§1 The multicategory of function replacement

This section is an essentially self-contained reworking of the construction of the entity named in the title, originally given in [H/M/P2]. Besides being better, because more conceptual, than the original treatment, its details are also needed in the subsequent development.

Higher arity assignments

$\mathbf{C}$ denotes a free multicategory. $O=O(\mathbf{C})$ is the set of objects, $A=A(\mathbf{C})$ the set of arrows of $\mathbf{C}$; $L \subset A$ is the set of generating arrows for $\mathbf{C}$.

We need the right way of defining the *higher arities* of arrows of $\mathbf{C}$. The "higher arity" of $\alpha \in A$ is what we denoted in [H/M/P] as $\langle \alpha \rangle$, the tuple of free generator arrows involved in α, except that we now need an "invariant" way of defining it. Recall that we have given up on restricting tuples to be functions on initial segments of the positive naturals. Thus, $|\langle \alpha \rangle|$ has to be allowed to be an *arbitrary finite set*.

What we have is a piece of additional structure on $\mathbf{C}$. Usual notation related to $\mathbf{C}$ is used without explanation.

(1) **Definition.** A *higher arity assignment* (HAA) consists of data as follows

(i) a function $\langle - \rangle : A \longrightarrow L^{*} : \alpha \longmapsto \langle \alpha \rangle$ ($\alpha \in A$)

(ii) an assignment of a coproduct diagram in $L^{\#}$:

$$\langle \alpha \rangle \xrightarrow{\ \sigma\ } \langle \alpha \circ_i \beta \rangle \xleftarrow{\ \rho\ } \langle \beta \rangle \ , \qquad \rho = \rho[\alpha, \beta, i] \ , \ \ \sigma = \sigma[\alpha, \beta, i] \ , \tag{1.1}$$

to each triple α, β, i such that $\alpha \circ_i \beta$ is well-defined;

which data are to satisfy the following conditions:

(iii) for $f \in L$, $|\langle f \rangle|$ is a singleton $\{*\}$, and $\langle f \rangle(*) = f$;

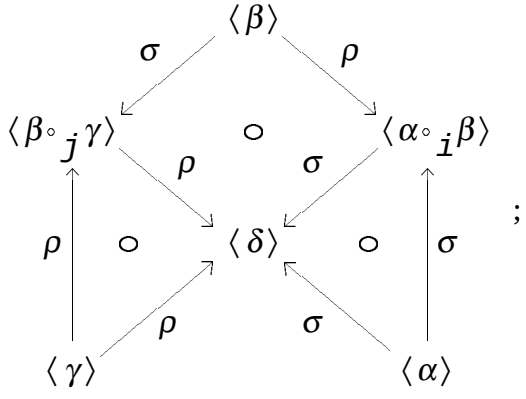
(iv) for $\alpha = 1_X$, $|\langle \alpha \rangle|$ is empty;

(v) when $\beta = 1_Y$, then in (1.1), $\sigma = 1_{\langle \alpha \rangle}$;

(vi) (associativity coherence): for each case of associativity

$$\delta \overset{\text{def}}{=} (\alpha \circ_i \beta) \circ_{\bar{j}} \gamma = \alpha \circ_i (\beta \circ_j \gamma),$$

we have the commutativities

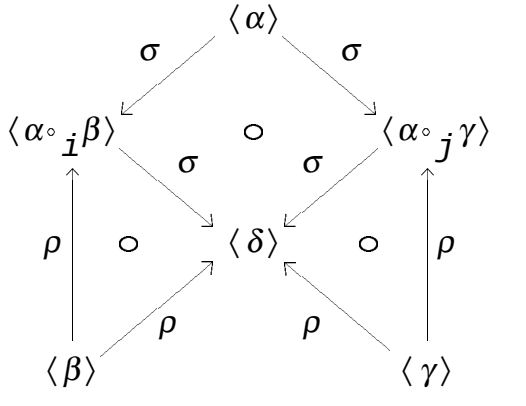


;

(vii) (commutativity coherence): for each case of commutativity

$$\delta \overset{\text{def}}{=} (\alpha \circ_i \beta) \circ_{\bar{j}} \gamma = (\alpha \circ_j \gamma) \circ_{\bar{i}} \beta,$$

we require



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(2) Proposition There is at least one HAA on any free multicategory.

Proof Recall that every $\alpha \in A$, other than an identity, is uniquely written as $\alpha = f(\alpha_1, \ldots, \alpha_n)$, with $f \in L$, the *canonical form* for α (this is "unique readability"). Here we are referring to simultaneous composition in $\boldsymbol{C}$; see [H/M/P]. When in particular, each α_h is an identity, we get $\alpha = f$, a generator.

Let us determine each $\langle f \rangle$ for $f \in L$, and each $\langle 1_X \rangle$, according to (iii) and (iv), otherwise arbitrarily. Furthermore, proceeding recursively, to each *positive* (non-identity) $\alpha = f(\alpha_1, \ldots, \alpha_n)$, $f \in L$, let us assign a tuple $\langle \alpha \rangle$, and a system of arrows

$$\langle \alpha_h \rangle \xrightarrow{\rho_h} \langle \alpha \rangle \quad (h = 1, \ldots, n)$$
$$\langle f \rangle \xrightarrow{\sigma} \langle \alpha \rangle$$
$$\rho_h = \rho_h[\alpha], \quad \sigma = \sigma[\alpha] \quad ,$$

making up a coproduct diagram in $L^{\#}$ (here, $\langle f \rangle$ is as in (iii)). Assuming inductively that we already have each tuple $\langle \alpha_h \rangle$, we can obviously create the additional items mentioned. In particular, the prescription determines $\langle \alpha \rangle$ up to isomorphism.

Here is how to get data of the kind in the definition of HAA from the ones just described.

Let $\alpha = \beta \circ_i \gamma$, and we want to define $\langle \beta \rangle \xrightarrow{\sigma} \langle \alpha \rangle \xleftarrow{\rho} \langle \gamma \rangle$; our definition will be a recursion on the size $\# |\langle \alpha \rangle|$ of α.

If β is not positive, i.e., it is an identity, then $\sigma = !$, the unique map from $\emptyset$, and ρ is the identity.

If β is positive, let the canonical form for β be $\beta = f(\beta_1, \ldots \beta_n)$. Then, that for α is $\alpha = f(\alpha_1, \ldots, \alpha_n)$, where $\alpha_k = \beta_k$ except for one value of $k = 1, \ldots, n$, say h, for which $\alpha_h = \beta_h \circ_j \gamma$, with a suitable j. Now, we have

$$\rho_k[\alpha] : \langle \alpha_k \rangle \longrightarrow \langle \alpha \rangle \quad (k = 1, \ldots, n)$$
$$\sigma[\alpha] : \langle f \rangle \longrightarrow \langle \alpha \rangle$$

$\rho_k[\beta] : \langle \beta_k \rangle \longrightarrow \langle \beta \rangle$ (k=1, . . . , n)
$\sigma[\beta] : \langle f \rangle \longrightarrow \langle \beta \rangle$

and, by the induction hypothesis,

$$\langle \beta_h \rangle \xrightarrow{\sigma} \langle \alpha_h \rangle \xleftarrow{\rho} \langle \gamma \rangle \qquad (\rho=\rho[\beta_h, \gamma, j] , \ \sigma=\sigma[\beta_h, \gamma, j]) .$$

We define

$$\langle \beta \rangle \xrightarrow{\bar{\sigma}} \langle \alpha \rangle \xleftarrow{\bar{\rho}} \langle \gamma \rangle \qquad (\bar{\rho}=\rho[\beta_h, \gamma, j] , \ \bar{\sigma}=\sigma[\beta_h, \gamma, j])$$

by

$\bar{\rho} = \rho_h[\alpha] \circ \rho$,
$\bar{\sigma} \circ \rho_k[\beta] = \rho_k[\alpha]$ ($k \neq h$) ,
$\bar{\sigma} \circ \rho_h[\beta] = \rho_h[\alpha] \circ \sigma$,
$\bar{\sigma} \circ \sigma[\beta] = \sigma[\alpha]$;

the last three lines define $\bar{\sigma}$ by $\langle \beta \rangle$ being a coproduct according to the canonical form of β .

The verification of the requisite properties is omitted.

There is a *flexibility* inherent in the notion of an HAA. By this we mean that one can arbitrarily rename the places in the tuples $\langle \alpha \rangle$. This is of course very similar to what we saw concerning places in the sources of (multi)arrows in the definition of multicategory (see the Introduction). Specifically, given an HAA $(\langle - \rangle, \sigma, \rho)$ on $\boldsymbol{C}$, we may prescribe an arbitrary family of isomorphisms $\tau_\alpha : \langle \alpha \rangle' \xrightarrow{\cong} \langle \alpha \rangle$ in $L^{\#}$, one for each $\alpha \in A$, and produce a new HAA whose tuples are the $\langle \alpha \rangle'$, rather than the original $\langle \alpha \rangle$. In fact, the new HAA is uniquely determined by the requirement that $(\mathrm{Id}_{\boldsymbol{C}}, \tau_{(-)})$ is a *morphism*, in the sense defined below shortly, from $(\boldsymbol{C}, \langle - \rangle, \ldots)$ to $(\boldsymbol{C}, \langle - \rangle', \ldots)$.

A *renaming morphism* $F:\mathbf{C}\longrightarrow\mathbf{C}'$ between *free* multicategories $\mathbf{C}$, $\mathbf{C}'$ is a morphism of multicategories that takes generators to generators ($\mathbf{C}=\mathcal{F}(\mathcal{L})$, $\mathbf{C}'=\mathcal{F}(\mathcal{L}')$, $\mathcal{L}=(O, L, s, t)$, $\mathcal{L}'=(O', L', s', t')$; for $f\in L$, $F(f)\in L'$). A renaming morphism involves an action $F:L\to L'$ on the generators involved.

Suppose $\mathbf{C}$ is equipped with a HAA $(\langle\text{-}\rangle, \rho,\sigma)$, $\mathbf{C}'$ with $(\langle\text{-}\rangle', \rho,\sigma)$. A morphism of the HAA-equipped multicategories (*HAA-morphism*, for short)

$$F:(\mathbf{C}, \langle\text{-}\rangle, \rho,\sigma)\longrightarrow(\mathbf{C}', \langle\text{-}\rangle, \rho,\sigma),$$

consists of a renaming morphism $F:\mathbf{C}\to\mathbf{C}'$, and besides, of the following data: for each $\alpha\in A(\mathbf{C})$, a specified isomorphism

$$F_\alpha: F\circ\langle\alpha\rangle\xrightarrow{\cong}\langle F\alpha\rangle\ ; \qquad (3^-)$$

(F in $F\circ\langle\alpha\rangle$ is the effect of F on the generating arrows; instead of $F_\alpha(p)$ we write $F(p)$) which is compatible with the structure involved:

$$\begin{array}{ccccc}
\langle F\alpha\rangle & \xrightarrow{\sigma} & \langle F\alpha\circ_i F\beta\rangle & \xleftarrow{\rho} & \langle F\beta\rangle \\
{\scriptstyle F}\uparrow & \circ & \uparrow{\scriptstyle F} & \circ & \uparrow{\scriptstyle F} \\
F\circ\langle\alpha\rangle & \xrightarrow[\sigma]{} & F\circ\langle\alpha\circ_i\beta\rangle & \xleftarrow[\rho]{} & F\circ\langle\beta\rangle\ .
\end{array} \qquad (3)$$

The morphisms (3^-) are called the *actions* of F on the places in the higher arities, or the "higher actions".

There is a natural composition of HAA-morphisms. Suppose $\mathbf{C}$, $\mathbf{C}'$, $\mathbf{C}''$ are free multicategories equipped with HAA's (whose mention we sometimes suppress), and $\mathbf{C}\xrightarrow{F}\mathbf{C}'\xrightarrow{F'}\mathbf{C}''$ are HAA morphisms. Then the HAA-morphism $F'F:\mathbf{C}\longrightarrow\mathbf{C}''$ has the composite of the underlying renaming transformations as the underlying renaming transformation, and it has the higher action

$$F'F\ :\ F'F\circ\langle\alpha\rangle\longrightarrow\langle F'F\alpha\rangle$$

defined as the composite

$$F'F \circ \langle \alpha \rangle \xrightarrow{F'F} F' \circ \langle F\alpha \rangle \xrightarrow{F'} \langle F'F\alpha \rangle$$

of, essentially, the given higher actions.

More precisely, here, the first arrow denoted $F'F$ is obtained as follows. With any map $F' : L' \to L''$ of sets, we have the functor $F' : (L')^{\#} \to (L'')^{\#}$ of the corresponding tuple categories, which acts on objects as $s \mapsto F' \circ s$, and on morphisms as, essentially, the identity. In our case, $F' : L' \to L''$ is the action on the generating arrows of the respective free multicategories. Thus, having the morphism $F : F \circ \langle \alpha \rangle \xrightarrow{\cong} \langle F\alpha \rangle$ of $(L')^{\#}$, we have the morphism $F'F : F'F \circ \langle \alpha \rangle \xrightarrow{\cong} F' \circ \langle F\alpha \rangle$ of $(L'')^{\#}$.

(4) Proposition The HAA that a free multicategory carries is unique up to isomorphism, in the sense that, for any two HAA's $(\langle - \rangle, \ldots)$, $(\langle - \rangle', \ldots)$ on $\boldsymbol{C}$, $\mathrm{Id} : \boldsymbol{C} \longrightarrow \boldsymbol{C}$ carries a *unique* HAA morphism structure $(\boldsymbol{C}, \langle - \rangle, \ldots) \longrightarrow (\boldsymbol{C}, \langle - \rangle', \ldots)$.

More generally, given the renaming morphism $F : \boldsymbol{C} \to \boldsymbol{C}'$, and HAA's $(\langle - \rangle, \ldots)$, $(\langle - \rangle', \ldots)$ on $\boldsymbol{C}$ and $\boldsymbol{C}'$, resp., there is a unique morphism $(\boldsymbol{C}, \langle - \rangle, \ldots) \longrightarrow (\boldsymbol{C}', \langle - \rangle', \ldots)$, based on the given F; naturally, this HAA morphism is also denoted by F.

Proof Define the auxiliary multicategory $\boldsymbol{C}^{@}$ as follows. The objects (both upper and lower level) of $\boldsymbol{C}^{@}$ are the same as those $\boldsymbol{C}$. An arrow of $\boldsymbol{C}^{@}$ is a pair (α, τ) such that $\alpha \in \mathrm{A}(\boldsymbol{C})$, and $\tau : F \circ \langle \alpha \rangle \xrightarrow{\cong} \langle F\alpha \rangle'$. We will have a forgetful map $U : \boldsymbol{C}^{@} \longrightarrow \boldsymbol{C}$ which is the identity on objects, maps (α, τ) to α, and is strict. This determines the structure of $\boldsymbol{C}^{@}$ to a large extent; what remains is to specify identities and composition within the limits given by the presence of U. There is no choice for the identities in fact: the τ-parts have to be "empty" maps.

Suppose $(\alpha, \tau_1) \circ_i (\beta, \tau_2)$ is to be well-defined; we should have $(\alpha, \tau_1) \circ_i (\beta, \tau_2) = (\alpha \circ_p \beta, \tau_3)$, where the question is what τ_3 is. It is defined by the commutativity of the diagram as in (3), where the three τ's are τ_1, τ_3 and τ_2 when read from left to right. This is legitimate since the (ρ, σ)-pairs are coproducts. One proves the laws of multicategory for $\mathbf{C}^{@}$ by directly using the coherence conditions for (ρ, σ).

We map a generator $f \in L$ of $\mathbf{C}$ to $(f, \tau : F \circ \langle f \rangle \xrightarrow{\cong} \langle Ff \rangle')$; τ here is determined as a map from a singleton to a singleton. We define $\Phi : \mathbf{C} \longrightarrow \mathbf{C}^{@}$ as being the identity on objects, mapping $f \in L$ as prescribed, and being strict on generators (that is, for every $f \in L$, $\Phi_f : |s(f)| \longrightarrow |s(\Phi(f))| = |s(f)|$ is an identity). The transition components of the required HAA-morphism $F : (\mathbf{C}, \langle - \rangle, \ldots) \longrightarrow (\mathbf{C}', \langle - \rangle', \ldots)$ are given by F_α = the τ-component of $\Phi(\alpha)$.

The uniqueness statement is clear since, conversely, specifying the required action-on-places structure is equivalent to specifying Φ as above.

With a given renaming morphism $F : \mathbf{C} \to \mathbf{C}'$, and given HAA's on $\mathbf{C}$ and $\mathbf{C}'$ (both referred to by the same symbols $\langle - \rangle$, ρ and σ), in view of (4), we can talk about *the higher action of* F, meaning the system of maps $F : F \circ \langle \alpha \rangle \xrightarrow{\cong} \langle F\alpha \rangle$ that make F into an HAA-morphism.

The $\square$-composition of arrows of a free multicategory

Fix $\mathbf{C}$, a free multicategory, equipped with a HAA $(\langle - \rangle, \rho, \sigma)$; notation is as before.

Let r be a new indeterminate arrow: a symbol not in L, but with declared source and target in $\mathbf{C}$: $sr \in O^*$, $tr \in O$. We then can take $\mathbf{C}[r]$, obtained by freely adjoining r to $\mathbf{C}$. $\mathbf{C}[r]$ may be defined by the obvious universal property. But also, $\mathbf{C}[r]$ is the same as $\mathcal{F}(\mathcal{L}[r])$, the free multicategory on the language $\mathcal{L}[r]$, where $\mathcal{L}[r]$ is obtained by simply adding r, with its source-and-target specs, to the language $\mathcal{L}$.

We have the canonical map $I:\mathbf{C}\to\mathbf{C}[r]$, usually regarded as an inclusion; certainly, I is bijective on objects, and injective on arrows. We can also safely assume that I is strict as a morphism of multicategories: its transition iso's are identities. Moreover, given any HAA on $\mathbf{C}$, we can define one on $\mathbf{C}[r]$ that extends, in the obvious strict sense, that on $\mathbf{C}$: this involves possibly renaming the HAA on $\mathbf{C}[r]$ on the part which is in the image of I to match that on $\mathbf{C}$. In what follows, our notation $\langle\text{-}\rangle$ for both HAA's will reflect these facts.

Suppose we are given $\beta\in A=\mathrm{A}(\mathbf{C})$ such that $\mathrm{t}\beta=\mathrm{t}r$, and we are given $\theta:\mathrm{s}r\xrightarrow{\cong}\mathrm{s}\beta$, an isomorphism in the category $O^{\#}$. In this case, we say that β is *θ-parallel to* r, and if θ is the identity, we simply say β is parallel to r. Notation is $\beta\|^{\theta}r$, $\beta\|r$.

By the universal property of $\mathbf{C}[r]$, there is a unique morphism $G:\mathbf{C}[r]\to\mathbf{C}$ such that G is the identity morphism when restricted to $\mathbf{C}$, G takes r to β, and the action $G:\mathrm{s}r\xrightarrow{\cong}\mathrm{s}\beta$ is the given θ. Let us denote G by $[\beta/r]^{\theta}$: "evaluating r as β"; and let us write, when $\mu\in\mathbf{C}[r]$, $\mu[\beta/r]^{\theta}$ for $G(\mu)$. When θ is the identity, we omit it from the notation.

Note that, in particular, $[\beta/r]^{\theta}$ is the identity on objects, it is strict on the generators of $\mathbf{C}$, and in fact, on all arrows of $\mathbf{C}$ (the actions $[\beta/r]^{\theta}:\mathrm{s}\alpha\to\mathrm{s}G(\alpha)$ are identities for all $\alpha\in A$).

If, in addition, $\beta=f\in L$ is a generator, and $\theta=\mathrm{id}$ (that is, $f\|r$), then, clearly, $[f/r]$ is a renaming morphism $\mathbf{C}[r]\to\mathbf{C}$ of free multicategories.

Let $\mu\in\mathrm{A}(\mathbf{C}[r])$ and $p\in|\langle\mu\rangle|$. Let's say that μ is *r/p-special* if $\langle\mu\rangle(p)=r$, and $\langle\mu\rangle(q)\in L$ (that is, $\langle\mu\rangle(q)$ is unequal to r) for all $q\in|\langle\mu\rangle|-\{p\}$.

Let $\alpha\in A$, $p\in|\langle\alpha\rangle|$. Let r be a new, indeterminate, arrow, declared to be parallel *to* $f\stackrel{\mathrm{def}}{=}\langle\alpha\rangle(p)$, $r\|f$.

We want to define $\alpha(r/p)$, the "result of replacing f with r at p". This is nothing but $\alpha\Box_p r$ -- however, we don't have $\Box$ yet; in fact, we want to use $\alpha\Box_p r$ to get at the operation $\Box$.

Here is our definition of $\alpha(r/p)$.

(5) Proposition For given α and p as above, there is a unique arrow $\alpha(r/p)$ in $\mathbf{C}[r]$ for which $\alpha(r/p)[f/r] = \alpha$, and which is r/p'-special, for p' such that $p=p'[f/r]$ (see (3$^-$) for $F=[f/r]$).

Proof. The proof uses the following lemma whose proof, using unique readability, is left to the reader.

(5') Lemma Assume $F:\mathbf{C}\longrightarrow\mathbf{C}'$ renaming, and assume that $F(\alpha)=\alpha'$, and $\alpha'=\beta'\circ_i\gamma'$. Then there is a unique decomposition $\alpha=\beta\circ_i\gamma$ such that $F(\beta)=\beta'$, $F(\gamma)=\gamma'$ and $F(i)=i'$.

The proof of both the existence and the uniqueness of $\alpha(r/p)$ is by induction on the *size of* α, which we define to be $\#|\langle\alpha\rangle|$.

Existence: The base case: $|\langle\alpha\rangle|$ is a singleton, $|\langle\alpha\rangle|=\{p\}$. Now, clearly, $\alpha(r/p) = r$ qualifies, and this is the only arrow that qualifies.

For the induction step, assume $\#|\langle\alpha\rangle|>1$. Then, α can be written as

$$\alpha=\alpha_1\circ_i\alpha_2\,, \tag{6}$$

with

$$\#|\langle\alpha_1\rangle|<\#|\langle\alpha\rangle|\,,\ \#|\langle\alpha_2\rangle|<\#|\langle\alpha\rangle|\,; \tag{7}$$

in fact, in most cases, we can write α in this form in more than one way. The relations (7) are used only because the logic of the induction requires it; most of what we say will use only (6), which is good to remember for the future.

Consider the HAA-associate to (6), a coproduct diagram in $L^{\#}$:

$$\langle \alpha_1 \rangle \xrightarrow{\sigma} \langle \alpha = \alpha_1 \circ_i \alpha_2 \rangle \xleftarrow{\rho} \langle \alpha_2 \rangle \ . \tag{8}$$

Given any $p \in |\langle \alpha \rangle|$,

either (case 1) there is $q \in |\langle \alpha_1 \rangle|$ such that $\sigma(q)=p$,
or (case 2) there is $q \in |\langle \alpha_2 \rangle|$ such that $\rho(q)=p$;

the cases are exclusive, and q is uniquely determined.

By induction hypothesis, we have $\alpha_1(r/q)$ (case 1) , $\alpha_2(r/q)$ (case 2). We claim that we can use the following compositions in $\boldsymbol{C}[r]$ as the definition of the required $\alpha(r/p)$:

$$\alpha(r/p) = \alpha_1(r/q) \circ_i , \alpha_2 \qquad \text{(case 1)} \tag{9.1}$$
$$\alpha(r/p) = \alpha_1 \circ_i (\alpha_2(r/q)) \qquad \text{(case 2).} \tag{9.2}$$

In (9.1), i' is such that $i=i'[f/r]$.

Take, for instance, case 1; case 2 is symmetric. For one thing, with $f=\langle \alpha \rangle(p)=\langle \alpha_1 \rangle(q)$, we have

$$\alpha(r/p)[f/r] = (\alpha_1(r/q) \circ_i , \alpha_2)[f/r]$$
$$= (\alpha_1(r/q)[f/r]) \circ_i (\alpha_2[f/r]) = \alpha_1 \circ_i \alpha_2 = \alpha \ .$$

For the other thing, the commutative diagram

$$\begin{array}{ccc} \langle \alpha_1(r/q) \rangle & \xrightarrow{\sigma} & \langle \alpha(r/p) \rangle \\ {\scriptstyle [f/r]}\downarrow & \circ & \downarrow{\scriptstyle [f/r]} \\ \langle \alpha_1 \rangle & \xrightarrow[\sigma]{} & \langle \alpha \rangle \end{array}$$

shows that $\sigma(q')=p'$. Together with the coproduct diagram

$$\langle \alpha_1(r/q) \rangle \xrightarrow{\ \sigma\ } \langle \alpha(r/p) \rangle \xleftarrow{\ \rho\ } \langle \alpha_2 \rangle \ ,$$

this shows, on the basis that $\alpha_1(r/q)$ is r/q'-special, that $\alpha(r/p)$ is r/p'-special. This completes the existence proof.

Uniqueness: Suppose both α' ad α'' qualify as $\alpha(r/p)$. Base: $|\langle \alpha \rangle|$ is a singleton, $|\langle \alpha \rangle| = \{p\}$. Seems easy.

Induction step: $\#|\langle \alpha \rangle| > 1$. We have, as before, (6), (7), and either "case 1" or "case 2". Applying the Lemma above, we have a decomposition $\alpha' = \alpha_1' \circ_i \alpha_2'$ such that $\alpha_1'[f/r] = \alpha_1$, $\alpha_2'[f/r] = \alpha_2$, and $i'[f/r] = i$, and similarly for double primes.

Let us assume case 1. Let $q' \in |\langle \alpha_1' \rangle|$ be such that $q'[f/r] = q$.

We claim that α_1' is r/q'-special, and α_2' is in $A = A(\boldsymbol{C})$. This is clear from the fact that α' is r/p'-special, and from the coproduct

$$\langle \alpha_1' \rangle \xrightarrow{\ \sigma\ } \langle \alpha' \rangle \xleftarrow{\ \rho\ } \langle \alpha_2' \rangle \qquad (10)$$

because (as before) $\sigma(q') = p'$.

Of course, the same can be said of the double primed items.

We have that both α_1' and α_1'' have the two properties that make them qualify as $\alpha_1(r/q)$; by the induction hypothesis, $\alpha_1' = \alpha_1''$. On the other hand, since α_2' , α_2'' are both mapped to α_2 by $[f/r]$, and are both in $\boldsymbol{C}$, we have $\alpha_2' = \alpha_2'' = \alpha_2$. We conclude that $\alpha' = \alpha_1' \circ_i \alpha_2' = \alpha_1'' \circ_i \alpha_2'' = \alpha''$ as promised.

We are ready to use the construction $\alpha(r/p)$ for the definition of the operation $\square$, albeit in a preliminary version, $\square^{\theta}$, with a parameter θ in it that later we will get rid of.

The idea is that, for $\beta \parallel^{\theta} \langle \alpha \rangle(p) \parallel r$,

$$\alpha\Box_p^\theta\beta \underset{\text{def}}{=} \alpha(r/p)[\beta/r]^\theta .$$

Since $\alpha = \alpha(r/p)[f/r]$, we have the actions

$$[f/r]:\mathrm{s}(\alpha(r/p)) \xrightarrow{\cong} \mathrm{s}(\alpha)$$
$$[\beta/r]^\theta:\mathrm{s}(\alpha(r/p)) \xrightarrow{\cong} \mathrm{s}(\alpha\Box_p\beta) ;$$

therefore, we have the induced isomorphism

$$\pi=\pi[\alpha, \beta, p,\theta] \underset{\text{def}}{=} [\beta/r]^\theta[f/r]^{-1}:\mathrm{s}(\alpha) \xrightarrow{\cong} \mathrm{s}(\alpha\Box_p^\theta\beta) . \tag{11}$$

The ψ-structure for the $\Box$-composition

Turning to the amalgamation structure for $\Box$, we deal with the " ψ-structure" first:

$$\langle\alpha\rangle\backslash p \xrightarrow{\psi} \langle\alpha\Box_p^\theta\beta\rangle , \quad \psi = \psi[\alpha, \beta, p, \theta] . \tag{11.1}$$

We start by fixing a free multicategory $\boldsymbol{C}$, a new indeterminate generator r with given source $\mathrm{s}r\in O^*$ and target $\mathrm{t}r\in O$, giving rise to $\boldsymbol{C}[r]$.

We define the auxiliary multicategory $\boldsymbol{C}^@$ (not the same as the one so denoted before).

The objects of $\boldsymbol{C}^@$ are those of $\boldsymbol{C}$. The arrows are entities of the form

$$\Lambda = (\gamma, \mu, \tau, t, \psi, \iota) = (\gamma, \mu, \tau:\mathrm{s}(\mu) \xrightarrow{\cong} \mathrm{s}(\gamma), \langle\mu\rangle \xleftarrow{\iota} t \xrightarrow{\dot{\psi}} \langle\gamma\rangle)$$

where, in addition,

$$\gamma\in\mathrm{A}(\boldsymbol{C}) , \quad \mu\in\mathrm{A}(\boldsymbol{C}[r]) , \quad \mathrm{t}_{\boldsymbol{C}}\gamma=\mathrm{t}_{\boldsymbol{C}[r]}\mu ;$$

and

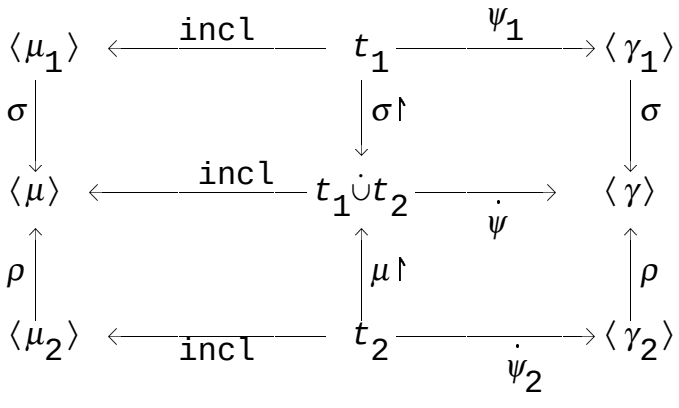
ι is an *inclusion arrow,* meaning that the function $|\iota| : |t| \longrightarrow |\langle\mu\rangle|$ is an inclusion of sets.

Here, both $\langle\mu\rangle$ and $\langle\gamma\rangle$ are regarded as objects of the category $(L\dot\cup\{r\})^{\#}$, and so is t; the arrows ι and $\dot\psi$ are arrows of the same category.

Much of the subsequent specification of $\boldsymbol{C}^{@}$ is contained in the statement that we have a forgetful map $\Gamma:\boldsymbol{C}^{@}\longrightarrow\boldsymbol{C}$ that is the identity on objects, takes $\Lambda = (\gamma, \mu, \ldots)$ to γ, and is strict. For instance, it means that we define

$$s_{\boldsymbol{C}^{\#}}((\gamma,\mu,\ldots)) = s_{\boldsymbol{C}}(\gamma)\,, \qquad t_{\boldsymbol{C}^{\#}}((\gamma,\mu,\ldots)) = t_{\boldsymbol{C}}(\gamma)\,.$$

Composition is defined in this way: let $\Lambda_j=(\mu_j, \gamma_j, \ldots)$ $(j=1, 2)$ and $i\in|s(\gamma)|$ such that $\gamma = \gamma_1 \circ_i \gamma_2$ is well-defined. Then, for $i'=\tau_1^{-1}(i)$, $\mu=\mu_1\circ_{i'}\mu_2$ is well-defined as well. The composite $\Lambda=\Lambda_1\circ_i\Lambda_2$ has (must have) its first two components γ and μ, resp.; $\Lambda=(\gamma, \mu, \ldots)$. Further, we take the commutative diagram



constructed as follows. First, we produce the middle vertical coproduct, but with the additional stipulation that the middle horizontal arrow marked "incl" is indeed an inclusion arrow. This makes everything in the diagram uniquely determined, including the arrow denoted ψ, which we make the ψ-component of the composite $\Lambda=\Lambda_1\circ_i\Lambda_2$. We put $t=t_1\dot\cup t_2$.

To define the τ-component τ of Λ, we use the commutative diagram

$$\begin{array}{ccccc} \mathrm{s}(\gamma_1)\backslash i & \xrightarrow{\psi} & \mathrm{s}(\gamma) & \xleftarrow{\varphi} & \mathrm{s}(\gamma_2) \\ \tau_1 \uparrow & & \uparrow \tau & & \uparrow \tau_2 \\ \mathrm{s}(\mu_1)\backslash i' & \xrightarrow{\psi} & \mathrm{s}(\mu) & \xleftarrow{\varphi} & \mathrm{s}(\mu_2) \end{array} \qquad (11')$$

The rest of the structure of $\boldsymbol{C}^{@}$ is determined by the presence of $\Gamma:\boldsymbol{C}^{@}\longrightarrow\boldsymbol{C}$ as described above.

Checking that the multicategory laws are satisfied involves the use of the coherence conditions in the definition of the HAA $(\langle\text{-}\rangle, \rho, \sigma)$ on both $\boldsymbol{C}[r]$ and $\boldsymbol{C}$, and also, the coherence conditions on the (φ, ψ)-maps. For these verifications, it is necessary to carry along the inclusion ι in the definition of an object in the category $\boldsymbol{C}^{@}$.

Next, with the given $\boldsymbol{C}[r]$, let us take an arbitrary arrow $\beta\in A$ and some appropriate θ such that is θ-parallel to r.

We define a morphism of multicategories $\langle\beta/r\rangle^{\theta}:\boldsymbol{C}[r]\longrightarrow\boldsymbol{C}^{@}$ by the stipulations

$$\begin{array}{ccc} \boldsymbol{C}[r] & \xrightarrow{\quad\langle\beta/r\rangle^{\theta}\quad} & \boldsymbol{C}^{@} \\ g & \longmapsto & (g,\ g,\ \mathrm{id}:\mathrm{s}(g)\to\mathrm{s}(g),\ \langle g\rangle \xleftarrow{\mathrm{id}} \langle g\rangle \xrightarrow{\mathrm{id}} \langle g\rangle) \qquad (g\in L) \\ r & \longmapsto & (\beta,\ r,\ \ \theta:\mathrm{s}(r)\to\mathrm{s}(\beta),\ \langle r\rangle \xleftarrow{!} \ \emptyset\ \xrightarrow{!} \langle\beta\rangle) \end{array}$$

and that $\Gamma\circ\langle\beta/r\rangle^{\theta}$ equal to $[\beta/r]^{\theta}$.

It is obvious that in $\mu\langle\beta/r\rangle^{\theta}$, the second component is μ itself.

In $\mu\langle\beta/r\rangle^{\theta}=(\mu,\ \gamma,\ \tau,\langle\mu\rangle \xleftarrow{\iota} t \xrightarrow{\dot{\psi}} \langle\gamma\rangle)$, the subset $|t|$ of $|\langle\mu\rangle|$ (via the inclusion ι) can be described by saying that it is obtained by deleting from $|\langle\mu\rangle|$ each p for which $\langle\mu\rangle(p)=r$. We use the shorthand notation $\langle\mu\rangle\backslash r$ for t to signify this fact.

$\dot{\psi}$ is always an injective map.

We note that

$$\{\langle\mu[\beta/r]^{\theta}\rangle(q):q\in(|\mu[\beta/r]^{\theta}|-\mathrm{range}(\psi))\}\subset\mathrm{range}\langle\beta\rangle\ ; \qquad (12)$$

that is, all generators that occur in $\mu[\beta/r]^{\theta}$ at places that are not in the image of ψ must occur in β.

All these remarks are easily proved by induction on the size of μ.

We have defined the so-called *ψ-structure* with components

$$\psi[\beta,\mu,\theta]:\langle\mu\rangle\backslash r\longrightarrow\langle\mu[\beta/r]^{\theta}\rangle\ ,$$

associated with the evaluation $[\beta/r]^{\theta}:\boldsymbol{C}\longrightarrow\boldsymbol{C}[r]$; $\psi[\beta,\mu,\theta]$ is the $\dot{\psi}$-component of $\mu\langle\beta/r\rangle^{\theta}$.

Next, let $\alpha\in A$, $p\in|\langle\alpha\rangle|$. Let $f=\langle\alpha\rangle(p)$. With the indeterminate r parallel to f, we have the morphisms $[f/r]:\boldsymbol{C}[r]\longrightarrow\boldsymbol{C}$, $\langle f/r\rangle:\boldsymbol{C}[r]\longrightarrow\boldsymbol{C}^{@}$ (in the definition of the last, the θ-component is an identity), and the arrow $\alpha(r/p)\in\mathrm{A}(\boldsymbol{C}[r])$. As before, let us write $p'=[f/r]^{-1}(p)\in|\langle\alpha(r/p)\rangle|$.

Let $\beta\in A$ be given θ-parallel to r and f. $\alpha(r/p)\langle\beta/r\rangle^{\theta}$ is some $\Lambda=(\gamma,\mu,\tau,t,\dot{\psi},\iota)$. We know that $\mu=\alpha(r/p)$, and therefore the subset $|t|=|\langle\mu\rangle\backslash r|$ of $|\mu|$ misses exactly one element, namely p'. We have the bijection

$$[f/r]:\langle\mu\rangle\backslash r\xrightarrow{\cong}\langle\alpha\rangle\backslash p\ .$$

We know that γ from Λ is $\alpha\square_{p}^{\theta}\beta$.

The "amalgamating map" ψ for the $\square$-composition in

$$\langle \alpha \rangle \backslash p \xrightarrow{\psi} \langle \alpha \square_p^{\theta} \beta \rangle \ ,$$

defined when

$$\theta : \mathrm{s}(\langle \alpha \rangle (p)) \xrightarrow{\cong} \mathrm{s}\beta \ , \ \mathrm{t}(\langle \alpha \rangle (p)) = \mathrm{t}\beta \ . \qquad (13)$$

$\psi = \psi[\alpha, \beta, p, \theta]$ is defined to be the composite

$$\langle \alpha \rangle \backslash p \xrightarrow{[f/r]^{-1}} \langle \mu \rangle \backslash r \xrightarrow{\dot{\psi}} \langle \alpha \square_p \beta \rangle$$

where $\dot{\psi}$ is the $\dot{\psi}$-component of $\alpha(r/p)\langle \beta / r \rangle^{\theta}$.