**Invariance of $(\Box, \psi)$ under renaming transformations**

Although it is trivial to prove, perhaps it's worth pointing out that the definition of the $(\Box, \psi)$-structure is well-defined in the sense that the items in (11.1) are independent of the choice of the particular copy of $\boldsymbol{C} \xrightarrow{\mathsf{can}} \boldsymbol{C}[r]$ (the latter is defined only up to isomorphism, whereas (11.1) is to be uniquely determined from $(\alpha, \beta, p)$ and the HAA-structure).

A useful generalization of the last remark is the invariance of the $([\beta/r]^{\theta}, \psi)$-structure under renaming transformations, generalizing the corresponding fact for the HAA-structure. Here is how this can be stated.

Suppose we have the renaming morphism $F:\boldsymbol{C} \longrightarrow \bar{\boldsymbol{C}}$, and we have $[\beta/r]^{\theta}:\boldsymbol{C}[r] \to \boldsymbol{C}$ as above. Then we can introduce a new indeterminate $\bar{r}$ over $\bar{\boldsymbol{C}}$ by requiring that $\bar{r}$ be parallel to $\bar{f}_{\mathrm{def}} Ff$. This gives us $\bar{F}:\boldsymbol{C}[r] \longrightarrow \bar{\boldsymbol{C}}[\bar{r}]$ that extends $F$, takes $r$ to $\bar{r}$, and has the action on places in $r$ given by

$$\bar{F}_r = F_f : F \circ \mathsf{s}r = F \circ \mathsf{s}f \xrightarrow{\cong} \mathsf{s}(Ff) = \mathsf{s}\bar{r} .$$

Any map $F:A \to \bar{A}$ of sets induces a functor $F:A^{\#} \longrightarrow \bar{A}^{\#}$ in an obvious way for which the action on objects is $s \longmapsto F \circ s$. Notice that for any $\mu \in A = \mathsf{A}(\boldsymbol{C}[r])$, we have that $\langle \mu \rangle \backslash r$ is an object in $A^{\#}$, and $\psi:\langle \mu \rangle \backslash r \to \langle \mu[\beta/r]^{\theta} \rangle$ is an arrow of $A^{\#}$. Therefore, with $F:\boldsymbol{C} \longrightarrow \bar{\boldsymbol{C}}$ as given, it makes sense to write $F\psi : F \circ \langle \mu \rangle \backslash r \to F \circ \langle \mu[\beta/r]^{\theta} \rangle$.

Let us indicate the effect of $\bar{F}$, and also that of $F$, by $\overline{(\text{-})}$; thus, for instance, we have $\bar{\beta} = F\beta$, $\bar{f} = Ff$. The following determines the isomorphism $\bar{\theta}:\mathsf{s}\bar{r} \xrightarrow{\cong} \mathsf{s}\bar{\beta}$:

$$\begin{array}{ccc} F \circ \mathsf{s}r & \xrightarrow{\quad F\theta \quad} & F \circ \mathsf{s}\beta \\ \bar{F} \downarrow \cong & \circlearrowright & \cong \downarrow \mathsf{F} \\ \mathsf{s}\bar{r} & \xrightarrow[\quad \bar{\theta} \quad]{} & \mathsf{s}\bar{\beta} \end{array} \qquad (14)$$

With this, we have $[\bar{\beta}/\bar{r}]^{\bar{\theta}}:\bar{\boldsymbol{C}}[\bar{r}]\longrightarrow\bar{\boldsymbol{C}}$, $\langle\bar{\beta}/\bar{r}\rangle^{\bar{\theta}}:\bar{\boldsymbol{C}}[\bar{r}]\longrightarrow\bar{\boldsymbol{C}}^{@}$.

We define the morphism $F^{@}:\boldsymbol{C}^{@}\longrightarrow\bar{\boldsymbol{C}}^{@}$. It is defined so that, among others, the following (with $\Gamma$, $\bar{\Gamma}$ the forgetful maps) commutes:

$$\begin{array}{ccc} \boldsymbol{C}^{@} & \xrightarrow{\quad\Gamma\quad} & \boldsymbol{C} \\ {\scriptstyle F^{@}}\downarrow & \circ & \downarrow{\scriptstyle F} \\ \bar{\boldsymbol{C}}^{@} & \xrightarrow[\quad\bar{\Gamma}\quad]{} & \bar{\boldsymbol{C}} \end{array} .$$

In particular, $F^{@}$ maps the arrow $\Lambda=(\gamma,\ \mu,\ \tau,\ t,\ \dot{\psi},\ \iota)$ to $(\bar{\mu},\bar{\gamma},\bar{\tau},\bar{\mathrm{t}},\dot{\bar{\psi}},\bar{\iota})$ where the last four items are determined by the commutative diagrams

$$\begin{array}{ccc} \mathrm{s}(\bar{\mu}) & \xrightarrow{\quad\bar{\tau}\quad} & \mathrm{s}(\bar{\gamma}) \\ \bar{F}\uparrow\cong & \circ & F\uparrow\cong \\ F\circ\mathrm{s}\mu & \xrightarrow[\quad F\tau\quad]{} & F\circ\mathrm{s}\gamma \end{array} \qquad \begin{array}{ccccc} \langle\bar{\mu}\rangle & \xleftarrow{\quad\bar{\iota}\quad} & \bar{t} & \xrightarrow{\quad\dot{\bar{\psi}}\quad} & \langle\bar{\gamma}\rangle \\ \bar{F}\uparrow\cong & \circ & \uparrow & \circ & \uparrow F \\ \bar{F}\circ\langle\mu\rangle & \xleftarrow[\quad\iota\quad]{} & t & \xrightarrow[\quad\dot{\psi}\quad]{} & F\circ\langle\gamma\rangle \end{array} \qquad (16)$$

and the fact that $\bar{\iota}$ is to be an inclusion. It is left to the reader to check that $F^{@}$ is thereby determined, and it is a morphism of multicategories.

We claim that the diagram

$$\begin{array}{ccc} \boldsymbol{C}[r] & \xrightarrow{\quad\langle\beta/r\rangle^{\theta}\quad} & \boldsymbol{C}^{@} \\ {\scriptstyle\bar{F}}\downarrow & \circ & \downarrow{\scriptstyle F^{@}} \\ \bar{\boldsymbol{C}}[\bar{r}] & \xrightarrow[\quad\langle\bar{\beta}/\bar{r}\rangle^{\bar{\theta}}\quad]{} & (\bar{\boldsymbol{C}})^{@} \end{array} \qquad (17)$$

commutes. The commutativity of the diagram on the generators of $\boldsymbol{C}$ is clear. The

commutativity on the generator $r$, including the action on places, is ensured by the definition of $\bar{\theta}$. The conclusions follow by the universal property of $\boldsymbol{C}[r]$.

The commutativity

$$\begin{array}{ccc} \boldsymbol{C}[r] & \xrightarrow{\quad [\beta/r]^{\theta} \quad} & \boldsymbol{C} \\ {\scriptstyle \bar{F}}\downarrow & \circ & \downarrow{\scriptstyle F} \\ \bar{\boldsymbol{C}}[\bar{r}] & \xrightarrow[\quad [\bar{\beta}/\bar{r}]^{\bar{\theta}} \quad]{} & \bar{\boldsymbol{C}} \end{array} \tag{18}$$

is a consequence.

Applying (18) to an arbitrary $\mu \in \mathsf{A}(\boldsymbol{C}[r])$, and any $\beta \in A$ parallel to $r$, we have

$$\overline{(\mu[\beta/r]^{\theta})} = \bar{\mu}[\bar{\beta}/\bar{r}]^{\bar{\theta}}; \tag{19}$$

whereas the application of (17) gives

$$\begin{array}{ccc} \langle \bar{\mu} \rangle \backslash \bar{r} & \xrightarrow{\quad \dot{\psi} \quad} & \langle \bar{\mu}[\bar{\beta}/\bar{r}]^{\bar{\theta}} \rangle \\ {\scriptstyle \bar{F}}\uparrow & \circ & \uparrow{\scriptstyle F} \\ F \circ (\langle \mu \rangle \backslash r) & \xrightarrow[\quad F\dot{\psi} \quad]{} & F \circ \langle \mu[\beta/r] \rangle^{\theta} \end{array} \tag{20}$$

Next, let $\alpha \in A$, $p \in |\langle \alpha \rangle|$, $f = \langle \alpha \rangle(p)$, and let the indeterminate $r$ be parallel to $f$. Continuing with our "bar" notation, we have

$$\overline{\alpha(r/p)} = \bar{\alpha}(\bar{r}/\bar{p}) \tag{21}$$

$$\overline{\alpha \square_p^{\theta} \beta} = \bar{\alpha} \square_{\bar{p}}^{\bar{\theta}} \bar{\beta} \tag{22}$$

and

$$\begin{array}{ccc} F\circ\langle\alpha\rangle\backslash p & \xrightarrow{\ \psi\ } & F\circ\langle\alpha\,\square\,{}^{\theta}_{p}\beta\rangle \\ \Big\downarrow{\scriptstyle \bar{F}} & \circ & \Big\downarrow{\scriptstyle F} \\ \langle\bar{\alpha}\rangle\backslash\bar{p} & \xrightarrow{\ \bar{\psi}\ } & \langle\bar{\alpha}\,\square\,{}^{\bar{\theta}}_{\bar{p}}\bar{\beta}\rangle \end{array} \qquad (23)$$

For (21), let $\mu=\alpha(r/p)$ ; we have to see that $\bar{\mu}$ is mapped by $[\bar{f}/\bar{r}]$ to $\bar{\alpha}$ , and that $\bar{\mu}$ is $\bar{p}'$-special, for $\bar{p}'=[\bar{f}/\bar{r}]^{-1}(p)$ . The first fact follows from (18) for $\beta=f$ and $\theta=\mathrm{id}$ . The same special case of (18), now applied to the actions on places, gives

$$\begin{array}{ccc} F\circ\langle\mu\rangle & \xrightarrow{\ [f/r]\ } & \langle\alpha\rangle \\ \Big\downarrow{\scriptstyle \bar{F}} & \circ & \Big\downarrow{\scriptstyle F} \\ \langle\bar{\mu}\rangle & \xrightarrow[\ [\bar{f}/\bar{r}]\ ]{} & \langle\bar{\alpha}\rangle \end{array} \quad . \qquad (24)$$

It follows that the isomorphism $\bar{F}:\bar{F}\circ\langle\mu\rangle \xrightarrow{\ \cong\ } \langle\bar{\mu}\rangle$ takes $p'$ to $\bar{p}'$ ; now, the desired second fact follows from the presence of the same isomorphism.

(22) is a special case of (19).

(23) follows from (20) and (24).

**The commutative law for the $\square$-composition**

Before we complete defining amalgamation for $\square$ with the $\varphi$-item, let us see the commutativity law for $\square$ : for $\alpha$ , $\beta$ , $\gamma$ in $\mathsf{A}(\boldsymbol{C})$ ,

$$(\alpha\,\square\,{}^{\theta}_{p}\beta)\,\square\,{}^{\tau}_{\bar{q}}\gamma \;=\; (\alpha\,\square\,{}^{\tau}_{q}\gamma)\,\square\,{}^{\theta}_{\bar{p}}\beta \qquad (25)$$

provided

$p, q\in |\langle\alpha\rangle|$ , $p\neq q$ ,

$$\theta : s(\langle\alpha\rangle(p)) \xrightarrow{\cong} s\beta\,,\ \ t(\langle\alpha\rangle(p)) = t\beta\,,$$
$$\tau : s(\langle\alpha\rangle(q)) \xrightarrow{\cong} s\gamma\,,\ \ t(\langle\alpha\rangle(q)) = t\gamma\,,$$
$$\bar{q} = \psi^{\Box}[\alpha, \beta, p, \theta](q)\,,\ \ \bar{p} = \psi^{\Box}[\alpha, \gamma, q, \tau](p) \qquad (25')$$

Note that $\langle\alpha\rangle(p) = \langle\alpha \Box_{q}^{\tau} \gamma\rangle(\bar{p})$ ; thus the $\Box_{\bar{p}}^{\theta}$-composition makes sense; similarly for $\Box_{\bar{q}}^{\tau}$.

We introduce two different indeterminates $r$ and $s$ over $\boldsymbol{C}$, with $r \| f$ $_{\mathrm{d}\bar{\mathrm{e}}\mathrm{f}}\alpha(p)$, $s \| g$ $_{\mathrm{d}\bar{\mathrm{e}}\mathrm{f}}\alpha(q)$, and consider the multicategory $\boldsymbol{C}[r, s]$ obtained by simultaneously adjoining $r$ and $s$ to $\boldsymbol{C}$. We have the inclusion $\boldsymbol{C} \longrightarrow \boldsymbol{C}[r, s]$, with the universal property that for any $F : \boldsymbol{C} \longrightarrow \boldsymbol{E}$, any $r'$, $s'$ in $\boldsymbol{E}$ and any $\theta$, $\tau$ such that $r' \|^{\theta} Ff$, $s' \|^{\tau} Fg$, there is a unique arrow $\bar{F} : \boldsymbol{C}[r, s] \longrightarrow \boldsymbol{E}$ extending $F$, mapping $r$ to $r'$, $s$ to $s'$, and having the actions $\bar{F} \circ sr \longrightarrow sr'$, $\bar{F} \circ ss \longrightarrow ss'$ equal to $\theta : F \circ sf \longrightarrow sr'$, $\tau : F \circ sg \longrightarrow ss'$, respectively.

The multicategories $\boldsymbol{C}[r]$, $\boldsymbol{C}[s]$ can be regarded as included in $\boldsymbol{C}[r, s]$ : the canonical morphisms $\boldsymbol{C}[r] \longrightarrow \boldsymbol{C}[r, s]$, $\boldsymbol{C}[s] \longrightarrow \boldsymbol{C}[r, s]$, given by the universal properties of $\boldsymbol{C}[r]$ and $\boldsymbol{C}[s]$, respectively, are inclusions. In fact, the inclusion $\boldsymbol{C}[r] \longrightarrow \boldsymbol{C}[r, s]$ endows $\boldsymbol{C}[r, s]$ with the the universal property of $\boldsymbol{C}[r][s]$. Similarly, we can regard $\boldsymbol{C}[r, s]$ as being $\boldsymbol{C}[s][r]$.

We fix an HAA structure on $\boldsymbol{C}[r, s]$, which may be, and is, assumed to extend those on $\boldsymbol{C}$, $\boldsymbol{C}[r]$ and $\boldsymbol{C}[s]$.

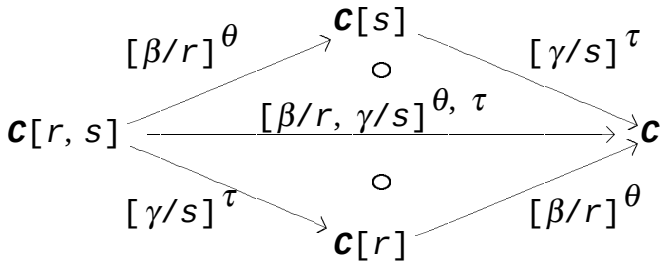
For any $\beta, \gamma \in \boldsymbol{C}$ and $\theta$, $\tau$ such that $\beta \|^{\theta} f$, $\gamma \|^{\tau} g$, we have the morphism

$$[\beta/r,\ \gamma/s]^{\theta,\ \tau} : \boldsymbol{C}[r, s] \longrightarrow \boldsymbol{C}$$

determined by the properties that it extends the identity $\boldsymbol{C} \longrightarrow \boldsymbol{C}$, takes $r$ to $\beta$, $s$ to $\gamma$, and has the actions $\theta : sr \longrightarrow s\beta$, $\gamma : ss \longrightarrow s\gamma$ on places.

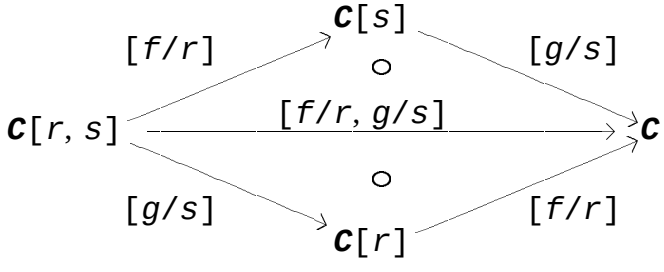
Note that, besides $[\beta/r]^{\theta}:\mathbf{C}[r]\longrightarrow\mathbf{C}$ and $[\gamma/s]^{\tau}:\mathbf{C}[s]\longrightarrow\mathbf{C}$, we also have $[\beta/r]^{\theta}:\mathbf{C}[r,s]\longrightarrow\mathbf{C}[s]$ and $[\gamma/s]^{\tau}:\mathbf{C}[r,s]\longrightarrow\mathbf{C}[r]$, by our identifications of $\mathbf{C}[r,s]$ with $\mathbf{C}[s][r]$ and $\mathbf{C}[r][s]$.

We have the following commutative diagram:




(26)

The commutativity is checked, as usual, on the generators. A special case is




(27)

$\alpha(r/p, s/q)$ is defined to be the unique $\mu\in\mathsf{A}(\mathbf{C}[r,s])$ for which $\mu[f/r, g/s]=\alpha$, and, for $p^{\#}, q^{\#}\in|\mathsf{s}\mu|$ defined by $[f/r, g/s](p^{\#})=p$, $[f/r, g/s](q^{\#})=q$, we have that $\mu$ is $(r/p^{\#}, s/q^{\#})$-special, meaning that $(\mathsf{s}\mu)(p^{\#})=r$, $(\mathsf{s}\mu)(q^{\#})=s$, and for all $n\in|\mathsf{s}\mu|-\{p^{\#}, q^{\#}\}$, $(\mathsf{s}\mu)(n)\in L$. The proof of the existence and uniqueness of $\alpha(r/p, s/q)$ is similar to that of (5).

With the actions $[f/r]:|\langle\alpha(r/p)\rangle|\xrightarrow{\cong}|\langle\alpha\rangle|$, $[g/s]:|\langle\alpha(s/q)\rangle|\xrightarrow{\cong}|\langle\alpha\rangle|$, we write, as before, $p'=[f/r]^{-1}(p)$, $q'=[g/s]^{-1}(q)$; but also, we put $q^{*}=[f/r]^{-1}(q)$, $p^{*}=[g/s]^{-1}(p)$.

Let $\mu=\alpha(r/p, s/q)$. By (27), we get

$$\mu[f/r][g/s]=\alpha. \qquad (28)$$

With $[g/s]: |\langle \mu[f/r] \rangle| \longrightarrow |\langle \alpha \rangle|$ , let

$$\tilde{q} = [g/s]^{-1}(q) \, . \tag{29}$$

Consider

$$\begin{array}{ccc} [f/r] \circ \langle \mu \rangle & \xrightarrow[\cong]{[f/r]} & \langle \mu[f/r] \rangle \\ q^{\#} & \longmapsto & \tilde{q} \end{array} \, . \tag{30}$$

The fact that $q^{\#} \mapsto \tilde{q}$ follows from (27). The fact that $\mu$ is $(r/p^{\#}, s/q^{\#})$-special, the existence of the isomorphism (30), and the fact $f \in L$ together immediately imply that $\mu[f/r]$ is $s/\tilde{q}$-special. This, with (28), (29), tells us that

$$\mu[f/r] = \alpha(s/q) \tag{31}$$

and $\tilde{q} = q'$ . Of course, similarly,

$$\mu[g/s] = \alpha(r/p) \, . \tag{32}$$

Notice that (31) is the first step towards proving the fact that $\mu = \alpha(s/q)(r/p^{*})$ . (27) implies that $p^{\#} \xmapsto{[f/r]} p^{*}$ . Thus, what remains is to show

$\mu$ is $(r/p^{\#})$-special over $\boldsymbol{C}[s]$ .

But this is obvious from $\mu$ being $(r/p^{\#}, s/q^{\#})$-special. We conclude that

$$\mu = \alpha(r/p, s/q) = \alpha(s/q)(r/p^{*}) = \alpha(r/p)(s/q^{*}) \tag{33}$$

We repeat that we have the following:

$$p^{\#} \xmapsto{[f/r]} p^{*} \, , \qquad q^{\#} \xmapsto{[f/r]} q'$$

$$p^{\#} \xmapsto{[g/s]} p' , \qquad q^{\#} \xmapsto{[g/r]} q^{*} .$$

The commutativity

$$\begin{array}{ccc} \boldsymbol{C}[r,s] & \xrightarrow{[\beta/r]^{\theta}} & \boldsymbol{C}[s] \\ {\scriptstyle [g/s]}\downarrow & \circ & \downarrow{\scriptstyle [g/s]} \\ \boldsymbol{C}[r] & \xrightarrow[{[\beta/r]^{\theta}}]{} & \boldsymbol{C} \end{array}$$

is a(nother) special case of (26). When applied to $\mu$, we obtain that

$$\mu[\beta/r]^{\theta}[g/s] = \mu[g/s][\beta/r]^{\theta} = \alpha(r/p)[\beta/r]^{\theta} \underset{\text{def}}{=} \alpha\Box_{p}^{\theta}\beta .$$

This is the first step towards showing the fact that $(\alpha\Box_{p}^{\theta}\beta)(s/\bar{q}) = \mu[\beta/r]^{\theta}$; here, $\bar{q}$ is as in (25). It remains to see that, for $\hat{q} \xmapsto{[g/s]} \bar{q}$, we have that $\mu[\beta/r]^{\theta}$ is $s/\hat{q}$-special. Because of (33), we may write $\mu[\beta/r]^{\theta}$ as $\alpha(s/q)\Box_{p^{*}}^{\theta}\beta$.

We apply (23) with $[g/s]:\boldsymbol{C}[s] \longrightarrow \boldsymbol{C}$ as $F$, and $[g/s]:\boldsymbol{C}[s][r] \longrightarrow \boldsymbol{C}[r]$ as $\bar{F}$. The $\bar{\theta}$ in (23) is determined by (14). Now, $\bar{r}=r$, and by the definition of $[g/s]$, the actions $[g/s]:\mathrm{s}r \to \mathrm{s}r$, $[g/s]:\mathrm{s}\beta \to \mathrm{s}\beta$ are identities. Therefore, $\bar{\theta}=\theta$. We obtain

$$\begin{array}{ccc} [g/s]\circ\langle\alpha(s/q)\rangle\backslash p^{*} & \xrightarrow[\cong]{[g/s]} & \langle\alpha\rangle\backslash p \\ \downarrow{\scriptstyle \psi^{\Box}} & \circ & \downarrow{\scriptstyle \psi^{\Box}} \\ [g/s]\circ\langle\alpha(s/q)\Box_{p^{*}}^{\theta}\beta\rangle & \xrightarrow[{[g/s]}]{\cong} & \langle\alpha\Box_{p}^{\theta}\beta\rangle \end{array}$$

From the last commutativity, it follows that the left vertical $\psi^{\Box}$ maps $q'$ to $\hat{q}$. Now, looking at (12), and knowing that $\alpha(s/q)$ is $q'$-special, we see that $\alpha(s/q)\Box_{p^{*}}^{\theta}\beta$ is $s/\hat{q}$-special.

We conclude, as was foreseen, that

$$(\alpha \square^{\theta}_{p} \beta)(s/\bar{q}) = \mu[\beta/r]^{\theta} \quad . \tag{34}$$

But then

$$(\alpha \square^{\theta}_{p} \beta) \square^{\tau}_{\bar{q}} \gamma = (\alpha \square^{\theta}_{p} \beta)(s/\bar{q})[\gamma/s]^{\tau} = \mu[\beta/r]^{\theta}[\gamma/s]^{\tau} \overset{*}{=} \mu[\beta/r,\ \gamma/s]^{\theta,\ \tau} ,$$

where the equality marked with $*$ is by (26). Of course, the equality

$$(\alpha \square^{\tau}_{q} \gamma) \square^{\theta}_{\bar{p}} \beta = \mu[\beta/r,\ \gamma/s]^{\theta,\ \tau}$$

is true also by reasons of symmetry. We have proved (25).

By the way, $\mu[\beta/r,\ \gamma/s]^{\theta,\ \tau}$ may be reasonably denoted as $\alpha(\square^{\theta}_{p}\beta,\ \square^{\tau}_{q}\gamma)$, and it amounts to having defined an instance of simultaneous $\square$-composition directly. We have

$$\alpha(\square^{\theta}_{p}\beta,\ \square^{\tau}_{q}\gamma) = (\alpha \square^{\theta}_{p} \beta) \square^{\tau}_{\bar{q}} \gamma = (\alpha \square^{\tau}_{q} \gamma) \square^{\theta}_{\bar{p}} \beta \quad .$$

We can extend the just-given arguments to prove the part of the coherence conditions relating to commutativity that refers to the $\psi$-maps alone (the quadrangle in 1.(xi) in [H/M/P]), by first defining a new "simultaneous" $\psi$-map $\psi\colon \langle \alpha \rangle \setminus \{p, q\} \longrightarrow \alpha(\square^{\theta}_{p}\beta,\ \square^{\tau}_{q}\gamma)$, and then showing that both composites of $\psi$'s in the quadrangle are equal to it. However, it is not really necessary to do this, because the arguments used in [H/M/P] for the coherence conditions via invariance remain available and easy.

**The $\varphi^{\square}$–structure**

Next, we deal with the $\varphi$-part of the amalgamation structure for the composition $\square$.

The definition is again a "multicategory recursion", except that the domain on which we define the assignment will be a proper subclass of the arrows in $\boldsymbol{C}[r]$. (We simply do not see how to define the $\varphi$-structure meaningfully on the whole of $\boldsymbol{C}[r]$; this is, of course, in contrast to the $\psi$-structure which was defined on the whole of $\boldsymbol{C}[r]$.)

The reader may see already now that every time we want to define recursively a partial function $F$ on the arrows, with $F$ having the domain $X$, we can do it as if we were defining a total function, provided $X$ is a *convex* class: whenever $\alpha \circ_i \beta \in X$, both $\alpha$ and $\beta$ belong to $X$ as well: we extend $F$ by assigning an arbitrary value to each argument not in $X$.

Let us call an arrow $\mu$ of $\boldsymbol{C}[r]$ *normal* if there is *at most one* $p \in |\langle \mu \rangle|$ for which $\langle \mu \rangle(p)=r$. Thus, the normal arrows are those that are in $\boldsymbol{C}$, and those that are $r$-*special*: $r/p$-special for some $p$. Clearly, the normal arrows form a convex class.

Let us fix $\beta \in A$ and $\theta$ so that $\theta$-parallel to $r$.

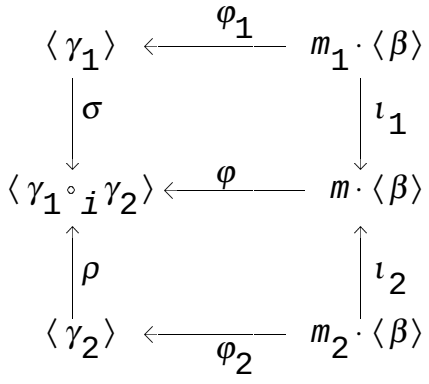
Define the multicategory $\boldsymbol{C}^{(\beta)}$ as having the same objects as $\boldsymbol{C}$, and having arrows of the form

$$\Xi = (\gamma, \mu, \tau : s\mu \xrightarrow{\cong} s\gamma, m, \langle \gamma \rangle \xleftarrow{\varphi} m \cdot \langle \beta \rangle)$$

where $\mu \in A(\boldsymbol{C}[r])$, $\gamma \in A(\boldsymbol{C})$, $m \in \{0, 1\}$; in fact, $m=0$ iff $\mu \in \boldsymbol{C}$ or $\mu$ is not normal, thus $m=1$ iff $\mu$ is $r$-special. We mean: $0 \cdot \langle \beta \rangle = \emptyset$, the empty tuple, $1 \cdot \langle \beta \rangle = \langle \beta \rangle$.

As before in the case of $\boldsymbol{C}^{@}$, we have a forgetful map $\Gamma : \boldsymbol{C}^{(\beta)} \longrightarrow \boldsymbol{C}$ that is the identity on objects, takes $\Xi = (\gamma, \mu, ...)$ to $\gamma$, and is strict. This implies, for instance, that $s\Xi = s\gamma$, $t\Xi = t\gamma$.

Composition in $\boldsymbol{C}^{(\beta)}$: given $\Xi_j = (\gamma_j, \mu_j, \tau_j, m_j, \varphi_j)$ $(j=1, 2)$, $j \in |s\gamma_1|$ such that $\gamma = \gamma_1 \circ_i \gamma_2$ is well-defined, to define $\Xi = \Xi_1 \circ_i \Xi_2 = (\gamma, \mu, \tau, m, \varphi)$. It follows that for $i' = \tau_1^{-1}(i)$, $\mu = \mu_1 \circ_{i'} \mu_2$ is well-defined. We calculate $\tau : s\mu \xrightarrow{\cong} s\gamma$ by the diagram (11'). We form the commutative diagram




(35)

with the right-hand vertical side being a coproduct; of course, this means $m=m_1+m_2$ . Case 1: $\mu$ is normal. In this case $m\leq 1$ , since both $\mu_1$ , $\mu_2$ have to be normal, and at most one can be $r$-special. We complete the definition by requiring that $\iota_j$ be the identity if $m=m_j=1$ ( $j=1, 2$ ); when $m_j=0$ , $\iota_j$ must be the initial map $\emptyset\to m\cdot\langle\beta\rangle$ ; this determines $\varphi$ as well. Now, we define $\Xi_1\circ_i\Xi_2$ to be $(\gamma, \mu, \tau, m, \varphi)$ . Case 2: $\mu$ is not normal. Now, we let $\Xi_1\circ_i\Xi_2$ be $(\gamma, \mu, \tau, 0, !)$ (somewhat arbitrarily).

When verifying either of the two multicategory laws for $\boldsymbol{c}^{(\beta)}$ : associativity or commutativity, we have two triple composites, call them $\Xi_{(1)}=(\mu_{(1)}, \ldots)$ , $\Xi_{(2)}=(\mu_{(2)}, \ldots)$ , and we want to show they are equal. Only the $\varphi$ map is in question. First note that $\mu_{(1)}$ is normal iff $\mu_{(2)}$ is normal iff all three factors are normal and at most one is $r$-special. When $\mu_{(1)}$ , $\mu_{(2)}$ are not normal, there is nothing to show. When they are normal, all the $m$'s and $\varphi$'s involved are defined "normally" according to the picture (35), with additional specifications on the $\iota$'s as stated above; using the coherence laws for the $\sigma$'s and $\rho$'s involved, the task of showing the $\varphi$'s to be equal is now easy.

We use $\boldsymbol{c}^{(\beta)}$ for defining a morphism

$$\begin{aligned}\langle\beta/r\rangle^{\theta} : \boldsymbol{c}[r] &\longrightarrow \boldsymbol{c}^{(\beta)}\\ g &\longmapsto (g, g, \mathrm{id}:sg\to sg, \langle g\rangle \xleftarrow{!} 0\cdot\langle\beta\rangle)\\ r &\longmapsto (r, \beta, \theta:s\beta\to sr, \langle\beta\rangle \xleftarrow{\mathrm{id}} 1\cdot\langle\beta\rangle)\end{aligned}$$

such that, also, $\langle\beta/r\rangle^{\theta}\circ\Gamma = [\beta/r]^{\theta}$ (as before), and let $\Phi$ act on $\alpha(r/p)$ (with $\alpha$ $p$-special as usual), to get

$$\alpha(r/p)\langle\beta/r\rangle^{\theta} = (\alpha\Box_p\beta,\ \alpha(r/p),\ \tau,\ 1,\ \langle\alpha\Box_p\beta\rangle \xleftarrow{\varphi[\alpha,\ \beta,\ p,\ \theta]} \langle\beta\rangle)$$

(it is obvious, by induction, that the first two components are as shown; that the codomain of $\varphi$ is $\langle\text{-}\rangle$ applied to the first item; and finally, that the domain of $\varphi$ is $\langle\beta\rangle$ and not $\emptyset$). This is our definition of the $\varphi$-component of the amalgamation structure for $\Box$.

**(36) Proposition** In

$$\langle\alpha\rangle\backslash p \xrightarrow{\psi} \langle\alpha\Box_p^{\theta}\beta\rangle \xleftarrow{\varphi} \langle\beta\rangle \qquad (\psi=\psi[\alpha,\ \beta,\ p,\theta],\ \varphi=\varphi[\alpha,\ \beta,\ p,\theta])$$

$\varphi$ and $\psi$ are the coprojections of a coproduct diagram in the category $L^{\#}$.

**Proof:** induction.