**The associative law for the $\square$-composition**

Let us verify the associative law:

$$(\alpha\square_p^{\theta}\beta)\square_{\bar{q}}^{\tau}\gamma \;=\; \alpha\square_p^{\bar{\theta}}(\beta\square_q^{\tau}\gamma) \tag{37}$$

Here, we are given that

$$p\in|\langle\alpha\rangle|\;,\;\; \beta\|^{\theta}f_{\mathrm{d\bar{e}f}}\langle\alpha\rangle(p)\;,\;\; q\in|\langle\beta\rangle|\;,\;\; \gamma\|^{\tau}g_{\mathrm{d\bar{e}f}}\langle\beta\rangle(q)\;.$$

We put

$$\bar{q}=\varphi[\alpha,\,\beta,\,p,\theta](q)\;. \tag{37.1}$$

$\bar{\theta}$ is the composite in

$$\begin{tikzcd} & \mathrm{s}\beta \arrow[dr, "\pi"] \arrow[d, phantom, "\circ"] & \\ \mathrm{s}(\langle\alpha\rangle(p)) \arrow[ur, "\theta"] \arrow[rr, "\bar{\theta}"'] & {} & \mathrm{s}(\beta\square_q^{\tau}\gamma) \end{tikzcd} \tag{38}$$

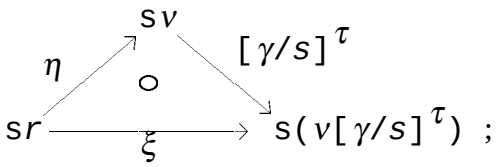
with $\pi=\pi[\beta,\,\gamma,\,q,\,\tau]$ as in (11).

The indeterminates $r$ and $s$ over $\boldsymbol{C}$ are parallel to $f$ and $g$, respectively, giving rise to $\boldsymbol{C}[r]$, $\boldsymbol{C}[s]$.

Let $v\in\mathrm{A}(\boldsymbol{C}[s])$ (soon to be taken to be the same as $\beta(q/s)$), and let $\eta$ be such that $v\|^{\eta}r$. This gives rise to the morphism

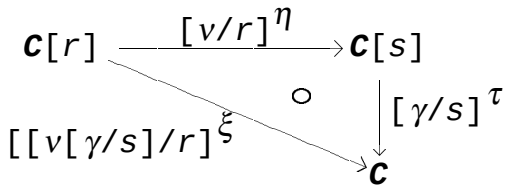
$$\begin{array}{rcll} [v/r]^{\eta}:\boldsymbol{C}[r] & \longrightarrow & \boldsymbol{C}[s] & \\ g & \longmapsto & g & (\,g\in L\,) \\ r & \longmapsto & v & \end{array}$$

such that, in addition, the action $[v/r]^{\eta}:\mathrm{s}r\xrightarrow{\cong}\mathrm{s}v$ is $\eta$. Let us take the composite $\xi$ in




(39)

thus, $\nu[\gamma/s]^{\tau} \|^{\xi} r$ . The following commutes:

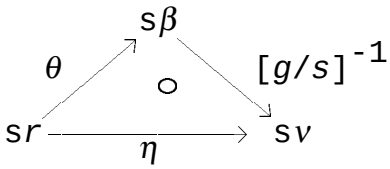



This is immediate when one looks at the effect on generators: on $\boldsymbol{C}$ , both morphisms from start to finish are identity; and on $r$ , they both give $\nu[\gamma/s]$ ; the place-actions are all right too. We have deduced the identity

$$\mu[\nu/r]^{\eta}[\gamma/s]^{\tau} = \mu[[\nu[\gamma/s]/r]^{\xi} \tag{40}$$

for $\mu \in A(\boldsymbol{C}[r])$ , $\nu \in A(\boldsymbol{C}[s])$ , $\nu \|^{\eta} r$ , $\gamma \in A$, $\gamma \|^{\tau} s$ , with $\xi$ from (39).

Let $\mu_{\overline{\text{def}}} \alpha(r/p) \in A(\boldsymbol{C}[r])$ , and $\nu_{\overline{\text{def}}} \beta(s/q) \in A(\boldsymbol{C}[s])$ . Let $\eta$ be the composite in




so that $\nu \|^{\eta} r$ .

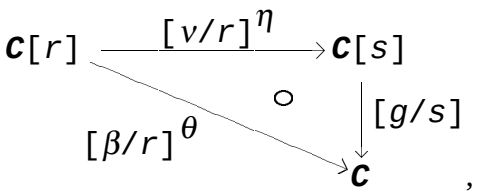
Let $\bar{q}_{\overline{\text{def}}} \varphi[\alpha, \beta, p, \theta](q)$ as in (37.1) . We have

$$\mu[\nu/r]^{\eta} = \mu[\beta/r]^{\theta}(s/\bar{q}) \ . \tag{41}$$

We show this equality by showing that the left-hand side meets the defining criteria (two of them) for the right-hand side. One of them is that

$$\mu[\nu/r]^{\eta} \xrightarrow[[g/s]]{} \mu[\beta/r]^{\theta} \ .$$

But this follows from the commutative diagram




,

which is easily verified.

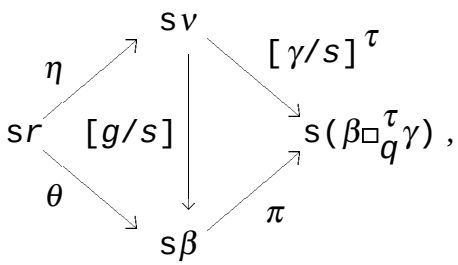
Let's write $\mu[\nu/r]^{\eta} = \alpha \square_p^{\eta} \nu$ , and $\nu[\gamma/s]^{\tau} = \beta \square_q^{\tau} \gamma$ .

The second thing we need is that $\mu[\nu/r]^{\eta}$ is $s/\bar{q}$-special. But this is immediate from an application of (36), i.e., the coproduct diagram:

$$\langle \alpha \rangle \backslash p \xrightarrow{\ \psi\ } \langle \alpha \square_p^{\eta} \nu \rangle \xleftarrow{\ \varphi\ } \langle \nu \rangle$$

This shows that $s$ gets into $\alpha \square_p^{\eta} \nu$ only from $\nu$ , and exactly at $\bar{q}=\varphi(q)$ , which is what we want for (41).

Comparing (38) and (39), we see that $\xi=\bar{\theta}$ ; this is because of the commutativity of




,

coming from the definitions of $\eta$ and $\pi$.

Combining (40) and (41), we get

$$\mu[\beta/r]^{\theta}(s/\bar{q})[\gamma/s]^{\tau} = \mu[[\nu[\gamma/s]/r]^{\bar{\theta}}$$

which is exactly

$$(\alpha \square_p^{\theta} \beta) \square_{\bar{q}}^{\tau} \gamma = \alpha \square_p^{\bar{\theta}} (\beta \square_q^{\tau} \gamma) \quad .$$

Here, we are treating a version of the $\square$-composition of [H/M/P] in which there are more parameters than in the original version (soon, we will revert to the original version). It is clear how to formulate the coherence conditions related to the commutative and associative laws in the present context; we omit the details.

To prove these laws, we argue basically as in the original treatment, which is to say, first, that these conditions are automatic for the separated case: when the arities $\langle \alpha \rangle$, ... are 1-1 functions, and they have disjoint ranges (as far as this is possible: see the original treatment); and second, that the conditions are inherited via renaming transformations (although not so called, the latter were also used in the original treatment [H/M/P]), because of the invariance of the $\psi$- and $\varphi$-structures on $\square$ under renaming transformations (we have not elaborated on the $\varphi$-structure in this regard, but the treatment is, of course, similar to that for $\psi$); and, finally, that any given situation in which we want to verify the coherence conditions can be "resolved" by a separated situation: this was done in the original treatment, but also see §3.(27).

These coherence conditions can also be verified by means that are more directly related to the arguments presented here.

**The multicategory of the $\square$-composition**

The simplification that is required to arrive at an honest multicategory in which $\square$ is the composition is achieved by restricting the domain of the composition operation, i.e., by

imposing a stricter condition of composability. (This was done also in [H/M/P], in an essentially identical way to the present one; see section 1.8. in Part 1 of [H/M/P]).

We *assume* that we are given, besides the free multicategory $\boldsymbol{C}$, another multicategory $\boldsymbol{E}$, together with a strict morphism $\mathsf{d}:\boldsymbol{C}\to\boldsymbol{E}$ which is the identity on upper level objects.

Suppose we have $\alpha\in\mathsf{A}(\boldsymbol{C})$, $p\in|\langle\alpha\rangle|$; let $f_{\mathsf{d}\bar{\bar{e}}\mathsf{f}}\langle\alpha\rangle p$. Let $\beta\in\mathsf{A}(\boldsymbol{C})$ such that

$$\mathsf{t}\beta=\mathsf{t}f\,,\qquad \text{and (!) } \mathsf{d}\beta=\mathsf{d}f\,. \tag{42}$$

Note that the last condition implies that $\mathsf{s}\beta=\mathsf{s}f$, since $\mathsf{d}$ is the identity on upper level objects and strict. Thus, we have $\beta\|f$. We define, as before

$$\alpha\square_p\beta = \alpha(r/p)[\beta/r]\ . \tag{43}$$

(that is, we consider $[\beta/r]^{\theta}$ only with $\theta$=identity; we will have to see, however, that we are not obliged to use other $\theta$'s!). The point is the precondition which is stronger than $\beta\|f$. We codify this by declaring a new multicategory $\boldsymbol{D}$ in which

$$\mathsf{A}(\boldsymbol{D})=\mathsf{A}(\boldsymbol{C})\,,\ \mathsf{O}(\boldsymbol{D})=L\,,\ \dot{\mathsf{O}}(\boldsymbol{D})=\mathsf{O}(\boldsymbol{E})\times\dot{\mathsf{O}}(\boldsymbol{C})\,,$$

with the map $\mathsf{O}(\boldsymbol{D})\to\dot{\mathsf{O}}(\boldsymbol{D})$ being

$$g\longmapsto \mathsf{T}g_{\mathsf{d}\bar{\bar{e}}\mathsf{f}}(\mathsf{d}f,\mathsf{t}f) : L\longrightarrow\mathsf{O}(\boldsymbol{E})\times\dot{\mathsf{O}}(\boldsymbol{C})\ .$$

Furthermore, for $\alpha\in A=\mathsf{A}(\boldsymbol{D})$, we put

$$\mathsf{s}_{\boldsymbol{D}}(\alpha)=\langle\alpha\rangle\,,\qquad \mathsf{t}_{\boldsymbol{D}}(\alpha)=\mathsf{T}\alpha_{\mathsf{d}\bar{\bar{e}}\mathsf{f}}(\mathsf{d}\alpha,\mathsf{t}_{\boldsymbol{C}}\alpha)\ .$$

Thereby, we have made sure that the *composability condition* for $\alpha\square_p\beta$ being well-defined is precisely (42) -- which was our goal! Needless to say, composition in $\boldsymbol{D}$, the $\square$-composition, is defined by (43). Also, the amalgamating maps $\psi^{\square}$, $\varphi^{\square}$ are defined as they were before, always with only identities for the $\theta$'s.

It remains to verify that $\mathbf{D}$ so defined is indeed a multicategory. The key to the proof of this statement is the fact that all morphisms of the form $[\beta/r]$ that we need are, or more precisely, may be assumed to be, strict.

Assume that $f \in L$, and let $r$ be an indeterminate arrow over $\mathbf{C}$ such that $r \| f$ (thus: $r$ is not just any indeterminate!). Consider $\mathbf{C}[r]$. Define the morphism $\mathrm{d}:\mathbf{C}[r] \longrightarrow \mathbf{E}$ by declaring that, on $\mathbf{C}$, $\mathrm{d}$ be the given map $\mathrm{d}:\mathbf{C} \to \mathbf{E}$, and $\mathrm{d}(r)=\mathrm{d}f$. The latter specification is legitimate: we have

$$\mathrm{s}_{\mathbf{E}}(\mathrm{d}f) = \mathrm{d} \circ \mathrm{s}_{\mathbf{C}[r]}(r), \qquad \mathrm{t}_{\mathbf{E}}(\mathrm{d}f) = \mathrm{d}(\mathrm{t}_{\mathbf{C}[r]}(r)),$$

since

$$\mathrm{s}_{\mathbf{C}[r]}(r) = \mathrm{s}_{\mathbf{C}[r]}(f) = \mathrm{s}_{\mathbf{C}}(f), \qquad \mathrm{t}_{\mathbf{C}[r]}(r) = \mathrm{t}_{\mathbf{C}[r]}(f) = \mathrm{t}_{\mathbf{C}}(f),$$

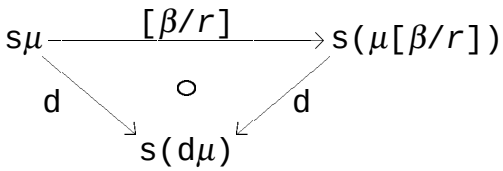
and $\mathrm{d}$ is a strict morphism $\mathrm{d}:\mathbf{C} \to \mathbf{E}$. In fact, we may require $\mathrm{d}:\mathbf{C}[r] \longrightarrow \mathbf{E}$ to be strict on $r$. Furthermore, we may require, even, that $\mathrm{d}:\mathbf{C}[r] \longrightarrow \mathbf{E}$ be strict altogether; this may require changing $\mathbf{C}[r]$ to an isomorphic copy in which all what that happens is that the sources of arrows undergo a "permutation" (isomorphic transformation); see [H/M/P], Part 2, section 2.

Having specified $\mathbf{C}[r]$ in this way, consider $\beta \in A$ such that $\mathrm{T}\beta = \mathrm{T}f$, and as a consequence, $\beta \| f$ and $\beta \| r$; and consider $[\beta/r]:\mathbf{C}[r] \longrightarrow \mathbf{C}$. The claim is that $[\beta/r]$ is a strict morphism. Consider the diagram

$$\begin{array}{ccc} \mathbf{C}[r] & \xrightarrow{\quad [\beta/r] \quad} & \mathbf{C} \\ & {\scriptstyle \mathrm{d}} \searrow \quad \swarrow {\scriptstyle \mathrm{d}} & \\ & \mathbf{E} & \end{array} \tag{44}$$

It commutes, by the usual argument; the key is the fact that $\mathrm{d}\beta = \mathrm{d}f = \mathrm{d}r$.

Let $\mu$ be any arrow in $\mathbf{C}[r]$. We have the commutativity




of the actions on places. Since the $\mathtt{d}$-actions are identities, so must be the $[\beta/r]$-action as well. The claim is shown.

The commutative law for $\boldsymbol{D}$ is (25) with $\theta$ and $\tau$ taken to be identities. This is simply a special case of (25), so it is true. However, the associative law, the special case of (37) with $\theta$ and $\tau$ being identities, requires that $\bar{\theta}$ used in (37) and defined in (38) be the same as $\theta$. This is true since $\pi$ in (38) is an identity: the latter follows from looking at the definition (11) of $\pi$, since the place-actions involved are identities.

Suppose given $\alpha$, $\beta$ and $p$ so that $\alpha \square_p \beta$ is well-defined (see (42)). Anticipating the notation used in a multitopic set, let us write $\mathtt{c}$ for $\mathtt{t}_{\boldsymbol{c}}$. The following are immediate consequences of the definition and (44):

$$\mathtt{d}(\alpha \square_p \beta) = \mathtt{d}\alpha\,, \tag{45.1}$$

$$\mathtt{c}(\alpha \square_p \beta) = \mathtt{c}\alpha\,. \tag{45.2}$$

We can restate the definition of the $\square$-composition in an elementary fashion.

**(46)** We have

$$\alpha \square_p \beta = \begin{cases} \beta & \text{if } |\langle \alpha \rangle| = \{p\} \\ (\alpha_1 \square_q \beta) \circ_i \alpha_2 & \text{if } \alpha = \alpha_1 \circ_i \alpha_2\,,\ \sigma[\alpha_1, \alpha_2, i](q) = p \text{ (**case 1**)} \\ \alpha_1 \circ_i (\alpha_2 \square_q \beta) & \text{if } \alpha = \alpha_1 \circ_i \alpha_2\,,\ \rho[\alpha_1, \alpha_2, i](q) = p \text{ (**case 2**)} \end{cases}$$

These facts are immediate from the definition. For instance, in "case 1",

$$\alpha \square_p \beta \overset{1}{=} \alpha(r/p)[\beta/r] \overset{2}{=} (\alpha_1(r/q) \circ_i \alpha_2)[\beta/r]$$

$$= (\alpha_1(r/q)[\beta/r]) \circ_i \alpha_2[\beta/r] \overset{3}{=} (\alpha_1 \Box_q \beta) \circ_i \alpha_2 \overset{4}{;}$$

1: definition of $\Box$ ; 2: (4.2) ; 3: $[\beta/r]$ preserves $\circ_i$ ; 4: definition of $\Box$ , and the fact that $[\beta/r]$ is the identity on $\boldsymbol{C} \subset \boldsymbol{C}[r]$ .

Note that (46) serves as a definition of $\Box$ , without talking about $\boldsymbol{C}[r]$ and the like, in the sense that it obviously fixes the meaning of $\Box$ . It could not, however, be given as a plain definition; it requires proof that the expressions on the right depend only on $\alpha$, $\beta$ and $p$ , and not on the decompositions chosen.

The definition of the amalgamation structure for $\Box$ can be restated in the style of (46) as follows. Throughout, we assume that $\alpha \Box_p \beta$ is well-defined.

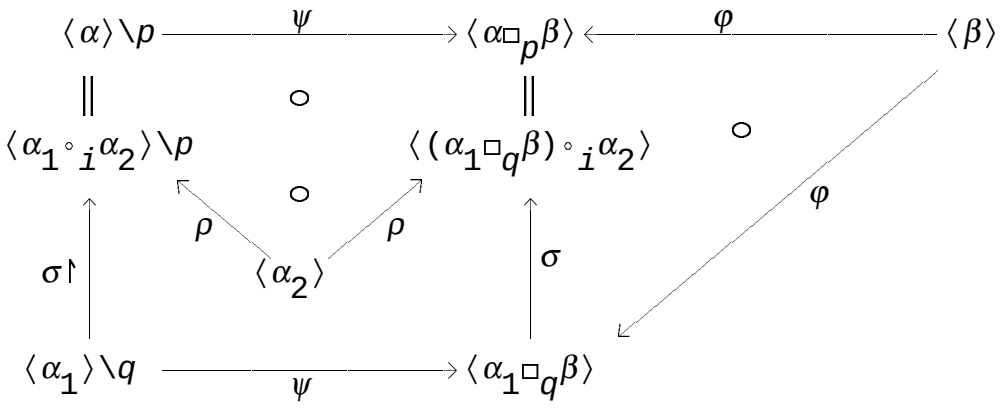
**(47.1)** If $|\langle \alpha \rangle| = \{p\}$ , then

$$\psi[\alpha, \beta, p] : \langle \alpha \rangle \backslash p \to \langle \alpha \Box_p \beta \rangle \text{ is } !: \emptyset \to \langle \beta \rangle ,$$

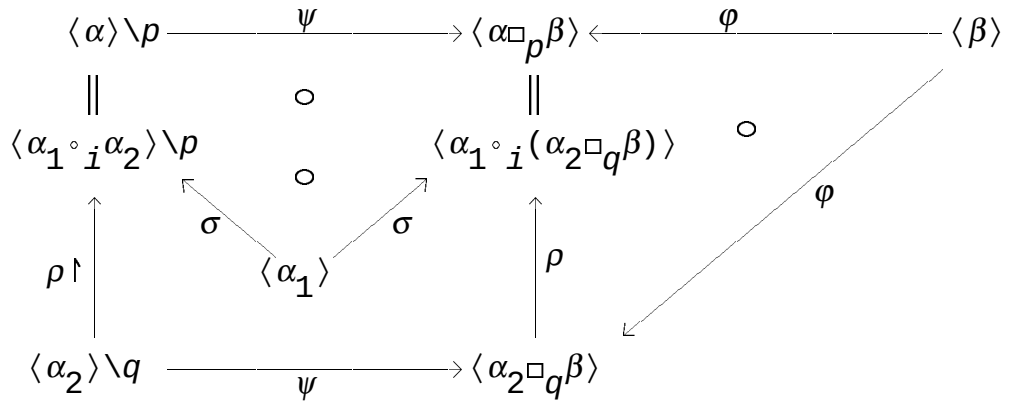
and

$$\varphi[\alpha, \beta, p] : \langle \beta \rangle \to \langle \alpha \Box_p \beta \rangle \text{ is } \mathrm{id}: \langle \beta \rangle \to \langle \beta \rangle .$$

**(47.2)** If $\alpha = \alpha_1 \circ_i \alpha_2$ , $\sigma[\alpha_1, \alpha_2, i](q) = p$ , then




**(47.3)** If $\alpha = \alpha_1 \circ_i \alpha_2$ , $\rho[\alpha_1, \alpha_2, i](q) = p$ , then




This completes the new treatment of the multicategory of function replacement.

In summary, let us note that the "multicategory of function replacement" is defined on the basis of any given free multicategory $\boldsymbol{C}$, together with a strict morphism $\mathsf{d}:\boldsymbol{C}\longrightarrow\boldsymbol{E}$ of multicategories which is the identity on upper-level objects (exactly as in [H/M/P]).

In item (viii) in the definition of "multitopic set", in section 1, part 3 of [H/M/P], now we should be referring to the construction given here, instead of the one in [H/M/P] (however, the present construction results in essentially the same structure as in [H/M/P]); $\mathsf{d}:\boldsymbol{C}\longrightarrow\boldsymbol{E}$ becomes $\mathsf{d}_k:\boldsymbol{C}_k\longrightarrow\boldsymbol{D}_{k-1}$, and the resulting multicategory of function replacement is $\boldsymbol{D}_k$.

**Invariance under morphisms**

Suppose we are given the following:

$\mathsf{d}:\boldsymbol{C}\to\boldsymbol{E}$, $\mathsf{d}:\bar{\boldsymbol{C}}\to\bar{\boldsymbol{E}}$, both strict and acting as the identity on upper level objects;

G:$\boldsymbol{E}\to\bar{\boldsymbol{E}}$;

the HAA-morphism $F:(\boldsymbol{C},\langle\text{-}\rangle,\rho,\sigma)\longrightarrow(\bar{\boldsymbol{C}},\langle\text{-}\rangle,\rho,\sigma)$;

such that the diagram