§2. The multitopic nerve of an ω-category

The purpose of this section is to define a functor

$$\begin{array}{ccc} \omega\mathtt{Cat} & \longrightarrow & \mathtt{multSet} \\ \boldsymbol{X} & \longmapsto & [\boldsymbol{X}] \end{array}$$

that we will call the *multitopic nerve* functor. Unfortunately, the definition is quite complicated, although not unnatural; it will be stated in the form of an elaborate theorem.

The multitopic nerve functor will be shown to be a right adjoint: it will have a left adjoint, the *free* functor. In motivating its definition, one should keep this circumstance in mind.

Let us look at how, for a given ω-category $\boldsymbol{X}$, one might naturally construct a corresponding multitopic set $\boldsymbol{M} = [\boldsymbol{X}]$, one whose cells are "the same as those of $\boldsymbol{X}$ to the extent possible".

There is no difficulty with identifying the 0- and the 1-cells of $\boldsymbol{M}$ with those of $\boldsymbol{X}$, and also, with identifying the domain and codomain operations on 1-cells in $\boldsymbol{M}$ with those for $\boldsymbol{X}$.

What about 2-cells of $\boldsymbol{M}$? A 2-cell of an ms may have a domain that is an arbitrary 1-pd: a composable string of 1-cells. This suggests that we define a 2-cell of our $\boldsymbol{M}$ to be a triple (u, α, b) where u is a 2-cell of $\boldsymbol{X}$, α is a 1-pd, b is a 1-cell. Of course, we need to add a condition: the domain of u as given in $\boldsymbol{X}$ should *agree* with α, and similarly for the codomain. This "agreement" has a natural sense: α, as a composable string of 1-cells in $\boldsymbol{X}$, has a composite $\llbracket\alpha\rrbracket$ in $\boldsymbol{X}$; we should require $\mathtt{d}u = \llbracket\alpha\rrbracket$ to hold. And of course, we also need that $\mathtt{c}u = b$.

Naturally, the domain and the codomain of $a=(u, \alpha, b)$ in the sense of $\boldsymbol{M}$ are defined as $\mathtt{d}a=\alpha$ and $\mathtt{c}a=b$.

The reader may want to look at item [4] below, where condition (b) reflects what was just said.

When we want to generalize the above to higher dimensions, we notice, first of all, that we

will need the operation $\alpha \longmapsto [\![\alpha]\!]$ for pd's in general; in particular, for $(n\text{-}1)$-pd's for the definition of " n-cell " of $\boldsymbol{M}$. $[\![\alpha]\!]$ may be called the "composite" of α; we will call it the *value* of α; it is an n-cell of the ω-category $\boldsymbol{X}$.

In the case $n=2$, this operation is easy; but in general, it is not immediate, although seems a natural enough thing to contemplate.

Secondly, we realize that there is another condition we have to impose on an n-cell (u, α, b). This is that α and b should be parallel in the multitopic set $\boldsymbol{M}$. For $n=2$, this condition was automatic from the condition "(b)" already stated; but in general, it has to separately required: this is item (a) in [4].

Let us return to the operation $\alpha \longmapsto [\![\alpha]\!]$. Here, α is an n-pd, for some n; that is, a (multi)arrow in the *free* multicategory $\boldsymbol{C}_n$ associated with the multitopic set $\boldsymbol{M}$. Notice that when α is a (free) generator of $\boldsymbol{C}_n$, that is, α is a single-cell-pd, $\alpha=a$, the value $[\![\alpha]\!]$ of α is naturally defined: when $a=(u, \alpha, b)$, we put $[\![a]\!] = u$. Now, it is an inevitable thought that $[\![\alpha]\!]$ should be defined by *multicategory recursion*: employing the freeness of $\boldsymbol{C}_n$ to define a morphism $\boldsymbol{C}_n \to \boldsymbol{E}_n$ to some suitably defined new multicategory $\boldsymbol{E}_n$ derived from the ω-category $\boldsymbol{X}$.

The definition of $\boldsymbol{E}_n$ is the heart of our procedure.

The inductive nature of the whole construction forces on us an interesting element in the definition of $\boldsymbol{E}_n$. Notice that what we saw above would indicate that the multiarrows of $\boldsymbol{E}_n$ themselves should have *values* that are n-cells of $\boldsymbol{X}$. It turns out that it is good, for the inductive argument to work, to define a bigger $\boldsymbol{E}_n$, in which the value of a multiarrow can be a cell in $\boldsymbol{X}$ of an arbitrary dimension greater or equal to n.

Notational conventions:

$\boldsymbol{X}$ is a fixed ω-category.

X_n = the set of n-cells of $\boldsymbol{X}$; $X_{\geq n} = \dot{\bigcup}_{m \geq n} X_m$.

$\circ_k$ is the symbol for composition at dimension k (of $>k$-cells) in $\boldsymbol{X}$.

$\mathtt{d}$ and $\mathtt{c}$ are used as domain and codomain in all the multitopic sets, without further qualification; if we must, we write $\mathtt{d}^{(\boldsymbol{M})}$ for the domain-map in the ms $\boldsymbol{M}$; etc. Also, $\mathtt{d}$ and $\mathtt{c}$ is used as domain and codomain in $\boldsymbol{X}$ itself. In any notation of the form $\mathtt{d}a$, $\mathtt{c}a$, it will be clear which $\mathtt{d}$ or $\mathtt{c}$ is meant from information on a.

When $\mathtt{d}$ is used as "domain" in $\boldsymbol{X}$, we have the derived notation $\mathtt{d}^{(n)}$ for $n\in\mathbb{N}$. $\mathtt{d}^{(n)}$ means the "n-dimensional domain": $\mathtt{d}$ applied as many times as needed to get down to dimension n. More precisely, we have

$$\mathtt{d}^{(n)}:X_{\geq n}\longrightarrow X_n$$

defined by: for any $m\geq n$, $a\in X_m$, $\mathtt{d}^{(n)}(a) \mathrel{\underline{\mathtt{def}}} \mathtt{d}^{m-n}(a)$. Of course, $\mathtt{d}^k(a) = \mathtt{d}(\mathtt{d}(\ldots\mathtt{d}(a)\ldots))$; in particular, $\mathtt{d}^0 a = a$.

$\mathtt{d}^{\overset{\uparrow}{1}}(\mathtt{d}^{\overset{\uparrow}{1}}(\ldots\mathtt{d}^{\overset{\uparrow}{1}}(a)\ldots))$

We similarly define $\mathtt{c}^{(n)}:X_{\geq n}\longrightarrow X_n$.

It is convenient to introduce a single entity (cell) $*$ of dimension -1 in every ω-category, and in every ms. We make $\mathtt{d}a=\mathtt{c}a=*$ for each 0-dimensional cell a. Further, we declare that $\mathtt{d}*=\mathtt{c}*=*$; thus, $*\,\|\,*$. In reading at least one of the definitions below, it is convenient to understand that the set of cells of dimension -2 is also $\{*\}$.

Recall that, with any ms $\boldsymbol{M}$, we have the maps $\mathtt{i}:M_n\longrightarrow P_n$, from n-cells to n-pd's. We pretend that $\mathtt{i}$ is an inclusion; whenever we have a map $\varphi:P_n\to T$, we automatically write $\varphi:M_n\to T$; the latter φ is in fact $\varphi\circ\mathtt{i}$. And of course, for $a\in M_n$, $\varphi(a)$ is $\varphi(\mathtt{i}a)$.

There is a general convention that abbreviates the notation of morphisms of any kind, and trusts the available info on the domain and codomain, or on the types of the arguments used in case of functions, to disambiguate the notation. An example is the formula

$$[\![a\diamond_p b]\!] \;=\; [\![a]\!]\circ_{n-1}[\![\mathtt{d}a\,\square_p\,\mathtt{B}b]\!]$$

in [8] below. Here, the first two instances of the function $[\![-]\!]$ refer to $[\![-]\!]:N_n \to X_{\geq n}$, the third one to $[\![-]\!]:N_{n-1} \to X_{\geq n-1}$. There are two more "ambiguous" functions, $\mathtt{d}$ and $\mathtt{B}$ in the formula.

With any ms at hand, $\circ_p$ is used for the placed composition in the free multicategories associated with the ms, and $\square_p$ for the placed composition in the fun-rep multicategories associated with the ms.

Theorem Let $\boldsymbol{X}$ be an ω-category. There is a uniquely determined system of entities given in [1]-[11], satisfying the conditions specified.

[1] The multitopic set $\boldsymbol{M}$.

Notation: $\boldsymbol{M}_n$ = the n-truncation of $\boldsymbol{M}$;
$M_n = \mathtt{C}_n(\boldsymbol{M})$ = the set of n-cells of $\boldsymbol{M}$;
$P_n = \mathtt{P}_n(\boldsymbol{M})$ = the set of n-pd's of $\boldsymbol{M}$;
$\boldsymbol{C}_n = \mathbf{C}_n(\boldsymbol{M})$ = the n-th free multicategory associated with $\boldsymbol{M}$;
$\boldsymbol{D}_n = \mathbf{D}_n(\boldsymbol{M})$ = the n-th fun-rep multicategory associated with $\boldsymbol{M}$.

[2] For each $n \in \mathbb{N}$, the multitopic set $\boldsymbol{N}_n$.

(1) $\boldsymbol{N}_n$ is n-truncated (or: n-dimensional): it has no cells of dimension $>n$.
(2) For $n \geq 1$, the $(n-1)$-truncation of $\boldsymbol{N}_n$ is the same as $\boldsymbol{M}_{n-1}$.
(3) $\boldsymbol{M}_n$ is a sub-multitopic set of $\boldsymbol{N}_n$.

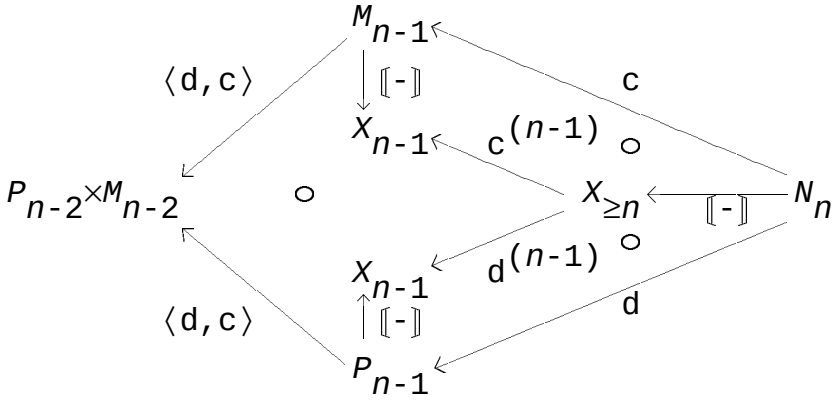
Notation: $N_n = \mathtt{C}_n(\boldsymbol{N}_n)$ = the set of n-cells of $\boldsymbol{N}_n$.
$Q_n = \mathtt{P}_n(\boldsymbol{N}_n)$ = the set of n-pd's of $\boldsymbol{N}_n$;
$\boldsymbol{F}_n = \mathbf{C}_n(\boldsymbol{N}_n)$ = the n-th free multicategory associated with $\boldsymbol{N}_n$;
$\boldsymbol{G}_n = \mathbf{D}_n(\boldsymbol{N}_n)$ = the n-th fun-rep multicategory associated with $\boldsymbol{N}_n$.

[3] For each $n\in\mathbb{N}$, the *evaluation map* $[\![-]\!]:Q_n\to X_{\geq n}$.

(1) For $a\in N_n$, $a\in M_n$ iff $[\![a]\!]\in X_n$.

(2) $[\![-]\!]:Q_n\to X_{\geq n}$ restricts to $[\![-]\!]:P_n\to X_n$.

[4] For each $n\geq 1$, the following diagram in $\mathtt{Set}$ gives N_n as a limit with projections $\mathtt{c}$, $[\![-]\!]$ and $\mathtt{d}$:



In other words: for $n\geq 1$, the elements of N_n are the $(n\text{-}1)$-*framed* cells of $\boldsymbol{X}$, whereby a "$(n\text{-}1)$-framed cell" of $\boldsymbol{X}$, for $n>0$, we mean a triple (u, α, b) where $u\in X_{\geq n}$, $\alpha\in P_{n-1}$ and $b\in M_{n-1}$ such that

(a) $\alpha \| b$ (i.e., $\mathtt{d}\alpha=\mathtt{d}b$, $\mathtt{c}\alpha=\mathtt{c}b$)

and

(b) $\mathtt{d}^{(n-1)}u=[\![\alpha]\!]$ and $\mathtt{c}^{(n-1)}u=[\![b]\!]$.

Writing a for $(u, \alpha, b)\in N_n$, we have $u=[\![a]\!]$, $\alpha=\mathtt{d}a$, $b=\mathtt{c}a$; i.e., $\alpha=\mathtt{d}a$ is the domain of a and $b=\mathtt{c}a$ is the codomain of $a\in N_n$ in the sense of the ms $\boldsymbol{N}_n$ (in the sense of the ms $\boldsymbol{M}$ if $a\in M_n$).

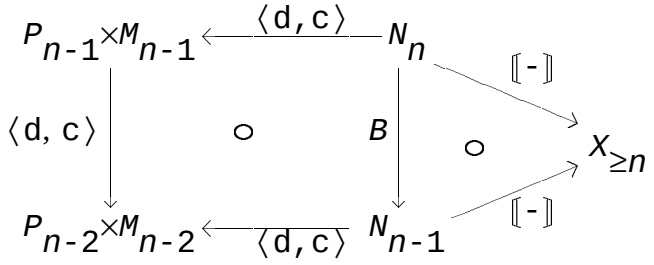
The above remain true, in essence, for $n=0$ as well. In fact, for $n=0$, $N_0 = X_{\geq 0}$; identifying $(u, *, *)$ with u makes the two determinations of N_0 the same.

Let us record (a) and (b) thus:

(a') $\mathtt{dd}a=\mathtt{dc}a$ and $\mathtt{cd}a=\mathtt{cc}a$

(b') $\mathtt{d}^{(n-1)}[\![a]\!]=[\![\mathtt{d}a]\!]$ and $\mathtt{c}^{(n-1)}[\![a]\!]=[\![\mathtt{c}a]\!]$

[5] For $n\geq 1$, the map $\mathtt{B}:N_n\longrightarrow N_{n-1}$ satisfying



[6] For each $n\geq 1$, the multicategory $\boldsymbol{E}_n$.

(1) $\boldsymbol{E}_n$ is one-level,

(2) $\mathtt{O}(\boldsymbol{E}_n)=M_{n-1}$,

(3) $\mathtt{A}(\boldsymbol{E}_n)=N_n$,

(4) $\mathtt{s}_{\boldsymbol{E}_n}(a)=\langle \mathtt{d}a\rangle$ and $\mathtt{t}_{\boldsymbol{E}_n}(a)=\mathtt{c}a$ for $a\in\mathtt{A}(\boldsymbol{E}_n)=N_n$.

[**Comment**s**:** for $a\in N_n$, a is an n-cell of $\boldsymbol{N}_n$; thus $\mathtt{d}a$ is an $(n-1)$-pd of $\boldsymbol{N}_n$; $\mathtt{d}a\in P_{n-1}$; reminder: $\langle \mathtt{d}a\rangle = \mathtt{s}_{\boldsymbol{D}_{n-1}}(\mathtt{d}a)$]

[7] For $n\geq 1$, the identities for $\boldsymbol{E}_n$: for $b\in M_{n-1}$, $1_b=(\mathtt{id}_{[\![b]\!]}, b, b)\in M_n$.

[8] **Notation**: $\Diamond_p$ is used for the placed composition in $\boldsymbol{E}_n$ ($n\geq 1$).

For $n\geq 1$, $a, b\in N_n$, $p\in |\mathtt{s}_{\boldsymbol{E}_n}(a)| = |\langle \mathtt{d}a\rangle|$ such that $a\Diamond_p b$ is well-defined ($\mathtt{t}_{\boldsymbol{E}_n}(b)=\mathtt{s}_{\boldsymbol{E}_n}(a)(p)$, i.e. $\mathtt{c}b=\langle \mathtt{d}a\rangle(p)$), we have

$$[\![a\Diamond_p b]\!] \;=\; [\![a]\!]\circ_{n-1}[\![\mathtt{d}a\,\square_p\,\mathtt{B}b]\!]$$

($\circ_{n-1}$ is composition in the ω-cat $\boldsymbol{X}$; $\square_p$ is composition in the multicategory $\boldsymbol{G}_{n-1}$).

[9] For each $n \geq 1$, the morphism of multicategories $\mathrm{d}: \boldsymbol{E}_n \longrightarrow \boldsymbol{D}_{n-1}$.

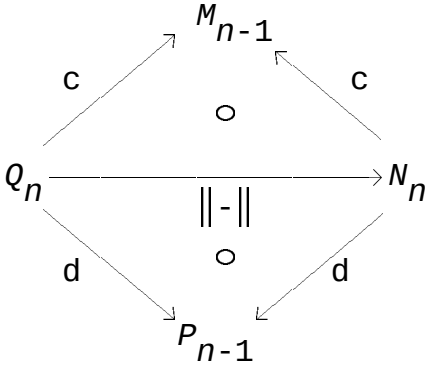
(1) $\mathrm{d}: \boldsymbol{E}_n \longrightarrow \boldsymbol{D}_{n-1}$ is strict, and it acts as the identity on higher-level objects.

(2) The effect on arrows: $\mathrm{d}: N_n \to P_{n-1}$, is the restriction of $\mathrm{d}^{(\boldsymbol{F}_n)}: Q_n \to P_{n-1}$.

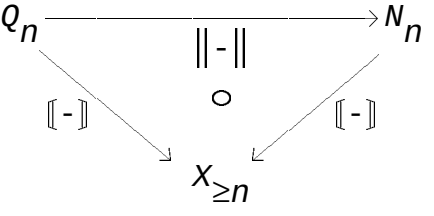
[10] For each $n \geq 1$, the morphism of multicategories $\| - \| : \boldsymbol{F}_n \longrightarrow \boldsymbol{E}_n$. $\| - \|$ is the identity on objects, and it is strict.

The effect on arrows, $\| - \| : Q_n \to N_n$, satisfies the commutativities:

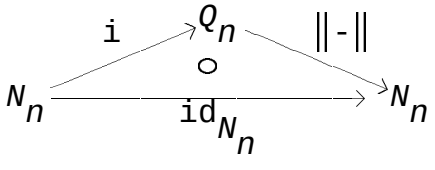
(1):



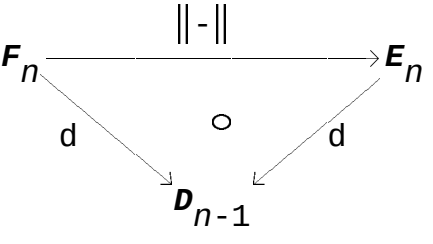
(2):



(3):



(4):



(Remember the morphism $\mathrm{d}: \boldsymbol{F}_n \to \boldsymbol{D}_{n-1}$ of multicategories associated with the ms $\boldsymbol{N}_n$.)

Equivalently, the first three can be stated as follows: for $\alpha \in Q_n$, and for $a \in N_n$,

$\mathrm{d}\|\alpha\| = \|\mathrm{d}\alpha\|$ **(1')**

$\mathrm{c}\|\alpha\| = \mathrm{c}\alpha$ **(1")**

$[\![\|\alpha\|]\!] = \alpha$ **(2')**

$\|a\| = a$. **(3')**

[11] Notation: For $u\in X_{\geq n}$, the map $\hat{\mathtt{d}}(=\hat{\mathtt{d}}_n):X_{\geq n}\longrightarrow X_{\geq n}$ is defined by

$$\hat{\mathtt{d}}u \stackrel{\mathtt{def}}{=} \begin{cases} \mathtt{d}u & \text{when } \mathtt{dim}(u)>n \\ u & \text{when } \mathtt{dim}(u)=n \end{cases}.$$

Similarly, $\hat{\mathtt{c}}:X_{\geq n}\longrightarrow X_{\geq n}$.

For $n\geq 1$, we have multicategory morphisms $\mathtt{D}:\boldsymbol{E}_n\longrightarrow\boldsymbol{E}_n$, $\mathtt{C}:\boldsymbol{E}_n\longrightarrow\boldsymbol{E}_n$ specified as follows. Both are identities on objects, and strict. The actions of $\mathtt{D}$ and $\mathtt{C}$ on arrows are:

$$\begin{array}{ccc} \mathtt{D}:N_n & \longrightarrow & N_n \\ a & \longmapsto & (\hat{\mathtt{d}}[\![a]\!],\mathtt{d}a,\mathtt{c}a) \end{array} \qquad \begin{array}{ccc} \mathtt{C}:N_n & \longrightarrow & N_n \\ a & \longmapsto & (\hat{\mathtt{c}}[\![a]\!],\mathtt{d}a,\mathtt{c}a) \end{array}$$

Moreover, the following hold:

for any $\alpha\in P_n$, $p\in|\langle\mathtt{d}\alpha\rangle|$, $a\in N_n$ such that $\alpha\square_p a$ is well-defined, we have

$$\mathtt{D}\|\alpha\square_p a\| = \|\alpha\square_p \mathtt{D}a\|, \qquad \textbf{(1)}$$
$$\mathtt{C}\|\alpha\square_p a\| = \|\alpha\square_p \mathtt{C}a\|, \qquad \textbf{(2)}$$

Proof The items listed depend on the parameter $n\in\mathbb{N}$, usually shown as a subscript (in the case of morphisms, this subscript is suppressed). The proof consists in defining the subscripted items by recursion, and proving the properties by induction on n. For $n=0$, all matters are trivial, or will be attended to later. Here, we assume that $n\geq 1$, and that all items depending on subscripts $k<n$ are defined, and they obey the rules stated that concern them.

According to the Proposition in Section 1 in Part 3 of [H/M/P], the n-truncated ms's $\boldsymbol{M}_n$ and $\boldsymbol{N}_n$ can be defined by supplying two sets of data:

(1) their $(n\text{-}1)$-truncations, which are the same, $\boldsymbol{M}_{n-1}$, and which is available by the induction hypothesis; and

(2) information on a set of n-cells as to their domains and codomains, which have to be consistently chosen (parallel).

Now, read [4]: this gives the definition of the set of n-cells for $\boldsymbol{N}_n$, the set N_n, together with the consistent domain and codomain information on them. [2](1) and [2](2) are satisfied.

Note that in [4] we have defined the map $[\![-]\!]:N_n\to X_{\geq n}$. This map will be extended to $[\![-]\!]:Q_n\to X_{\geq n}$ later.

[2](3) and [3] give us the definition of $\boldsymbol{M}_n$; this refers to $[\![-]\!]:N_n\to X_{\geq n}$ only.

Next, we turn to item $[5]_n$, the function $\mathrm{B}:N_n\longrightarrow N_{n-1}$. The definition of B is contained in the conditions stated for it in [5], by what we know about N_{n-1} according to $[4]_{n-1}$. In simple terms, the definition amounts to the following.

For $a=([\![a]\!],\mathrm{d}a,\mathrm{c}a)\in N_n$, we let $\mathrm{B}a \mathrel{\overline{\mathrm{def}}} ([\![a]\!],\mathrm{dd}a,\mathrm{cc}a)$. We do have that $\mathrm{B}a\in N_{n-1}$: $\mathrm{dd}a\|\mathrm{cc}a$, i.e.,

$$\mathrm{d}^{(n-2)}[\![a]\!]=[\![\mathrm{dd}a]\!], \tag{1}$$

$$\mathrm{c}^{(n-2)}[\![a]\!]=[\![\mathrm{cc}a]\!], \tag{2}$$

seen as follows. We have

$$[\![\mathrm{d}^{(n-2)}[\![\|\mathrm{d}a\|]\!]]\!] \overset{1}{=} [\![\mathrm{d}\|\mathrm{d}a\|]\!] \overset{2}{=} [\![\mathrm{dd}a]\!], \tag{3}$$

1 by $[4](\mathrm{b}')_{n-1}$, 2 by $[10](1)_{n-1}$. Also,

$$[\![\|\mathrm{d}a\|]\!] \overset{3}{=} [\![\mathrm{d}a]\!] \overset{4}{=} \mathrm{d}^{(n-1)}[\![a]\!], \tag{4}$$

3 by $[10](2)_{n-1}$, 4 by $[4](\mathrm{b}')_n$. Applying $\mathrm{d}^{(n-2)}$ to (4),

$$\mathrm{d}^{(n-2)}[\![a]\!] = \mathrm{d}^{(n-2)}[\![\|\mathrm{d}a\|]\!] \tag{5}$$

From (3) and (5), (1) follows. (2) is similar.

The main point of the whole construction is the definition of $[\![-]\!] : N_n \to X_{\geq n}$ through the multicategory $\boldsymbol{E}_n$; this latter item is which we construct now.

The construction is, essentially, contained in [6], [9](1) and [8] .

After the basics in [6], the main thing is the (placed) composition $\Diamond$ in $\boldsymbol{E}_n$, essentially given in [8]. Let's give the definition the composition in detail.

Given $a, b \in N_n$, $p \in |s_{\boldsymbol{E}_n}(a)| = |\langle \mathrm{d}a \rangle|$ such that $t_{\boldsymbol{E}_n}(b) = s_{\boldsymbol{E}_n}(a)(p)$, i.e.

$$\mathrm{c}b = \langle \mathrm{d}a \rangle (p) , \tag{6}$$

we define

$$a \Diamond_p b = ([\![a \Diamond_p b]\!] , \mathrm{d}(a \Diamond_p b), \mathrm{c}(a \Diamond_p b)) \tag{7}$$

by

$$[\![a \Diamond_p b]\!] \ \bar{\mathrm{def}} \ [\![a]\!] \circ_{n-1} [\![\mathrm{d}a \Box_p \mathrm{B}b]\!] \tag{8}$$

($\circ_{n-1}$ is composition in the ω-cat $\boldsymbol{X}$; $\Box_p$ is composition in the multicategory $\boldsymbol{G}_{n-1}$: item $[2]_{n-1}$),

$$\mathrm{d}(a \Diamond_p b) \ \bar{\mathrm{def}} \ \mathrm{d}a \Box_p \mathrm{d}b \ , \tag{9}$$
$$\mathrm{c}(a \Diamond_p b) \ \bar{\mathrm{def}} \ \mathrm{c}a \ . \tag{9'}$$

In order to see that the above make sense, we have to establish the truth of the following assertions (10)-(16).

(10) The expression on the right in (9) is well-defined.

The expression on the right in (8) is well-defined as an element of X_n :

$$t_{\boldsymbol{G}_{n-1}}(\mathrm{B}b) = (\langle \mathrm{d}a \rangle (p))^{\cdot} \text{ , i.e.,}$$

$$\mathrm{dB}b \;=\; \mathrm{d}(\langle \mathrm{d}a\rangle(p)) \;, \tag{11}$$

$$\mathrm{cB}b \;=\; \mathrm{c}(\langle \mathrm{d}a\rangle(p)) \;; \tag{12}$$

and

$$\mathrm{d}^{(n-1)}[\![a]\!] = \mathrm{c}^{(n-1)}[\![\mathrm{d}a \Box_p \mathrm{B}b]\!] \;. \tag{13}$$

The entity in (7) is in N_n, i.e.,

$$\mathrm{d}a \Box_p \mathrm{d}b \;\|\; \mathrm{c}a \;, \tag{14}$$

$$\mathrm{d}^{(n-1)}([\![a]\!] \circ_{n-1} [\![\mathrm{d}a \Box_p \mathrm{B}b]\!]) = [\![\mathrm{d}a \Box_p \mathrm{d}b]\!] \tag{15}$$

and

$$\mathrm{c}^{(n-1)}([\![a]\!] \circ_{n-1} [\![\mathrm{d}a \Box_p \mathrm{B}b]\!]) = [\![\mathrm{c}a]\!] \;. \tag{16}$$

Proofs for (10)-(16):

First of all, let's remark that for $b \in N_n$, and $\mathrm{B}b \in N_{n-1}$, we have

$$\|\mathrm{d}b\| \;=\; \mathrm{DB}b \;, \tag{17}$$

$$\|\mathrm{c}b\| \;=\; \mathrm{c}b \;=\; \mathrm{CB}b \;. \tag{18}$$

These equalities result from $[10]_{n-1}$ applied $\mathrm{B}b$ as a, and the definition of D in $[11]_{n-1}$.

(10):
$$\begin{array}{ccccc} \mathrm{t}_{\boldsymbol{D}_{n-1}}(\mathrm{d}b) & \overset{?}{=} & \mathrm{s}_{\boldsymbol{D}_{n-1}}(a)(p) & & \\ \| & & \| & & \\ (\mathrm{dd}b,\, \mathrm{cd}b) & & \langle \mathrm{d}a\rangle(p) & = (\mathrm{c}b) & = (\mathrm{dc}b,\, \mathrm{cc}b) \\ 1 & & & \uparrow & 2 \end{array}$$

$\uparrow$ holds by the assumption in (6); items 1 and 2 are equal because of condition [4](a') for b.

(11): $\mathtt{dB}b \stackrel{?}{=} \mathtt{d}(\langle \mathtt{d}a \rangle(p))$

$\| \qquad\qquad \| \ (29.0)$

$\mathtt{dd}b = \mathtt{dc}b$

[8](a)'

(12): $\mathtt{cB}b \stackrel{?}{=} \mathtt{c}(\langle \mathtt{d}a \rangle(p))$: similar to previous.

(13): $\mathtt{d}^{(n-1)}[\![a]\!] \stackrel{?}{=} \mathtt{c}^{(n-1)}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!]$:

We have

$\mathtt{C}\|\mathtt{d}a \Box_p \mathtt{B}b\| = \|\mathtt{d}a \Box_p \mathtt{CB}b\|$ (by [11](2'))

$= \|\mathtt{d}a \Box_p \mathtt{c}b\|$ (by (18))

$= \|\mathtt{d}a \Box_p (\langle \mathtt{d}a \rangle(p))\|$ (by (6))

$= \|\mathtt{d}a\|$ (basic!)

Taking $[\![-]\!]$ of both sides:

$$[\![\mathtt{d}a]\!] = \mathtt{d}^{(n-1)}[\![a]\!]$$

by $[9]_{n-1}$(b'); and

$$[\![\mathtt{C}(\mathtt{d}a \Box_p \mathtt{B}b)]\!] = \mathtt{c}^{(n-1)}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!]$$

by the definition of $\mathtt{C}$ ($[11]_{n-1}$).

This does it.

(14): $\mathtt{d}a \Box_p \mathtt{d}b \stackrel{?}{\|} \mathtt{c}a$: this is obvious, since $\mathtt{d}a \Box_p \mathtt{d}b \| \mathtt{d}a$, and $\mathtt{d}a \| \mathtt{c}a$.

(15): $\mathtt{d}^{(n-1)}([\![a]\!] \circ_{n-1} [\![\mathtt{d}a \Box_p \mathtt{B}b]\!]) \stackrel{?}{=} [\![\mathtt{d}a \Box_p \mathtt{d}b]\!]$:

The left-hand side equals $\mathtt{d}^{(n-1)}[\![\mathtt{d}a \Box_p \mathtt{B}b]\!]$, by facts of life concerning ω-cats. On the

other hand,

$$\mathtt{D}\|\mathtt{d}a\square_p \mathtt{B}b\| = \|\mathtt{d}a\square_p \mathtt{DB}b\| \qquad \text{(by } [11]_{n-1}(1')\text{)}$$
$$= \|(\mathtt{d}a\square_p \|\mathtt{d}b\|)\| \qquad \text{(by (17))}\ .$$

Taking $[\![-]\!]$ of both sides:

$$[\![\mathtt{d}a\square_p \|\mathtt{d}b\|]\!] = [\![\mathtt{d}a\square_p \mathtt{d}b]\!] \qquad \text{(by } [11]_{n-1}(3)\text{)},$$
$$[\![\mathtt{D}\|\mathtt{d}a\square_p \mathtt{B}b\|]\!] = \mathtt{d}^{(n-1)}[\![\mathtt{d}a\square_p \mathtt{B}b]\!] \qquad \text{(by the definition of D)}.$$

This suffices.

(16): this follows from $[4]_{n-1}$(b') , and the identity $\mathtt{c}^{(n-1)}(u\circ_{n-1}v)=\mathtt{c}^{(n-1)}(u)$ for $u, v\in X_{\geq n}$.

Before we treat the required properties of the multicategory composition $\Diamond$, and the identities for it, we deal with the amalgamation structure for $\Diamond$. This is taken care of by the statement of $[9]_n(1)$, which implies that that structure is identical to the same structure for $\boldsymbol{D}_{n-1}$. We establish $\mathtt{d}:\boldsymbol{E}_n \longrightarrow \boldsymbol{D}_{n-1}$ satisfying $[9]_n(1)$: $\mathtt{d}:\boldsymbol{E}_n \longrightarrow \boldsymbol{D}_{n-1}$ is strict, and it acts as the identity on higher-level objects.

Keeping in mind that $\boldsymbol{E}_n$ is a 1-level multicategory, $\boldsymbol{D}_{n-1}$ is a 2-level one, here is the action of $\mathtt{d}:\boldsymbol{E}_n \to \boldsymbol{D}_{n-1}$ on objects:

$$\begin{array}{ccc} \mathtt{O}(\boldsymbol{E}_n) & \xrightarrow{\quad(-)^{\cdot}\quad} & \dot{\mathtt{O}}(\boldsymbol{E}_n) \\ \mathtt{d}\downarrow & & \downarrow d \\ \mathtt{O}(\boldsymbol{D}_{n-1}) & \xrightarrow[\quad(-)^{\cdot}\quad]{} & \dot{\mathtt{O}}(\boldsymbol{D}_{n-1}) \end{array} \qquad \text{is} \qquad \begin{array}{ccc} M_{n-1} & \xrightarrow{\quad\mathrm{id}\quad} & M_{n-1} \\ \mathrm{id}\downarrow & & \downarrow b\mapsto(\mathtt{d}b,\,\mathtt{c}b) \\ M_{n-1} & \xrightarrow[b\mapsto(\mathtt{d}b,\mathtt{c}b)]{} & P_{n-2}\times M_{n-2} \end{array}$$

Thus, in particular, $\mathtt{d}:\boldsymbol{E}_n \to \boldsymbol{D}_{n-1}$ is the identity on higher-level objects.

The action of $\mathtt{d}:\boldsymbol{E}_n \longrightarrow \boldsymbol{D}_{n-1}$ on arrows is defined to be $\mathtt{d}:N_n \longrightarrow P_{n-1}$, given in the

definition of N_n, as the second component of $a \in N_n$.

It is immediate that $d : \boldsymbol{E}_n \to \boldsymbol{D}_{n-1}$ preserves "source" *strictly*, with the transition bijections of §2 of Part 2 of [H/M/P] being identities.

For preserving "target":

$$\begin{array}{ccc} "d"(t_{\boldsymbol{E}_n}(a)) & \overset{?}{=} & t_{\boldsymbol{D}_{n-1}}("d"a) \\ \| & & \| \\ (dca, cca) & = & (dda, dca) \\ & \uparrow & \end{array}$$

$\uparrow$ holds by condition (a') in [4].

It is clear that $d : \boldsymbol{E}_n \to \boldsymbol{D}_{n-1}$ preserves identities.

The fact that $d : \boldsymbol{E}_n \to \boldsymbol{D}_{n-1}$ preserves composition is (9):

$$d(a \Diamond_p b) = da \Box_p db \ .$$

The amalgamation structure for the composition $\Diamond$ has not been defined yet; but it is uniquely determined by the requirements that $d : \boldsymbol{E}_n \to \boldsymbol{D}_{n-1}$ be strict and identity on upper-level objects: d acts as the identity on the (φ,ψ)-maps of the amalgamation structure. Note also that, even before we know that the laws for $\Diamond$ hold, the associated coherence laws concerning the amalgamation structure are assured: they are inherited from $\boldsymbol{D}_{n-1}$.