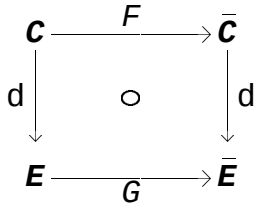


(48)

commutes. One has a function-replacement multicategory $(\boldsymbol{D}, \square, \varphi, \psi)$ associated with $(\boldsymbol{C}, \langle - \rangle, \rho, \sigma)$, and another one, $(\bar{\boldsymbol{D}}, \square, \varphi, \psi)$, associated with $(\bar{\boldsymbol{C}}, \langle - \rangle, \rho, \sigma)$.

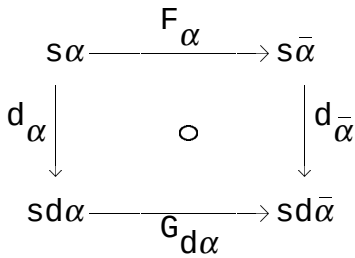
We claim that there is a natural induced map $F^{\$}: (\boldsymbol{D}, \square, \varphi, \psi) \longrightarrow (\bar{\boldsymbol{D}}, \square, \varphi, \psi)$ of multicategories; moreover, the correspondence $F \mapsto F^{\$}$ is a functor.

The action of $F^{\$}$ on arrows is the same as that of F; and the transition isomorphisms of $F^{\$}$ are the actions $F_{\alpha}: \langle \alpha \rangle \xrightarrow{\cong} \langle F\alpha \rangle$ of the HAA-morphism F.

Since F is a renaming morphism, we have $F: L \to \bar{L}$ acting on the (free) generators; this is the action of $F^{\$}$ on upper-level objects. On lower-level objects, $F^{\$}: O^{*} \times O \longrightarrow \bar{O}^{*} \times \bar{O}$ acts in the expected way.

We are going to use bars for the effect of F, just as in (22).

Recall that the assumed qualities of $\mathsf{d}: \boldsymbol{C} \to \boldsymbol{E}$ ensure that, for $\alpha, \beta \in \mathsf{A}(\boldsymbol{C})$, $\mathsf{d}\alpha = \mathsf{d}\beta$ implies that $\mathsf{s}\alpha = \mathsf{s}\beta$. In the situation (48), we have that the same assumption, $\mathsf{d}\alpha = \mathsf{d}\beta$, implies that the actions $F_{\alpha}: \mathsf{s}\alpha \to \mathsf{s}\bar{\alpha}$, $F_{\beta}: \mathsf{s}\beta \to \mathsf{s}\bar{\beta}$ coincide. Indeed, (48) induces the following commutative diagram:



of place-actions. Since the d's are (identities on objects and) strict, the verticals are identities.

We obtain that $F_\alpha = G_{\mathtt{d}\alpha}$; this shows that F_α depends on $\mathtt{d}\alpha$ only.

$F^{\$}$ preserves composition: assuming that $\alpha \Box_p \beta$ is well-defined in $\boldsymbol{D}$, that is, $\mathtt{d}\beta = \mathtt{d}(\langle\alpha\rangle p)$, $\mathtt{c}\beta = \mathtt{c}(\langle\alpha\rangle p)$, we have, first, that $\bar{\alpha} \Box_{\bar{p}} \bar{\beta}$ is well-defined in $\bar{\boldsymbol{D}}$, and, second, that $\overline{\alpha \Box_p \beta} = \bar{\alpha} \Box_{\bar{p}} \bar{\beta}$.

The first assertion follows (directly) from the commutativity of (48).

The second is a special case of (22). Now, θ is an identity, and $\bar{\theta}$, determined by (14), is an identity as a consequence of the fact that, in (14), the two verticals are the same: the left-hand side one is F_f, the right-hand-side one is F_β; since $\mathtt{d}f = \mathtt{d}\beta$, we have $F_f = F_\beta$!

We also need that the diagram

$$\begin{array}{ccccc} \langle\alpha\rangle\backslash p & \xrightarrow{\psi^\Box} & \langle \alpha \Box_p \beta \rangle & \xleftarrow{\varphi^\Box} & \langle\beta\rangle \\ \downarrow F & \circ & \downarrow F & \circ & \downarrow F \\ \langle\bar{\alpha}\rangle\backslash p & \xrightarrow{\psi^\Box} & \langle \bar{\alpha} \Box_{\bar{p}} \bar{\beta} \rangle & \xleftarrow{\varphi^\Box} & \langle\beta\rangle \end{array}$$

commutes; the verification is left to the reader.

Multitopic sets and their morphisms

It is worth looking at the definition of "multitopic set" and "morphism of multitopic sets" in the light of the changed notion of "multicategory".

We adopt the useful convention that, in every multitopic set, there is a single cell of dimension $\mathtt{-1}$, denoted $*$; that, for every $\mathtt{0}$-cell a, $\mathtt{d}a = \mathtt{c}a = *$; and also that $d* = c* = *$. Therefore, any two zero cells are parallel; and $*$ is parallel to itself.

A basic "singularity" of $\mathtt{0}$-cells remains: for $\mathtt{0}$-cells a and b, just because we have $\mathtt{d}b = \mathtt{c}a$ (which is right), it does *not* mean that we can form a $\mathtt{0}$-pd, or formal composite,

$b \circ_* a$ that would be another 0-cell. This circumstance is best reflected in the definition of "multitopic set" by the fact that the item $\boldsymbol{C}_0$ *should not be defined at all*. This reflects a *change* from the definition of "multitopic set" from [H/M/P], albeit a minor one.

We make a set of small corrections in the definition of "multitopic set" in [H/M/P], part 3, concerning the lowest dimensions.

Presently, there is a statement after the definition: $P_k = \mathrm{A}(\boldsymbol{C}_k) = \mathrm{A}(\boldsymbol{D}_k)$, which is invalid, under the given definitions, for $k=0$; we remove this error.

The first point to make is that there is no need for the item $\boldsymbol{C}_0$; it is not used anywhere. On the other hand, the item $\boldsymbol{D}_0$ is used, and its definition should be changed. References to items (i) to (viii) are made to those items in the starting definition of section 1, part 3.

Discard item (iv), the definition of $\boldsymbol{C}_0$.

In (vi), the definition of $\boldsymbol{D}_0$ should be this: $\boldsymbol{D}_0$ is 2-level, $\mathrm{O}(\boldsymbol{D}_0) = \mathrm{A}(\boldsymbol{D}_0) = C_0$, $\dot{\mathrm{O}}(\boldsymbol{D}_0) = \{*\}$; $\mathrm{s}_{\boldsymbol{D}_0}(a) = \langle a \rangle$ and $1_a = a$ for $a \in \mathrm{A}(\boldsymbol{D}_0) = C_0$; for $a, b \in C_0$, and with $\#$ the unique element of $|\langle b \rangle|$, $b \circ_{\#} a \stackrel{\mathrm{def}}{=} a$.

In (ii), change $\langle \boldsymbol{C}_k \rangle_{k \in \mathbb{N}}$ to $\langle \boldsymbol{C}_k \rangle_{k \in \mathbb{N} - \{0\}}$.

In the sentence right after (viii), add "for $k \geq 1$ " before " $P_k = \mathrm{A}(\boldsymbol{C}_k) = \mathrm{A}(\boldsymbol{D}_k)$ ", and add " $P_0 = \mathrm{A}(\boldsymbol{D}_0)$ ".

Now that tuples are defined on arbitrary finite sets, rather than on initial segments of (positive) integers as in the original treatment, the idea of a strict morphism of multicategories is largely, but not completely, obsolete.

In building a multitopic set $\boldsymbol{M}$, the first step is the specification of an arbitrary set M_0 of 0-cells.

When we build up a multitopic set $\boldsymbol{M}$ recursively, and we are done with $\leq n$-cells, where $n \geq 0$, the next step is

(49) prescribing $(n+1)$-cells a with given $\mathrm{d}a$ and $\mathrm{c}a$ such that $\mathrm{dd}a=\mathrm{dc}a$, $\mathrm{cd}a=\mathrm{cc}a$.

(Note that (49) makes sense even for $n=0$, by the conventions stated above.)

The data thus gained give us generators for a free multicategory $\boldsymbol{C}^0_{n+1}$ with objects the n-cells, and generating arrows the $(n+1)$-cells, with source and target determinations $\mathrm{s}(a)=\langle \mathrm{d}a \rangle$, $\mathrm{t}(a)=\mathrm{c}a$.

Further, we have $\boldsymbol{D}_n$, and in $\boldsymbol{D}_n$, we have, for each $(n+1)$-cell a , the arrow $\mathrm{d}a$, with source $\mathrm{s}_{\boldsymbol{D}_n}(\mathrm{d}a)=\langle \mathrm{d}a \rangle$, and $\mathrm{t}_{\boldsymbol{D}_n}(\mathrm{d}a)=(\mathrm{dd}a,\mathrm{cd}a)$.

(When $n=0$, these statements are not quite right with the original specifications in [H/M/P]; there, an arrow of $\boldsymbol{D}_0$ is a pair of arbitrary 0-cells; now, after the change, $\mathrm{d}a$ is both a 0-cell and an arrow of $\boldsymbol{D}_0$. When $n=0$, the formula above for $\mathrm{t}_{\boldsymbol{D}_n}(\mathrm{d}a)$ makes sense if we identify $(*, *)$ with $*$.)

We notice that $\mathrm{t}_{\boldsymbol{D}_n}(\mathrm{d}a)=(\mathrm{dc}a, \mathrm{cc}a)=\mathrm{t}_{\boldsymbol{D}_n}(\mathrm{c}a)$. Therefore, it is meaningful to make the assignment $a \longmapsto \mathrm{d}a$, from the generating arrows of $\boldsymbol{C}^0_{n+1}$ to the arrows of $\boldsymbol{D}_n$, in a first step towards a map $\mathrm{d}^0 : \boldsymbol{C}^0_{n+1} \longrightarrow \boldsymbol{D}_n$ that acts on the upper level objects as the identity, and as $b \longmapsto \mathrm{T}b = \mathrm{t}_{\boldsymbol{D}_n}(b)=(\mathrm{d}b, \mathrm{c}b)$ on the lower level objects: we have $\mathrm{s}_{\boldsymbol{C}^0_{n+1}}(a)=\mathrm{s}_{\boldsymbol{D}_n}(\mathrm{d}a)$ and $\mathrm{T}(\mathrm{t}_{\boldsymbol{C}^0_{n+1}}(a))=\mathrm{t}_{\boldsymbol{D}_n}(\mathrm{d}a)$. We take the identities as $\theta_a : \mathrm{s}_{\boldsymbol{C}^0_{n+1}}(a) \xrightarrow{\cong} \mathrm{s}_{\boldsymbol{D}_n}(\mathrm{d}a)$ as the initial components of the system of transition isomorphisms for the map d^0 .

By the freeness of $\boldsymbol{C}^0_{n+1}$ now applied, we have our map $\mathrm{d}^0 : \boldsymbol{C}^0_{n+1} \longrightarrow \boldsymbol{D}_n$. In this case, we "force" this map to be strict, by twisting $\boldsymbol{C}^0_{n+1}$ to a suitable $\boldsymbol{C}_{n+1}$ isomorphic to $\boldsymbol{C}^0_{n+1}$, and getting the strict $\mathrm{d} : \boldsymbol{C}_{n+1} \longrightarrow \boldsymbol{D}_n$ (see end of section 2 of part 2 in [H/M/P]).

Note that in this process, the only arbitrary, or selectable, parameters are the ones under the

heading (49).

To complete the recursion building up the ms, we must construct $\boldsymbol{D}_{n+1}$. It is here that another choice enters: the HAA on the free multicategory $\boldsymbol{C}_{n+1}$, which gives rise to the source-structure of $\boldsymbol{D}_{n+1}$. Of course, the HAA is determined up to isomorphism, but still, it is not uniquely determined. In [H/M/P], there was a definite choice: the left-to-right enumeration of the function symbols. But we do not need to stick to that. Up to *isomorphism*, we do not get any new ms in the present way -- but to appreciate this fact, we first need to re-examine the notion of *morphism* of ms's.

The definition given in [H/M/P] of morphisms of ms's, although it remains correct for the old type of ms, has to be reworded in our new context of abstract places.

To understand the concept, the best thing to do is describe a morphism $\Phi:\boldsymbol{M}\longrightarrow\bar{\boldsymbol{M}}$ recursively, defining the components $\Phi_n:M_n\longrightarrow\bar{M}_n$ by recursion (M_n, $\bar{M}_n$ are the sets of n-cells of $\boldsymbol{M}$, $\bar{\boldsymbol{M}}$, resp.)

$\Phi_0:M_0\longrightarrow\bar{M}_0$ is just any map of sets.

Suppose $n\geq 0$, and we have defined the components Φ_k for $k\leq n$. To define the set-map $\Phi_{n+1}:M_{n+1}\longrightarrow\bar{M}_{n+1}$, we look at each $a\in M_{n+1}$, and its $\mathrm{d}a$ and $\mathrm{c}a$. Assume recursively that each Φ_k, $k\leq n$, induces a *renaming* morphism $\Psi_k:\boldsymbol{C}_k\longrightarrow\bar{\boldsymbol{C}}_k$ (mapping free generators to free generators) whose action on objects is $\Phi_{k-1}:M_{k-1}\longrightarrow\bar{M}_{k-1}$, and whose action on generating arrows is $\Phi_k:M_k\longrightarrow\bar{M}_k$.

(50) Choose, for each $a\in M_{n+1}$, some $\bar{a}\in\bar{M}_{n+1}$ such that $\mathrm{d}\bar{a}=\Psi_n(\mathrm{d}a)$ and $\mathrm{c}\bar{a}=\Psi_n(\mathrm{c}a)$, and define Φ_{n+1} on M_{n+1} by setting $\Phi_{n+1}(a)=\bar{a}$.

We now have the data to induce a morphism $\Psi_{n+1}:\boldsymbol{C}_{n+1}\longrightarrow\bar{\boldsymbol{C}}_{n+1}$. Namely, we map the objects by $\Phi_n:M_n\to\bar{M}_n$. The generator a is mapped to the generator $\bar{a}=\Phi_{n+1}(a)$. This

is legitimate, since, on the one hand, $s_{\boldsymbol{C}_{n+1}}(a)=\langle \mathrm{d}a\rangle$, $s_{\bar{\boldsymbol{C}}_{n+1}}(\bar{a})=\langle \mathrm{d}\bar{a}\rangle$, and we have, by (4), and by the fact that $\mathrm{d}\bar{a}=\Psi_n(\mathrm{d}a)$, an isomorphism

$$\Psi_n=(\Psi_n)_{\mathrm{d}a}:\Psi_n{}^{\circ}\langle \mathrm{d}a\rangle \xrightarrow{\ \cong\ } \langle \mathrm{d}\bar{a}\rangle \tag{51}$$

induced by the morphism $\Psi_n:\boldsymbol{C}_n\longrightarrow\bar{\boldsymbol{C}}_n$ of multicategories; and on the other hand, we have $t_{\bar{\boldsymbol{C}}_{n+1}}(\bar{a})=\Psi_n(t_{\boldsymbol{C}_{n+1}}(a))$ too. In particular, the action of Ψ_{n+1} on the source places of a is the action of Ψ_n on the higher places of $\mathrm{d}a$.

By our discussion above, $\Psi_n:\boldsymbol{C}_n\longrightarrow\bar{\boldsymbol{C}}_n$ induces the map $\Psi_n^{\$}:\boldsymbol{D}_n\longrightarrow\bar{\boldsymbol{D}}_n$. We claim that the diagram

$$\begin{array}{ccc} \boldsymbol{D}_n & \xleftarrow{\ \mathrm{d}\ } & \boldsymbol{C}_{n+1} \\ {\scriptstyle\Psi_n^{\$}}\downarrow & & \downarrow{\scriptstyle\Psi_{n+1}} \\ \bar{\boldsymbol{D}}_n & \xleftarrow[\ \mathrm{d}\ ]{} & \bar{\boldsymbol{C}}_{n+1} \end{array} \tag{52}$$

commutes. This is right since it works on the generators of $\boldsymbol{C}_{n+1}$: $a\in M_{n+1}$ is mapped by both composites to $\mathrm{d}\bar{a}$; and the θ-components at a also come out equal, the d's being strict and by what we said about the θ of Ψ_{n+1} , and by what we know about the θ of $\Psi_n^{\$}$.

In this process of building up a morphism $\Phi:\boldsymbol{M}\to\bar{\boldsymbol{M}}$ the only (somewhat) free choices to make are the ones in (50), for each $n\in\{-1\}\cup\mathbb{N}$.

We can summarize by saying that morphism $\Phi:\boldsymbol{M}\to\bar{\boldsymbol{M}}$ is a system of maps $\langle\Phi_n:M_n\longrightarrow\bar{M}_n\rangle_{n\in\mathbb{N}}$ such that there is a, necessarily unique, system of renaming (multicategory) morphisms $\Psi_n:\boldsymbol{C}_n\longrightarrow\bar{\boldsymbol{C}}_n$ such that Ψ_n acts on objects as Φ_{n-1} , on generating arrows as Φ_n , and the induced diagrams (52) commute.

Miscellanious

Some additional material to be used later.

Let $\Phi:\mathbf{M}\longrightarrow\bar{\mathbf{M}}$ be a map of ms's. We say that Φ is a *cover at* (*dimension*) n if

for any $\bar{a}\in\bar{M}_n$, $\beta\in P_{n-1}$ and $c\in M_{n-1}$ such that $\beta\|c$, $\beta\overset{\Phi}{\longmapsto}\mathrm{d}\bar{a}$ and $c\overset{\Phi}{\longmapsto}\mathrm{c}\bar{a}$, there is $a\in M_n$ such that $\mathrm{d}a=\beta$, $\mathrm{c}a=c$ and $a\overset{\Phi}{\longmapsto}\bar{a}$.

When $n=0$, β and c are the unique (-1)-dimensional elements, and their mention is superfluous. Thus, "Φ is a cover at 0" means that the induced map $\Phi:M_0\longrightarrow\bar{M}_0$ is surjective.

We say that Φ is an ∞-*cover at* (*dimension*) n if

for any $\bar{a}\in\bar{M}_n$, $\beta\in P_{n-1}$ and $c\in M_{n-1}$ such that $\beta\|c$, $\beta\overset{\Phi}{\longmapsto}\mathrm{d}\bar{a}$ and $c\overset{\Phi}{\longmapsto}\mathrm{c}\bar{a}$, there are infinitely many distinct $a\in M_n$ such that $\mathrm{d}a=\beta$, $\mathrm{c}a=c$, and $a\overset{\Phi}{\longmapsto}\bar{a}$.

A remark similar to the one above is to be made when $n=0$.

(53) Lemma Suppose $\Phi:\mathbf{M}\longrightarrow\bar{\mathbf{M}}$ is a cover at n. Then, for any $\bar{\gamma}=\bar{\alpha}\underset{\bar{p}}{\square}\bar{\beta}$ in $\bar{P}_n$ and any $\gamma\in P_n$ such that $\gamma\overset{\Phi}{\longmapsto}\bar{\gamma}$, there are $\alpha,\beta\in P_n$ and $p\in|\langle\alpha\rangle|$ such that $\gamma=\alpha\underset{p}{\square}\beta$, $\alpha\overset{\Phi}{\longmapsto}\bar{\alpha}$, $\beta\overset{\Phi}{\longmapsto}\bar{\beta}$ and $p\overset{\Phi}{\longmapsto}\bar{p}$.

Proof. This is analogous to (5'), and the proof uses (5'). Note also that we did not assert (as we did in (5')) a unique existence.

The proof is an induction on $\#|\langle\bar{\alpha}\rangle|$.

Now, the minimal value for $\#|\langle\bar\alpha\rangle|$ is 1. In this case, $\bar\alpha$ is in $\bar M_n$, $\bar\alpha=\bar a$, $\beta\|\bar a$ and $\bar\gamma=\bar\beta$. We take $\beta \underline{\underline{\mathrm{def}}} \gamma$. Using the cover condition to $\mathrm{d}\gamma$ and $\mathrm{c}\gamma$ as β and c, since we have $\mathrm{d}\gamma \overset{\Phi}{\longmapsto} \mathrm{d}\bar a$, $\mathrm{c}\gamma \overset{\Phi}{\longmapsto} \mathrm{c}\bar a$, there is $\alpha=a\in M_n$ such that $a\|\gamma=\beta$ and $a \overset{\Phi}{\longmapsto} \bar a$. For $\{p\}=|\langle a\rangle|$, $a\square_p\beta$ is well-defined and equals γ.

Let $\#|\langle\bar\alpha\rangle|\geq 2$. We apply (46). Let $\bar\alpha=\bar\alpha_1\circ_{\bar i}\bar\alpha_2$ with $\#|\langle\bar\alpha_\varepsilon\rangle|<\#|\langle\bar\alpha\rangle|$ $(\varepsilon=1,2)$. Using the notation of (46) "with bars", we have two cases, "case 1" and "case 2". The two are similar; we deal with case 1. In case 1, we have $\bar\gamma=(\bar\alpha_1\square_{\bar q}\bar\beta)\circ_i\bar\alpha_2$. By (5'), we have γ_1, α_2 and i such that, for $\bar\gamma_1 \underline{\underline{\mathrm{def}}} \bar\alpha_1\square_{\bar q}\bar\beta$, we have $\gamma=\gamma_1\circ_i\alpha_2$, $\gamma_1 \overset{\Phi}{\longmapsto} \bar\gamma_1$, $\alpha_2 \overset{\Phi}{\longmapsto} \bar\alpha_2$ and $i \overset{\Phi}{\longmapsto} \bar i$. By the induction hypothesis applied to $\bar\alpha_1$, there are α_1, β and q such that $\gamma_1=\alpha_1\square_q\beta$, $\alpha_1 \overset{\Phi}{\longmapsto} \bar\alpha_1$, $\beta \overset{\Phi}{\longmapsto} \bar\beta$ *and* $q \overset{\Phi}{\longmapsto} \bar q$.

Note that since $\alpha_1\square_q\beta \,\|\, \alpha_1$, $\alpha \underline{\underline{\mathrm{def}}} \alpha_1\circ_i\alpha_2$ is well-defined. Let $p=\sigma[\alpha_1,\alpha_2,i](q)$. Then

$$\alpha\square_p\beta=(\alpha_1\circ_i\alpha_2)\square_p\beta=(\alpha_1\square_q\beta)\circ_i\alpha_2=\gamma_1\circ_i\alpha_2=\gamma,$$

and $\alpha \overset{\Phi}{\longmapsto} \bar\alpha$.

We say that a pd $\alpha\in P_n$ is *separated* if the function $\langle\alpha\rangle:|\langle\alpha\rangle|\to M_n$ is 1-1.

(54) **(i)** Suppose $\Phi:\boldsymbol{M}\longrightarrow\bar{\boldsymbol{M}}$ is a cover at $n\text{-}1$ and at n. Then "the cover condition becomes true for n-pd's":

for any $\bar\alpha\in\bar P_n$, $\beta\in P_{n-1}$ and $c\in M_{n-1}$ such that $\beta\|c$, $\beta \overset{\Phi}{\longmapsto} \mathrm{d}\bar\alpha$ and $c \overset{\Phi}{\longmapsto} \mathrm{c}\bar\alpha$, there is $\alpha\in M_n$ such that $\mathrm{d}\alpha=\beta$, $\mathrm{c}\alpha=c$ and $\alpha \overset{\Phi}{\longmapsto} \bar\alpha$.

(ii) Suppose $\Phi:\boldsymbol{M}\longrightarrow\bar{\boldsymbol{M}}$ is a cover at $n\text{-}1$ and it is an ∞-cover at n. Then for any $\bar\alpha\in\bar P_n$, $\beta\in P_{n-1}$ and $c\in M_{n-1}$ such that $\beta\|c$, $\beta \overset{\Phi}{\longmapsto} \mathrm{d}\bar\alpha$ and

$c \overset{\Phi}{\longmapsto} \mathrm{c}\bar{\alpha}$, there is a separated $\alpha \in M_n$ such that $\mathrm{d}\alpha = \beta$, $\mathrm{c}\alpha = c$ and $\alpha \overset{\Phi}{\longmapsto} \bar{\alpha}$.

Proof The proofs of the two parts can be given together. In what follows, the parts in [...] are needed only for (ii).

Assume $\bar{\alpha}$, β and c are given as stated. The proof is by induction on $\#|\langle\bar{\alpha}\rangle|$. [For (ii), we prove that, for any given finite set I, there is α as stated and in addition, $\mathrm{range}(\langle\alpha\rangle) \cap I = \emptyset$.]

When $\#|\langle\bar{\alpha}\rangle| = 0$, the assertion is the same as the hypothesis. [When $\#|\langle\alpha\rangle| = 0$, α is automatically separated.]

Let $\#|\langle\bar{\alpha}\rangle| \geq 1$. We can write $\bar{\alpha}$ in the form $\bar{\alpha} = \bar{a}(\langle \circ_{\bar{p}} \bar{\alpha}_{\bar{p}} \rangle)_{\bar{p} \in |\langle \mathrm{d}\bar{a} \rangle|}$ (simultaneous composition), with $\bar{a} \in \bar{R}_n$ and $\bar{\alpha}_p \in \bar{S}_n$ ($p \in |\langle \mathrm{d}\bar{a} \rangle|$).

We have, for $\bar{\beta} \overset{\mathrm{def}}{=} \mathrm{d}\bar{\alpha}$, $\bar{\gamma} \overset{\mathrm{def}}{=} \mathrm{d}\bar{a}$, $\bar{\beta}_p \overset{\mathrm{def}}{=} \mathrm{d}\bar{\alpha}_p$, that $\bar{\beta} = \bar{\gamma}(\langle \square_{\bar{p}} \bar{\beta}_{\bar{p}} \rangle)_{\bar{p} \in |\langle \bar{\beta} \rangle|}$. Since $\beta \overset{\Phi}{\longmapsto} \bar{\beta}$, with an induced bijective mapping $p \in |\langle\beta\rangle| \overset{\cong}{\longmapsto} \bar{p} \in |\langle\bar{\beta}\rangle|$, and Φ is a cover at $n\text{-}1$, by (repeated use of) (53), there are γ and β_p ($p \in |\langle\beta\rangle|$) such that $\beta = \gamma(\langle \square_p \beta_p \rangle)_{p \in |\langle\beta\rangle|}$, $\gamma \overset{\Phi}{\longmapsto} \bar{\gamma}$ and $\beta_p \overset{\Phi}{\longmapsto} \bar{\beta}_{\bar{p}}$.

Since Φ is a cover at n, $c \| \beta \| \gamma$, $\gamma \overset{\Phi}{\longmapsto} \bar{\gamma} = \mathrm{c}\bar{a}$ and $c \overset{\Phi}{\longmapsto} \mathrm{c}\bar{\alpha} = \mathrm{c}\bar{a}$, there is $a \in M_n$ such that $\mathrm{d}a = \gamma$, $\mathrm{c}a = c$ and $a \overset{\Phi}{\longmapsto} \bar{a}$ [and $a \notin I$].

Since $\gamma(\langle \square_p \beta_p \rangle)_{p \in |\langle\beta\rangle|}$ is well-defined, we have $\beta_p \| \langle\gamma\rangle(p)$. Also, $\beta_p \overset{\Phi}{\longmapsto} \mathrm{d}\alpha_{\bar{p}}$, $\langle\gamma\rangle(p) \overset{\Phi}{\longmapsto} \mathrm{c}\alpha_{\bar{p}}$. Therefore, since $\#|\langle\bar{\alpha}_{\bar{p}}\rangle| < \#|\langle\bar{\alpha}\rangle|$, by the induction hypothesis, for each $p \in |\langle\gamma\rangle|$, there is $\alpha_p \in P_n$ such that $\mathrm{d}\alpha_p = \beta_p$, $\mathrm{c}\alpha_p = \langle\gamma\rangle(p)$ and $\alpha_p \overset{\Phi}{\longmapsto} \bar{\alpha}_{\bar{p}}$.

[For a repetition-free enumeration $i \mapsto p_i$ of the set $|\langle\gamma\rangle|$, with i ranging over integers

$<\ell$ for some $\ell\in\mathbb{N}$, find, recursively in $i<\ell$, $\alpha_{p_i}\in P_n$ so that $\mathrm{d}\alpha_{p_i}=\beta_{p_i}$, $\mathrm{c}\alpha_{p_i}=\langle\gamma\rangle(p_i)$, $\alpha_{p_i}\overset{\Phi}{\longmapsto}\bar{\alpha}_{\bar{p}_i}$ and

$$|\langle\alpha_{p_i}\rangle|\cap\Big(I\cup\{a\}\cup\bigcup_{j<i}|\langle\alpha_{p_j}\rangle|\Big)=\emptyset\ .]$$

We have $\alpha\ \overline{\mathrm{def}}\ a(\langle\circ_p\alpha_p\rangle)_{p\in|\langle\mathrm{d}a\rangle|}$ is well-defined, $\mathrm{d}\alpha=\beta$, $\mathrm{c}\alpha=c$ and $\alpha\overset{\Phi}{\longmapsto}\bar{\alpha}$ as desired. [By the construction, and the fact that $\langle\alpha\rangle$ is a coproduct of $\langle a\rangle$ and the $\langle\alpha_{p_i}\rangle$, α is separated and $\mathrm{range}(\langle\alpha\rangle)\cap I=\emptyset$.]

Note the following corollary of (54)(i), proved by induction on n:

(55) Let $n\in\mathbb{N}$. If $\Phi:\boldsymbol{M}\longrightarrow\bar{\boldsymbol{M}}$ is a cover at each $k\leq n$, then the induced map $\Phi:M_n\longrightarrow\bar{M}_n$ is surjective.

The steps in the proof of (55) are similar to the ones that will be seen in the proof of the next statement, which is the one we will actually use.

(56) Suppose $\Phi:\boldsymbol{M}\longrightarrow\bar{\boldsymbol{M}}$ is a cover at each $k\leq n-2$, an ∞-cover at $n-1$, and a cover at n. Then, for any $\bar{\alpha}\in\bar{M}_n$, there is $\alpha\in M_n$ such that $\alpha\overset{\Phi}{\longmapsto}\bar{\alpha}$ and $\mathrm{d}\alpha$ is separated.

Proof Let $\bar{\beta}\ \overline{\mathrm{def}}\ \mathrm{d}\bar{\alpha}\in\bar{M}_{n-1}$. Then, $d\bar{\beta}$ and $c\bar{\beta}$ are parallel. Find, by (55) at $n-2$, $c\in M_{n-2}$ such that $c\mapsto\mathrm{c}\bar{\beta}$. We have $\mathrm{d}c\overset{\Phi}{\longmapsto}\mathrm{dc}\bar{\beta}=\mathrm{dd}\bar{\beta}$, $\mathrm{c}c\overset{\Phi}{\longmapsto}\mathrm{cc}\bar{\beta}=\mathrm{cd}\bar{\beta}$; thus, by (54)(i) at $n-2$, there is $\gamma\|c$ such that $\gamma\overset{\Phi}{\longmapsto}\mathrm{d}\bar{\beta}$. Now we can apply (54)(ii) at $n-1$, to get β separated with $\mathrm{d}\beta=\gamma$, $\mathrm{c}\beta=c$ and $\beta\overset{\Phi}{\longmapsto}\bar{\beta}$. By Φ being a cover at $n-1$, there is $\hat{c}\in M_{n-1}$ such that $\hat{c}\|\beta$ and $\hat{\mathrm{c}}\overset{\Phi}{\longmapsto}\mathrm{d}\bar{\alpha}$. Finally, by (54)(i) at n, find $\alpha\in P_n$ such that $\mathrm{d}\alpha=\beta$, $\mathrm{c}\alpha=\hat{c}$ and $\alpha\overset{\Phi}{\longmapsto}\bar{\alpha}$.