We turn to the proofs of the multicategory laws in $\boldsymbol{E}_n$. Remember: n is fixed, and we work with the induction hypothesis that "all is well up to $n\text{-}1$".

First, we make a purely notational move.

Let $\alpha \in P_{n-1}$, $p \in |\langle \alpha \rangle|$. For $u \in X_{\geq n-1}$, u *fits into* α *at* p if $\mathsf{d}^{(n-2)}u = [\![\mathsf{d}(\langle \alpha \rangle(p))]\!]$, $\mathsf{c}^{(n-2)}u = [\![\mathsf{c}(\langle \alpha \rangle(p))]\!]$. In this case,

$$b \operatorname{d\bar{e}f} (u, d(\langle \alpha \rangle(p)), c(\langle \alpha \rangle(p)))$$

defines an element $b \in N_{n-1}$, and $\alpha \square_p b \in Q_{n-1}$ is well-defined. When we want to be specific we write $b[u, \alpha, p]$ for b.

Since b is determined from α and u, we may write $\alpha \square_p u$ to mean $\alpha \square_p b$; for $u \in X_{\geq n-1}$, $\alpha \square_p u \in Q_{n-1}$ is well-defined iff u fits into α at p.

Note that if $b \in N_{n-1}$ is any element, and $\alpha \square_p b \in Q_{n-1}$ is well-defined, then, for $u = [\![b]\!]$, we have $b = b[u, \alpha, p]$; thus, we have the formula

$$\alpha \square_p b = \alpha \square_p [\![b]\!] . \qquad (19)$$

As a consequence of 1.(9.1),(9.2), we have, under the hypothesis that u fits into α at p, and with the notation of 1.(9.1),(9.2), that

$$\alpha \square_p u = (\alpha_1 \square_q u) \circ_i \alpha_2 \qquad \text{(case 1 of 1.(9.1))} \qquad (20)$$
$$\alpha \square_p u = \alpha_1 \circ_i (\alpha_2 \square_q u) \qquad \text{(case 2 of 1.(9.2))} \qquad (21)$$

Let $\alpha \in P_{n-1}$, $p \in |\langle \alpha \rangle|$, $u \in X_{>n-1}$ such that u fits into α at p. Then $\mathsf{d}u$, $\mathsf{c}u$ fit into α at p as well (obviously), and we have

$$\mathsf{d}\,[\![\alpha \square_p u]\!] = [\![\alpha \square_p \mathsf{d}u]\!] , \qquad (22)$$
$$\mathsf{c}\,[\![\alpha \square_p u]\!] = [\![\alpha \square_p \mathsf{c}u]\!] . \qquad (23)$$

These are immediate consequences of $[11]_{n-1}(1)$ and (2), together with the definition of D

and the relation $[10]_{n-1}$(2'); observe that $\mathtt{D}b[u, \alpha, p] = b[\mathtt{d}u, \alpha, p]$.

Here is a useful convention: in an ω-category $\boldsymbol{X}$, we define $x \circ_n y$ for all $x, y \in X_{\geq n}$ (thus, x or y or both may belong to X_n) under the condition $\mathtt{d}^{(n)} x = \mathtt{c}^{(n)} y$ -- and of course, when (e.g.) $x \in X_n$, then $\mathtt{d}^{(n)} x = x$. If in fact $x \in X_n$, then $x \circ_n y = y$; when both $x, y \in X_n$, the condition of composability is that $x = y$, and the composite $x \circ_n y = x = y$.

(24) Lemma ("Functoriality of generalized whiskering")

Let $u, v \in X_{\geq n-1}$, $\alpha \in Q_{n-1}$, $p \in |\langle \alpha \rangle|$. Assume that u fits into α at p , and $u \circ_{n-1} v$ is well-defined ($\mathtt{c}^{(n-1)} v = \mathtt{d}^{(n-1)} u$). Then u and $u \circ_{n-1} v$ fit into α at p as well (obviously), and we have

$$[\![\alpha \square_p u]\!] \circ_{n-1} [\![\alpha \square_p v]\!] = [\![\alpha \square_p (u \circ_{n-1} v)]\!] \quad . \tag{25}$$

Proof The proof is by induction on $\# |\langle \alpha \rangle|$. We deal with the case $n \geq 2$.

When $\# |\langle \alpha \rangle| = 1$, $|\langle \alpha \rangle| = \{p\}$, we have, for $b = b[\alpha, u, p]$, that $\alpha \square_p b = b$ and $[\![\alpha \square_p u]\!] = u$; and similarly for v and $u \circ_{n-1} v$ in place of u ; thus, (25) is just $u \circ_{n-1} v = u \circ_{n-1} v$.

When $\# |\langle \alpha \rangle| > 1$, we have two cases as in 1.(9.1),(9.2); we deal with the first one. By (20), we obtain, for w any one of u , v , $u \circ_{n-1} v$,

$$\begin{aligned} [\![\alpha \square_p w]\!] &= [\![\alpha_1 \square_q w)]\!] \circ_{n-2} [\![\mathtt{d}(\alpha_1 \square_q w)) \square_i \mathtt{B}\|\alpha_2\|]\!] \quad . \\ &= [\![\alpha_1 \square_q w)]\!] \circ_{n-2} [\![d\alpha_1 \square_i \mathtt{B}\|\alpha_2\|]\!] \\ &= [\![\alpha_1 \square_q w)]\!] \circ_{n-2} x \end{aligned}$$

with $x = [\![\mathtt{d}\alpha_1 \square_i \mathtt{B}\|\alpha_2\|]\!]$, independently of w .

By the induction hypothesis,

$$[\![\alpha_1 \square_q (u \circ_{n-1} v)]\!] = [\![\alpha_1 \square_q u)]\!] \circ_{n-1} [\![\alpha_1 \square_q v)]\!] \quad .$$

Thus,

$$
\begin{aligned}
[\![\alpha \Box_p (u \circ_{n-1} v)]\!] &= [\![\alpha_1 \Box_q (u \circ_{n-1} v))]\!] \circ_{n-2} x = \\
&= ([\![\alpha_1 \Box_q u)]\!] \circ_{n-1} [\![\alpha_1 \Box_q v)]\!]) \circ_{n-2} x \\
&= ([\![\alpha_1 \Box_q u)]\!] \circ_{n-2} x) \circ_{n-1} ([\![\alpha_1 \Box_q v)]\!] \circ_{n-2} x) \\
&\uparrow \\
&= [\![\alpha \Box_p u]\!] \circ_{n-1} [\![\alpha \Box_p v]\!] ,
\end{aligned}
$$

$\uparrow$ by the law of functoriality of whiskering in the ω-cat $\boldsymbol{X}$.

Proof of the associative law for $\boldsymbol{E}_n$:

$$
(a \Diamond_p b) \Diamond_{\bar{q}}^{?} c = a \Diamond_p (b \Diamond_q c)
$$

$$
(\, a, b, c \in N_n \,,\ p \in |\langle \mathrm{d}a \rangle| \,,\ q \in |\langle \mathrm{d}b \rangle| \,,\ \varphi : \langle \mathrm{d}b \rangle \to \langle \mathrm{d}(a \Diamond_p b) \rangle \,,\ \bar{q} = \varphi(q) \,,
$$
$$
\mathrm{c}b = \langle \mathrm{d}a \rangle (p) \,,\ \mathrm{c}c = \langle \mathrm{d}b \rangle (q) \,)
$$

It suffices to show

$$
[\![(a \Diamond_p b) \Diamond_{\bar{q}} c]\!] = [\![a \Diamond_p (b \Diamond_q c)]\!] \ .
$$

We have

$$
\begin{aligned}
[\![(a \Diamond_p b) \Diamond_{\bar{q}} c]\!] &= [\![a \Diamond_p b]\!] \circ_{n-1} [\![\mathrm{d}(a \Diamond_p b) \Box_{\bar{q}} \mathrm{B}c]\!] \\
&= ([\![a]\!] \circ_{n-1} [\![\mathrm{d}a \Box_p \mathrm{B}b]\!]) \circ_{n-1} [\![(\mathrm{d}a \Box_p \mathrm{d}b) \Box_{\bar{q}} \mathrm{B}c]\!] \,, \qquad (26)
\end{aligned}
$$

and

$$
[\![a \Diamond_p (b \Diamond_{\bar{q}} c)]\!] = [\![a]\!] \circ_{n-1} [\![\mathrm{d}a \Box_p \mathrm{B}(b \Diamond_q c)]\!] \qquad (27)
$$

Using the associative law for $\circ_{n-1}$ in $\boldsymbol{X}$, and by

$$
(\mathrm{d}a \Box_p \mathrm{d}b) \Box_{\bar{q}} \mathrm{B}c = \mathrm{d}a \Box_p (\mathrm{d}b \Box_q \mathrm{B}c)
$$

(associative law in $\boldsymbol{G}_{n-1}$), we see that, for the equality of (26) and (27), it is enough to

show:

$$[\![\mathtt{d}a\Box_p \mathtt{B}(b\Diamond_q c)]\!] \overset{?}{=} [\![\mathtt{d}a\Box_p \mathtt{B}b]\!] \circ_{n-1} [\![\mathtt{d}a\Box_p (\mathtt{d}b\Box_q \mathtt{B}c)]\!] \ . \qquad (28)$$

But this becomes a case of the last Lemma as follows. Let $\alpha=\mathtt{d}a\in P_{n-1}$, $u=[\![b]\!]$, $v=[\![\mathtt{d}b\Box_q \mathtt{B}c]\!]$. Then,

$$[\![\mathtt{B}(b\Diamond_q c)]\!] = [\![b\Diamond_q c]\!] \overset{\uparrow}{=} [\![b]\!]\circ_{n-1}[\![\mathtt{d}b\Box_q \mathtt{B}c]\!] = u\circ_{n-1}v\ ,$$

$\uparrow$ by the definition of composition in $\boldsymbol{E}_n$. Now, also using (19) (three times), (28) becomes

$$[\![\alpha\Box_p(u\circ_{n-1}v)]\!] = [\![\alpha\Box_p u]\!] \circ_{n-1} [\![\alpha\Box_p v]\!] \ ,$$

which is true by the Lemma.

Generalized middle exchange

We next prepare the way for the verification of the commutative law for $\boldsymbol{E}_n$. Our immediate goal is to produce some identities that generalize the middle exchange law in $\boldsymbol{X}$.

Let $\alpha\in Q_{n-1}$, $p\neq q$, $p, q\in |\langle\alpha\rangle|$, $u, v\in X_{\geq n-1}$. Assume that u fits into α at p, v fits into α at q. Since u and v are generating arrows for the free multicategory $\boldsymbol{F}_{n-1}$, the amalgamating maps $\psi_1:\langle\alpha\rangle\backslash p\to\langle\alpha\Box_p u\rangle$, $\psi_2:\langle\alpha\rangle\backslash q\to\langle\alpha\Box_q v\rangle$ are inclusion maps of the corresponding sets; $\psi_1(q)=q$, $\psi_2(p)=p$.

We fix a number ℓ with $n\text{-}1\leq\ell$, $\ell<\dim(u)$, $\ell<\dim(v)$; we are mostly interested in the case $\ell=n\text{-}1$. We will abbreviate the operators $\mathtt{d}^{(\ell)}$, $\mathtt{c}^{(\ell)}$ by Δ, Γ, respectively.

Using the notation of simultaneous composition, we write

$$\alpha(u, v) = \alpha(\Box_p u, \Box_q v) = (\alpha\Box_p u)\Box_q v = (\alpha\Box_q v)\Box_p u\ ;$$

and similarly with either Δu or Γu replacing u, and also similarly for v. As a further abbreviation, instead of $[\![\alpha(u, v)]\!]$, we write $A(u, v)$, and similarly for other

expressions like this. Now, consider the following diagrams:

$$A(\Delta u,\ \Delta v) \xrightarrow{\ A(u,\ \Delta v)\ } A(\Gamma u,\ \Delta v) \xrightarrow{\ A(\Gamma u,\ v)\ } A(\Gamma u,\ \Gamma v) \qquad (29)$$

$$A(\Delta u,\ \Delta v) \xrightarrow{\ A(\Delta u,\ v)\ } A(\Delta u,\ \Gamma v) \xrightarrow{\ A(u,\ \Gamma v)\ } A(\Gamma u,\ \Gamma v) \qquad (30)$$

$$A(\Delta u,\ \Delta v) \xrightarrow{\qquad A(u,\ v)\qquad} A(\Gamma u,\ \Gamma v)\ . \qquad (31)$$

First of all, they are correct in the sense that for each arrow, say w, the tail-end of w shown is Δw, the head is Γw: this is seen by (22) and (23).

The main claim is that the composites in (29) and (30) are both equal to (31):

$$A(\Gamma u,\ v) \circ_{\ell} A(u,\ \Delta v) = A(u,\ \Gamma v) \circ_{\ell} A(\Delta u,\ v) = A(u,\ v)\ . \qquad (32)$$

It is enough to show that

$$A(\Gamma u,\ v) \circ_{\ell} A(u,\ \Delta v) = A(u,\ v)\ ; \qquad (32')$$

the other equality will then hold for reasons of symmetry.

The proof of (32') is similar to that of (24), but no induction is needed. We start by writing α in the form

$$\alpha = \alpha_1 \circ_i \alpha_2 \qquad (33)$$

such that, for the associated amalgamating maps $\langle \alpha_1 \rangle \xrightarrow{\ \sigma\ } \langle \alpha \rangle \xleftarrow{\ \rho\ } \langle \alpha_2 \rangle$, we have

$$p \in \mathrm{Im}(\sigma) \ \text{ and } \ q \in \mathrm{Im}(\rho)\ , \qquad (34)$$

or the other way around:

$$q \in \mathrm{Im}(\sigma) \ \text{ and } \ p \in \mathrm{Im}(\rho)\ ; \qquad (35)$$

the places p and q are "separated". We will argue only informally that this is possible. Thinking of α as a term (in the universal-algebraic sense), we have in it the particular

occurrences of the function symbols $f=\langle\alpha\rangle(p)$ and $g=\langle\alpha\rangle(q)$ singled out by p and q, respectively; we talk about these occurrences simply as f and g. Note that either f is not in the scope of g, or g is not in the scope of f (possibly both alternatives hold). In the first case, we take the subterm of α, call it α_2, whose principal (head-) operator is g, and remove it from α; the remainder is α_1. In this way we have done the desired decomposition (33) of α with (34) because f must be in α_1, and g is in α_2. The second case is symmetric.

Let us then assume (33) with (34); $p=\sigma(p')$, $q=\rho(q')$. We write $\alpha_1(u)$ for $\alpha_1\Box_p,u$, $A_1(u)$ for $[\![\alpha_1\Box_p,u]\!]$, and similarly for other expressions appearing below.

Note that, by two applications of 1.(9.1),(9.2), we have

$$\alpha(u, v) = \alpha_1(u)\circ_i\alpha_2(v),$$

and similarly when u and/or v is replaced by Δu, Γu, resp. by Δv, Γv. Using

$$\mathrm{d}(\alpha_1\Box_p,u) = \mathrm{d}(\alpha_1\Box_p,\Delta u) = \mathrm{d}(\alpha_1\Box_p,\Gamma u) = \mathrm{d}\alpha_1,$$

applying the morphism $\|\text{-}\|:\boldsymbol{F}_{n-1}\longrightarrow\boldsymbol{E}_{n-1}$, and the definition of composition in $\boldsymbol{E}_{n-1}$, one gets

$$\begin{aligned}
A(u, v) &= A_1(u)\circ_{n-2}[\![\mathrm{d}\alpha_1\Box_i A_2(v)]\!]\\
A(\Delta u, \Delta v) &= A_1(\Delta u)\circ_{n-2}[\![\mathrm{d}\alpha_1\Box_i A_2(\Delta v)]\!]\\
A(\Gamma u, \Gamma v) &= A_1(\Gamma u)\circ_{n-2}[\![\mathrm{d}\alpha_1\Box_i A_2(\Gamma v)]\!]\\
A(u, \Delta v) &= A_1(u)\circ_{n-2}[\![\mathrm{d}\alpha_1\Box_i A_2(\Delta v)]\!]\\
A(\Gamma u, v) &= A_1(\Gamma u)\circ_{n-2}[\![\mathrm{d}\alpha_1\Box_i A_2(v)]\!]\\
A(\Gamma u, \Delta v) &= A_1(\Gamma u)\circ_{n-2}[\![\mathrm{d}\alpha_1\Box_i A_2(\Delta v)]\!]
\end{aligned}$$

Now, $A_1(\Delta u)=\Delta A_1(u)$; and we have other analogous identities. Furthermore,

$$[\![\mathrm{d}\alpha_1\Box_i\Delta A_2(v)]\!] = \Delta[\![\mathrm{d}\alpha_1\Box_i A_2(v)]\!].$$

Writing $x_1 = A_1(u)$, $x_2 = [\![\mathrm{d}\alpha_1\Box_i A_2(v)]\!]$, we obtain

$$\begin{aligned}
A(u, v) &= x_1\circ_{n-2}x_2\\
A(\Delta u, \Delta v) &= \Delta x_1\circ_{n-2}\Delta x_2
\end{aligned}$$

$$A(\Gamma u,\, \Gamma v) = \Gamma a_1 \circ_{n-2} \Gamma a_2$$
$$A(u,\, \Delta v) = x_1 \circ_{n-2} \Delta x_2$$
$$A(\Gamma u,\, v) = \Gamma x_1 \circ_{n-2} x_2$$
$$A(\Gamma u,\, \Delta v) = \Gamma x_1 \circ_{n-2} \Delta x_2$$

The protagonists line up in the diagram:

$$\begin{array}{ccc} \Delta x_2 & \overset{x_2}{\Longrightarrow} & \Gamma x_2 \\ \downarrow & & \downarrow \\ \Delta x_1 & \underset{x_1}{\Longrightarrow} & \Gamma x_1 \end{array}$$

and (29), (31) become

$$\Delta x_1 \circ_{n-2} \Delta x_2 \xrightarrow{x_1 \circ_{n-2} \Delta x_2} \Gamma x_1 \circ_{n-2} \Delta x_2 \xrightarrow{\Gamma x_1 \circ_{n-2} x_2} \Gamma x_1 \circ_{n-2} \Gamma x_2$$

and

$$\Delta x_1 \circ_{n-2} \Delta x_2 \xrightarrow{x_1 \circ_{n-2} x_2} \Gamma x_1 \circ_{n-2} \Gamma x_2 \quad .$$

We now see that the assertion (32) (i.e., that the composite of (29) is (31)) is an instance of the $(n-2,\, \ell)$-middle exchange law for the ω-cat $\boldsymbol{X}$.

Proof of the commutative law for the composition in $\boldsymbol{E}_n$:

$$(a \Diamond_p b) \Diamond_{\bar{q}} c \overset{?}{=} (a \Diamond_q c) \Diamond_{\bar{p}} b$$

under the conditions

$$a\,,\ b\,,\ c \in N_n\,,$$
$$p,\, q \in |\langle \mathrm{d}a \rangle|\,,\ p \neq q\,,\ \mathrm{c}b = \langle \mathrm{d}a \rangle(p)\,,\ \mathrm{c}c = \langle \mathrm{d}a \rangle(q)\,,$$

$\psi_1 : \langle \mathrm{d}a \rangle \backslash p \longrightarrow \langle \mathrm{d}(a \Diamond_p b) \rangle$, $\psi_2 : \langle \mathrm{d}a \rangle \backslash q \longrightarrow \langle \mathrm{d}(a \Diamond_q c) \rangle$,
$\bar{q} = \psi_1(q)$, $\bar{p} = \psi_2(p)$.

It is enough to see that the $[\![-]\!]$-values of the two sides are equal. We have

$$\begin{aligned} [\![(a \Diamond_p b) \Diamond_{\bar{q}} c]\!] &= [\![a \Diamond_p b]\!] \circ_{n-1} [\![\mathrm{d}(a \Diamond_p b) \Box_{\bar{q}} \mathrm{B}c]\!] \\ &= [\![a]\!] \circ_{n-1} [\![\mathrm{d}a \Box_p \mathrm{B}b]\!] \circ_{n-1} [\![(\mathrm{d}a \Box_p \mathrm{d}b) \Box_{\bar{q}} \mathrm{B}c]\!] \end{aligned} \qquad (36)$$

and

$$[\![(a \Diamond_q c) \Diamond_{\bar{p}} b]\!] = [\![a]\!] \circ_{n-1} [\![\mathrm{d}a \Box_q \mathrm{B}c]\!] \circ_{n-1} [\![(\mathrm{d}a \Box_q \mathrm{d}c) \Box_{\bar{p}} \mathrm{B}b]\!] \ . \qquad (37)$$

Let $u = [\![b]\!]$, $v = [\![c]\!]$, $\alpha = \mathrm{d}a$; and let us use the notation introduced above, with $\ell = n-1$: $\Delta = \mathrm{d}^{(n-1)}$, $\Gamma = \mathrm{c}^{(n-1)}$, $A(u, v) = [\![\alpha(u, v)]\!]$, etc. Then we have

$$\begin{aligned} &[\![\mathrm{B}b]\!] = u \ , \ [\![\mathrm{B}c]\!] = v \ , \ [\![\mathrm{d}b]\!] = \Delta u \ , \text{ etc,;} \\ &[\![(\mathrm{d}a \Box_p \mathrm{d}b) \Box_{\bar{q}} \mathrm{B}c]\!] = A(\Delta u, v) \ , \\ &[\![(\mathrm{d}a \Box_q \mathrm{d}c) \Box_{\bar{p}} \mathrm{B}b]\!] = A(u, \Delta v) \ . \end{aligned}$$

Because of the relations $\mathrm{c}b = \langle \mathrm{d}a \rangle(p)$, $\mathrm{c}c = \langle \mathrm{d}a \rangle(q)$, we have $\alpha \Box_p \Gamma b = \alpha \Box_q \Gamma c = \alpha$; this is the reason why

$$\begin{aligned} [\![\mathrm{d}a \Box_p \mathrm{B}b]\!] &= A(u, \Gamma v) \ , \\ [\![\mathrm{d}a \Box_q \mathrm{B}c]\!] &= A(\Gamma u, v) \ . \end{aligned}$$

Now we see that the equality of the expressions on (36) and (37) reduces to

$$A(u, \Gamma v) \circ_{n-1} A(\Delta u, v) = A(\Gamma u, v) \circ_{n-1} A(u, \Delta v) \ ,$$

which is (32) above, with $\ell = n-1$.

This completes the proof that $\boldsymbol{E}_n$ is indeed a multicategory.

We turn to $[10]_n$: the definition of $\|-\| : \boldsymbol{F}_n \to \boldsymbol{E}_n$.

The definition is obvious. $\boldsymbol{F}_n$ is the nth free multicategory associated with $\boldsymbol{N}_n$. It is freely generated by the multiarrows of $\boldsymbol{E}_n$, that is, the elements of N_n. To specify $\|-\| : \boldsymbol{F}_n \to \boldsymbol{E}_n$, we first make its action

$$M_{n-1}=O(\boldsymbol{F}_n) \longrightarrow M_{n-1}=O(\boldsymbol{E}_n)$$

on objects the identity map. The identity map

$$N_n=L(\boldsymbol{F}_n) \longrightarrow N_n=A(\boldsymbol{E}_n)$$

respects source and target. The universal property of $\boldsymbol{F}_n$ provides a unique morphism $\|-\| : \boldsymbol{F}_n \to \boldsymbol{E}_n$ which is the identity on objects, and for which $[10]_n(3)$ holds.

The commutativity $[10]_n(4)$ holds, since it holds on the generators of $\boldsymbol{F}_n$ that are the multiarrows of $\boldsymbol{E}_n$. Since both d's in $[10]_n(4)$ are both strict [in the case of $\boldsymbol{F}_n$, this is because of the dispensation made in the definition of "multitopic set"], it follows that $\|-\| : \boldsymbol{F}_n \to \boldsymbol{E}_n$ is also strict.

The item in $[3]_n$: the evaluation map $[\![-]\!] : Q_n \to X_{\geq n}$ can be defined only now (although its restriction to $[\![-]\!] : N_n \to X_{\geq n}$ has already been introduced). We define, for $\alpha \in Q_n$, by [10](2'), read in the reverse order: $[\![\alpha]\!] \stackrel{\text{def}}{=} [\![\|\alpha\|]\!]$. By $[10]_n(3)$, $[\![-]\!] : Q_n \to X_{\geq n}$ so defined indeed extends $[\![-]\!] : N_n \to X_{\geq n}$. Since the last map restricts to $[\![-]\!] : M_n \to X_n$, it is trivial to show that $[\![-]\!] : Q_n \to X_{\geq n}$ restricts to $[\![-]\!] : P_n \to X_n$.

The lower commutativity in $[10]_n(1)$ is contained in $[10]_n(4)$. The upper part of $[10]_n(1)$ is true since $\|-\| : \boldsymbol{F}_n \to \boldsymbol{E}_n$ preserves target, and on both $\boldsymbol{F}_n$ and $\boldsymbol{E}_n$ target is just applying c.

Finally, we establish [11] for n.

The specification of D, C is fully stated in $[11]_n$; it remains to show that they preserve identities and composition.

Identities: this is obvious, since $\mathtt{D}$ and $\mathtt{C}$ act as the identity on the subset M_n of N_n.

Composition:

$$\mathtt{D}(a \Diamond_p b) \stackrel{?}{=} (\mathtt{D}a) \Diamond_p (\mathtt{D}b) .$$

We have

$$[\![\mathtt{D}(a \Diamond_p b)]\!] = \tilde{\mathtt{d}}([\![a]\!] \circ_{n-1} [\![\mathtt{d}a \Box_p \mathtt{B}b]\!]) = (\tilde{\mathtt{d}}[\![a]\!]) \circ_{n-1} \tilde{\mathtt{d}}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!] \quad (38)$$

$$[\![(\mathtt{D}a) \Diamond_p (\mathtt{D}b)]\!] = [\![\mathtt{D}a]\!] \circ_{n-1} [\![\mathtt{d}\mathtt{D}a \Box_p \mathtt{B}\mathtt{D}b]\!] = (\tilde{\mathtt{d}}[\![a]\!]) \circ_{n-1} [\![\mathtt{d}a \Box_p \mathtt{B}\mathtt{D}b]\!] \quad (39)$$

Note that $\dim [\![\mathtt{d}a \Box_p \mathtt{B}b]\!] = \dim [\![b]\!]$.

There are two cases: $\dim [\![b]\!] = n$ (case 1), and $\dim [\![b]\!] > n$ (case 2). In case 1, $\tilde{\mathtt{d}}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!] = [\![\mathtt{d}a \Box_p \mathtt{B}b]\!]$ and $\mathtt{D}b = b$; thus (38) and (39) are equal. In case 2, $\mathtt{B}\mathtt{D}b = \mathtt{D}_{n-1}\mathtt{B}b$, and

$$\tilde{\mathtt{d}}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!] = \mathtt{d}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!] = \tilde{\mathtt{d}}_{n-1}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!] .$$

Therefore, the equality

$$\tilde{\mathtt{d}}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!] = [\![\mathtt{d}a \Box_p \mathtt{B}\mathtt{D}b]\!]$$

is equivalent to

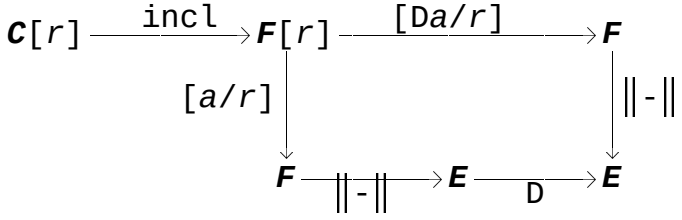
$$\tilde{\mathtt{d}}_{n-1}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!] = [\![\mathtt{d}a \Box_p \mathtt{D}_{n-1}\mathtt{B}b]\!] ,$$

which is true by $[11]_{n-1}$.

Next, we show $[11]_n(1)$ and $[11]_n(2)$. Of course, the two are similar; we treat the first one.

Consider the following diagram of multicategories and their morphisms; the indeterminate r

is chosen parallel to $\langle\alpha\rangle(p)$; we have dropped the subscript n :



I claim that the two composites $\boldsymbol{C}[r] \longrightarrow \boldsymbol{E}$ are equal. It suffices to check this on the generators on $\boldsymbol{C}[r]$, which are those of $\boldsymbol{C}$ and r . On a generator of $\boldsymbol{C}$, all morphisms in the diagram act as the identity. On r , one of the composites results in $\|\mathtt{D}a\|$, on the other $\mathtt{D}\|a\|$. But on $a \in N_n$, and on $\mathtt{D}a \in N_n$, $\|\text{-}\|$ acts as the identity ($[10]_n(3)$). Therefore, the two values are equal. The claim is proved.

When we apply said commutativity on $\alpha(r/p)$ (remember $\alpha \in P_n = \mathtt{A}(\boldsymbol{C}_n)$), we obtain the desired equality [11](1).

This completes the proof of the Theorem.

The Theorem defines the object function $\boldsymbol{X} \mapsto \boldsymbol{M} = [\boldsymbol{X}]$ of the functor $[\text{-}] : \omega\mathtt{Cat} \longrightarrow \mathtt{mltSet}$. The action of the functor on arrows is now easy to specify.

Suppose we have $\Gamma : \boldsymbol{X}^1 \longrightarrow \boldsymbol{X}^2$; we define the morphism $\Phi = [\Gamma] : \boldsymbol{M}^1 \longrightarrow \boldsymbol{M}^2$ as follows.

Recall from section 1 that to specify Φ , we define the components $\Phi_n : M^1_n \longrightarrow M^2_n$ recursively. For the purposes of the induction, we define the multicategory morphism $\Phi \mid n : \boldsymbol{N}^1_n \longrightarrow \boldsymbol{N}^2_n$ by recursion on n ; the component $(\Phi \mid n)_n : N^1_n \longrightarrow N^2_n$ will give rise, by restriction, to $\Phi_n : M^1_n \longrightarrow M^2_n$.

We will write simply Φb for $(\Phi \mid n)_k(b)$ ($b \in (\bigcup_{k<n} M^1_k) \cup N^1_n$) , and employ other similar abbreviations.

Assume $\Phi : \boldsymbol{N}^1_n \longrightarrow \boldsymbol{N}^2_n$ has been defined; as an induction hypothesis, we assume the following:

(i)$_n$ For $a \in N^1_n$, $[\![\Phi a]\!] = \Gamma [\![a]\!]$, $d\Phi a = \Phi da$ and $c\Phi a = \Phi ca$;

(ii)$_n$ Φ induces a morphism of multicategories $\Lambda : \boldsymbol{E}^1_n \longrightarrow \boldsymbol{E}^2_n$; its action on objects is $\Phi : M^1_{n-1} \to M^2_{n-1}$, its action on arrows is $\Phi : N^1_n \to N^2_n$; and we have

$$\begin{array}{ccc} \boldsymbol{E}^1_n & \xrightarrow{\ \Lambda\ } & \boldsymbol{E}^2_n \\ {\scriptstyle d}\downarrow & \circ & \downarrow{\scriptstyle d} \\ \boldsymbol{D}^1_{n-1} & \xrightarrow[\ \Psi\ ]{} & \boldsymbol{D}^2_{n-1} \end{array} \qquad (40)$$

(The only information in the diagram (40), in addition to Λ being a multicategory morphism and not contained already in (i), concerns the place-actions of Λ. In section 1, we denoted the morphism $\boldsymbol{D}^1_{n-1} \to \boldsymbol{D}^2_{n-1}$ induced by Φ by Ψ_{n-1}; here we write Ψ instead.)

We specify $\Phi = \Phi \mid (n+1) : \boldsymbol{N}^1_{n+1} \longrightarrow \boldsymbol{N}^2_{n+1}$.

The restriction of Φ to $(\leq n)$-cells of the multitopic set $\boldsymbol{N}^1_{n+1}$ is given by $\Phi \mid n$.

Let $a \in N^1_{n+1}$, to define $\Phi a = \Phi_{n+1}(a) = \Phi \mid (n+1) \in N^2_{n+1}$. We put

$$\Phi a \ \overline{\text{def}} \ (\Gamma [\![a]\!], \Phi(da), \Phi(ca)) . \qquad (40')$$

We need to check two things (see 2.[4](a) and (b)):

(a?): $\Phi(da) \parallel \Phi(ca)$;

this follows since $da \parallel ca$; and

(b?): $d^{(n)} \Gamma [\![a]\!] = [\![\Phi(da)]\!]$, $c^{(n)} \Gamma [\![a]\!] = [\![\Phi(ca)]\!]$.

To see (b?), recall that the morphism $\Phi : \boldsymbol{N}^1_n \longrightarrow \boldsymbol{N}^2_n$ of multitopic sets induces a map

$\Phi : \boldsymbol{F}^1_n \longrightarrow \boldsymbol{F}^2_n$ of multicategories, since $\boldsymbol{F}^i_n = \mathbf{C}_n[\boldsymbol{N}^i_n]$. With Λ taken from $(\mathrm{ii})_n$, the diagram

$$\begin{array}{ccc} \boldsymbol{F}^1_n & \xrightarrow{\ \Phi\ } & \boldsymbol{F}^2_n \\ \Big\| \text{-} \Big\| \downarrow & \circ & \downarrow \Big\| \text{-} \Big\| \\ \boldsymbol{E}^1_n & \xrightarrow[\ \Lambda\ ]{} & \boldsymbol{E}^2_n \end{array} \qquad (41)$$

commutes, since it does on the generators of $\boldsymbol{F}^1_n$; in turn, this latter fact follows from (40).

Applied to $\alpha \in \mathrm{A}(\boldsymbol{F}^1_n) = N^1_n$, (i) and the commutativity (41) implies that

$$[\![\Phi\alpha]\!] = \Gamma [\![\alpha]\!] ,$$

or more completely,

$$\Phi\alpha = (\Gamma [\![\alpha]\!], \Phi \mathrm{d}\alpha, \Phi \mathrm{c}\alpha) \qquad (41')$$

Now (b?) follows: $[\![\Phi(\mathrm{d}a)]\!] = \Gamma [\![\mathrm{d}a]\!] = \Gamma \mathrm{d}^{(n)} [\![a]\!] = \mathrm{d}^{(n)} \Gamma [\![a]\!]$; and similarly for the other equality.

We have shown that the definition of Φ_{n+1} is legitimate.

To complete the induction, we need to show $(\mathrm{i})_{n+1}$ and $(\mathrm{ii})_{n+1}$. $(\mathrm{i})_{n+1}$ is immediate from the definition.

For $(\mathrm{ii})_{n+1}$, we need

$$\Phi(a \Diamond_p b) \stackrel{?}{=} \Phi a \Diamond_{\Phi p} \Phi b \qquad (a, b \in N^1_n,\ a \Diamond_p b \text{ is well-defined}) \qquad (42)$$

Taking the $[\![\text{-}]\!]$-values of each side, we want (among others)

$$\Gamma([\![a]\!] \circ_n [\![\mathrm{d}a \,\square_p\, \mathrm{B}b]\!]) \stackrel{?}{=} \Gamma [\![a]\!] \circ_n [\![\mathrm{d}\Phi a \,\square_p\, \mathrm{B}\Phi b]\!] \ . \qquad (43)$$

But, for $\alpha=\mathtt{d}a$, $\beta=\mathtt{B}b$, both in $Q^1_n=\mathtt{Arr}(\boldsymbol{G}^1_n)$, we have, by the morphism $\Psi:\boldsymbol{G}^1_n\longrightarrow\boldsymbol{G}^2_n$ of multicategories induced by the morphism $\Phi\,|\,n:\boldsymbol{N}^1_n\longrightarrow\boldsymbol{N}^2_n$ of multitopic sets, that $\Phi(\alpha\square_p\beta)=\Phi\alpha\square_p\Phi\beta$, which implies, by also using that $\Phi\mathtt{d}a=\mathtt{d}\Phi a$ and $\Phi\mathtt{B}b=\mathtt{B}\Phi b$, that $\Gamma\,[\![\mathtt{d}a\square_p\mathtt{B}b]\!]=[\![\mathtt{d}\Phi a\square_p\mathtt{B}\Phi b]\!]$; this is sufficient for (43).

The rest of the verification is left to the reader.

It is immediate that $[\text{-}]:\omega\mathtt{Cat}\longrightarrow\mathtt{mltSet}$ so defined is a functor.

We make a remark used later.

(44) For $\alpha,\,\beta\in Q_n$, if $\alpha\circ_j\beta$ is well-defined, we have

$$[\![\alpha\circ_j\beta]\!]=[\![\alpha]\!]\circ_{n-1}[\![\mathtt{d}\alpha\square_j\mathtt{B}\|\beta\|]\!]\ .$$

Proof $[\![\alpha\circ_j\beta]\!]=[\![\|\alpha\circ_j\beta\|]\!]=[\![\|\alpha\|\diamond_j\|\beta\|]\!]=[\![\|\alpha\|]\!]\circ_{n-1}[\![\mathtt{d}\|\alpha\|\square_j\mathtt{B}\|\beta\|]\!]$
$=[\![\alpha]\!]\circ_{n-1}[\![\mathtt{d}\alpha\square_j\mathtt{B}\|\beta\|]\!]$.