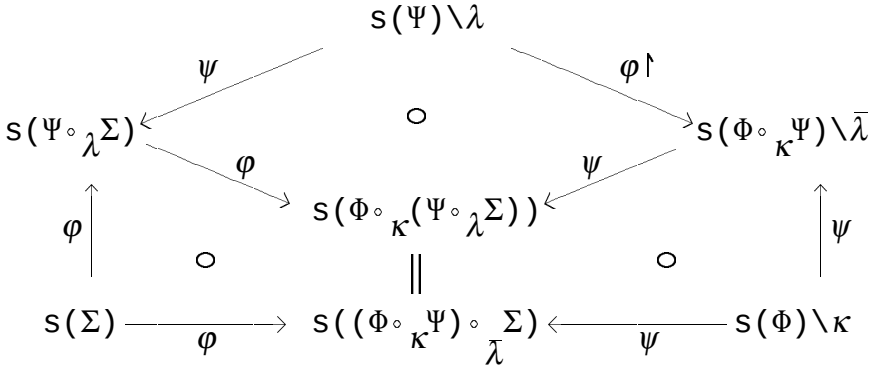
We give a *generalized associative law* (GAL).

Suppose Φ, Ψ and Σ given so that $\Phi\circ_{\kappa}\Psi$ and $\Psi\circ_{\lambda}\Sigma$ are well-defined.

Define $\bar{\lambda}$, a placement of Σ in $\Phi\circ_{\kappa}\Psi$, as $\bar{\lambda}=\dot{\varphi}\circ\lambda$, where $\varphi\colon \mathsf{s}\Psi\to\mathsf{s}(\Phi\circ_{\kappa}\Psi)$ is the canonical φ-map; $(\Phi\circ_{\kappa}\Psi)\circ_{\bar{\lambda}}\Sigma$ is well-defined. I claim that

$$\Phi\circ_{\kappa}(\Psi\circ_{\lambda}\Sigma) \;=\; (\Phi\circ_{\kappa}\Psi)\circ_{\bar{\lambda}}\Sigma$$

and



The proof is easy.

When, in the definition of $\Phi\circ_{\iota}\Psi$, the map $\iota\colon |\Psi| \longrightarrow |\mathsf{s}\Phi|$ is an inclusion (a map that acts as an identity on its domain, although its codomain may be a larger set than the domain), which will happen repeatedly, we omit it from the notation, and we write $\Phi\circ\Psi$. This means that the expression $\Phi\circ\Psi$ is meaningful under the following conditions: $|\mathsf{s}\Phi| \subset |\Psi|$, and $\mathsf{t}\circ\Psi = \mathsf{s}\Phi\circ\iota$ for the inclusion $\iota\colon |\mathsf{s}\Phi| \longrightarrow |\Psi|$. Of course, similar conventions apply in case of an expression $\alpha\circ_{\iota}\Psi$.

There is an obvious *renaming law* that can be stated as

$$\Phi\circ_{\kappa}\Psi \;=\; \Phi\circ_{\kappa\circ\mu}(\Psi\circ\mu)\;;$$

here, μ is any bijection $\mu : |\mu| \xrightarrow{\cong} |\Psi|$.

Part 2: Simultaneous composition in a free multicategory and in the associated multicategory of function replacement.

Assume now that $\boldsymbol{C}$ is a free multicategory; we use the usual notation associated with a free multicategory.

Recall the HAA structure associated with $\boldsymbol{C}$. We can extend it to incorporate polycomposition. First of all, we can define

$$\langle \Phi \rangle : |\langle \Phi \rangle| \longrightarrow A$$

by

$$|\langle \Phi \rangle| = \bigsqcup_{i \in |\Phi|} |\langle \Phi i \rangle| = \{(i, p) : i \in |\Phi| , p \in |\langle \Phi i \rangle| \}$$

and

$$\langle \Phi \rangle ((i, p)) = \langle \Phi i \rangle (p) .$$

$\langle \Phi \rangle$ is a coproduct of the $\langle \Phi i \rangle$, but a *definite* coproduct: tuples have to be dealt with up to equality, not just up to isomorphism.

With this definition, we have

$$\langle \Phi \rangle \xrightarrow{\sigma} \langle \Phi \circ_{\kappa} \Psi \rangle \xleftarrow{\rho} \langle \Psi \rangle ,$$

a coproduct, derived in a straightforward way from the original (σ,ρ)-structure. Also, the extended (σ,ρ)-structure satisfies analogs of the coherence conditions originally postulated for σ and ρ .

There is an obvious variant, in fact, special case, in the form

$$\langle\alpha\rangle \xrightarrow{\ \sigma\ } \langle\alpha\circ_\kappa\Psi\rangle \xleftarrow{\ \rho\ } \langle\Psi\rangle \quad .$$

We let $\boldsymbol{D}$ be the function replacement multicategory associated with $\boldsymbol{C}$ (see Sec 1). We have simultaneous composition in both $\boldsymbol{C}$ and $\boldsymbol{D}$. We establish a distributive law connecting the two.

Of course, the (poly)arrows of $\boldsymbol{C}$ and $\boldsymbol{D}$ are the same; $\circ$ is for (poly)composition in $\boldsymbol{C}$, $\square$ is for (poly)composition in $\boldsymbol{D}$.

Suppose $(\Phi_1\circ_\kappa\Phi_2)\square_\lambda\Psi$ is well-defined. Here, we have $\lambda:T\Gamma\longrightarrow T\circ\langle\Phi_1\circ_\kappa\Phi_2\rangle$; in particular, λ is a function $\lambda:|\Gamma|\longrightarrow|\langle\Phi_1\circ_\kappa\Phi_2\rangle|$. Construct any diagram

$$\begin{array}{ccccc}
|\langle\Phi_1\rangle| & \xrightarrow{\ \sigma\ } & |\langle\Phi_1\circ_\kappa\Phi_2\rangle| & \xleftarrow{\ \rho\ } & |\langle\Phi_2\rangle| \\
\uparrow{\scriptstyle\lambda_1} & \square & \uparrow{\scriptstyle\lambda} & \square & \uparrow{\scriptstyle\lambda_2} \\
|\Psi_1| & \xrightarrow[\ \nu_1\ ]{} & |\Psi| & \xleftarrow[\ \nu_2\ ]{} & |\Psi_2|
\end{array} \tag{3}$$

in $\mathtt{Set}$, with the two pullbacks as shown. At the moment, we only have the sets $|\Psi_1|$, $|\Psi_2|$, but no Ψ_1, Ψ_2; but we can define $\Psi_1=\Psi\restriction|\Psi_1|$, $\Psi_2=\Psi\restriction|\Psi_2|$. We then clearly have the diagram

$$\begin{array}{ccccc}
T\circ\langle\Phi_1\rangle| & \xrightarrow{\ \sigma\ } & T\circ\langle\Phi_1\circ_\kappa\Phi_2\rangle| & \xleftarrow{\ \rho\ } & T\circ\langle\Phi_2\rangle| \\
\uparrow{\scriptstyle\lambda_1} & \square & \uparrow{\scriptstyle\lambda} & \square & \uparrow{\scriptstyle\lambda_2} \\
T\Psi_1 & \xrightarrow[\ \nu_1\ ]{} & T\Psi & \xleftarrow[\ \nu_2\ ]{} & T\Psi_2
\end{array} \tag{4}$$

in $(\dot{\mathtt{O}}(\boldsymbol{D}))^{\#}$; in particular $\Phi_1\square_{\lambda_1}\Psi_1$, $\Phi_2\square_{\lambda_2}\Psi_2$ are well-defined. The relations 1.(45.1), (45.2) obviously generalize to give $\Phi_j\square_{\lambda_j}\Psi_j \parallel \Phi_j$; therefore, $\kappa:t\Phi_2\to s\Phi_1$ gives $\kappa:t(\Phi_2\square_{\lambda_2}\Psi_2)\longrightarrow s(\Phi_1\square_{\lambda_1}\Psi_1)$; thus the right-hand-side in

$$(\Phi_1 \circ_\kappa \Phi_2) \square_\lambda \Psi = (\Phi_1 \square_{\lambda_1} \Psi_1) \circ_\kappa (\Phi_2 \square_{\lambda_2} \Psi_2) \qquad (5)$$

is well-defined; we claim that the equality holds. In fact, it is an easy consequence of 1.(9.1), (9.2).

We say that $\alpha \in A = \mathsf{A}(\boldsymbol{C})$ is *positive* if $|\langle \alpha \rangle|$ is non-empty.

The so-called "unique readability" property of multiarrows in a free multicategory can be expressed by saying that

(6) every positive α can be written in a unique way as $\alpha = f \circ \Phi$ with f a generator ($f \in L$), with the additional stipulation $|\Phi| = |\mathsf{s}f|$.

Recall that the notation $f \circ \Phi$ implies only that the placement of Φ in f is an inclusion; here, we required that it be even an identity.

We write $\mathsf{P}\alpha$ for f, and $\hat{\alpha}$ for Φ as in (6). Thus, we have, for positive α, that

$$\alpha = \mathsf{P}\alpha \circ \hat{\alpha}, \qquad \mathsf{t}\hat{\alpha} = \mathsf{sP}\alpha .$$

We may refer to (6) as the *canonical decomposition* of a positive multiarrow.

We will need to use decompositions $\alpha = f \circ_\kappa \Phi$ with $f \in L$ where κ is not necessarily the identity placement. Of course, in every such case, we must have that $f = \mathsf{P}\alpha$, and that the places of Φ can be renamed so that, for the resulting Φ', we have $\alpha = f \circ \Phi'$, with the implied placement ι being an inclusion; of course, we cannot always ensure that ι is an identity; however, we must have that $\Phi' = \hat{\alpha} \restriction |\Phi'|$.

Assume α positive, and we have

$$\alpha = \mathsf{P}\alpha \circ_\kappa \Phi . \qquad (7)$$

Consider the standard (ψ,φ)-diagram

$$\mathrm{s}P\alpha\backslash \mathrm{Im}\kappa \xrightarrow{\psi} \mathrm{s}\alpha \xleftarrow{\varphi} \mathrm{s}\Phi\,. \qquad (8)$$

When κ is the identity map and $\Phi=\tilde{\alpha}$, the left-hand side object is empty, and we get the isomorphism

$$\mathrm{s}\alpha \xleftarrow[\cong]{\varphi} \mathrm{s}\tilde{\alpha} \qquad (9)$$

Still under (7), and with reference to the standard coproduct diagram

$$\langle P\alpha\rangle \xrightarrow{\sigma} \langle\alpha\rangle \xleftarrow{\rho} \langle\Phi\rangle\ ,$$

we have $|\langle P\alpha\rangle| = \{*\}$; we write $p_\alpha \overline{\mathrm{def}}\ \sigma(*)$. Let us write $\langle\alpha\rangle^-$ for $\langle\alpha\rangle\backslash p_\alpha$. We conclude that we have the isomorphism

$$\langle\alpha\rangle^- \xleftarrow[\cong]{\rho} \langle\Phi\rangle \qquad (10)$$

every time (7) holds; in particular,

$$\langle\alpha\rangle^- \xleftarrow[\cong]{\rho} \langle\tilde{\alpha}\rangle$$

Let us note that if α is positive, and $\alpha\circ_\lambda\Psi$ is well-defined, then in the canonical (σ,ρ)-diagram

$$\langle\alpha\rangle \xrightarrow{\sigma} \langle\alpha\circ_\lambda\Psi\rangle \xleftarrow{\rho} \langle\Psi\rangle\ ,$$

we have

$$\sigma(p_\alpha) = p_{\alpha\circ_\lambda\Psi}\ ; \qquad (11)$$

this is easily proved, say, by induction.

An immediate consequence is that

$$P(\alpha \circ_\lambda \Psi) = P\alpha \,. \qquad (12)$$

(13) Let $\alpha,\beta \in A = A(\boldsymbol{D})$; suppose both α and β are positive. If $p = p_\alpha$, and $\alpha \square_p \beta$ is well-defined (in $\boldsymbol{D}$), then $\alpha \square_p \beta$ is positive, and, for $\varphi = \varphi_\square[\alpha, \beta, p]$, we have $\varphi(p_\beta) = p_{\alpha \square_p \beta}$; and as a consequence, $P_{\alpha \square_p \beta} = P_\beta$.

This is seen by an induction on $\ell = \#|\langle \alpha \rangle|$. When $\ell = 1$, $\alpha \square_p \beta = \beta$ and φ is the identity; so the assertion is clear. For $\ell > 1$, let $\alpha = \alpha_1 \circ_i \alpha_2$ with $\#|\langle \alpha_1 \rangle| < \ell$, $\#|\langle \alpha_2 \rangle| < \ell$. We apply 1.(9.1), (9.2) with its notation. In "case 1", by 1.(9.1), we have

$$\begin{array}{ccc} \langle (\alpha_1 \square_q \beta) \circ \alpha_2 \rangle & = & \langle \alpha \square_p \beta \rangle \\ \sigma \uparrow & \circ & \uparrow \varphi \\ \langle \alpha_1 \square_q \beta \rangle & \xleftarrow[\varphi']{} & \langle \beta \rangle \end{array}$$

Since $\sigma(q) = p_\alpha$, by (11) and σ being 1-1, we have $q = p_{\alpha_1}$. Therefore, by the induction hypothesis, $\varphi'(p_\beta) = p_{\alpha_1 \square_q \beta}$. By (11) again,

$$\sigma(p_{\alpha_1 \square_q \beta}) = p_{(\alpha_1 \square_q \beta) \circ_i \alpha_2} = p_{\alpha \square_q \beta} \;;$$

thus

$$\varphi(p_\beta) = \sigma\varphi'(p_\beta) = p_{\alpha \square_q \beta}$$

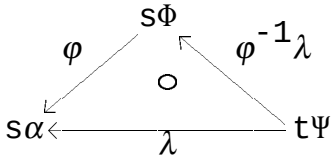
as promised.

Next, we assume (7) , and we investigate the canonical decomposition of polycomposites both in ***C*** and ***D*** in which α is the first factor.

(14) Assume (7), $\alpha\circ_\lambda\Psi$ is well-defined (in ***C***), and $\mathrm{Im}\lambda\cap\mathrm{Im}\psi=\emptyset$, with ψ from (8). Then

$$\alpha\circ_\lambda\Psi = \mathrm{P}\alpha\circ_\kappa(\Phi\circ_{\varphi^{-1}\lambda}\Psi)$$

with $\varphi^{-1}\lambda$ determined from



Of course, this is proved by using the GAL. (8) being a coproduct, the last condition implies that $\mathrm{Im}\lambda\subset\mathrm{Im}\varphi$ (with φ from (8)). Since the functions φ and λ are 1-1, this easily implies that there is $\varphi^{-1}\lambda$ as in the diagram. By the GAL, this $\varphi^{-1}\lambda$ is suitable for the required equality.

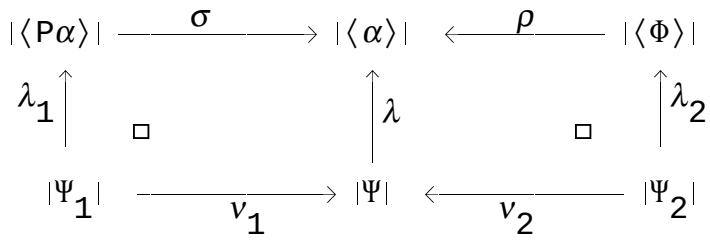
(15) Assume (7), $\alpha\square_\lambda\Psi$ is well-defined (in ***D***), and $\mathrm{p}_\alpha\notin\mathrm{Im}(\lambda)$. Then, with ρ taken from (10), we have

$$\alpha\square_\lambda\Psi = (\mathrm{P}\alpha)\circ_\kappa(\Phi\square_{\rho^{-1}\lambda}\Psi) \ .$$

Proof By the distributive law (5)**,**

$$\alpha\square_\lambda\Psi = (\mathrm{P}\alpha\circ_\kappa\Phi)\square_\lambda\Psi = (\mathrm{P}\alpha\square_{\lambda_1}\Psi_1)\circ_\kappa(\Phi\square_{\lambda_2}\Psi_2)$$

with any



in particular, we may take ν_1 , ν_2 to be inclusions. We have $p_\alpha \notin \mathrm{Im}(\lambda)$, $|\langle P\alpha\rangle| = \{*\}$, $\sigma(*)=p_\alpha$, and that (σ, ρ) is a coproduct. These facts have three consequences. One is that ρ indeed gives an isomorphism (10) -- as we already know. The second is that σ and λ have disjoint images in $|\langle \gamma\rangle|$, hence, $\Psi_1=\emptyset$, and $P\alpha \sqcup_{\lambda_1} \Psi_1 = P\alpha$. The third is that (ν_1, ν_2) is a coproduct. It follows that ν_2 is the identity map, $\Phi_2=\Phi$, and $\lambda_2=\rho^{-1}\circ\lambda$.

The claimed equality follows.