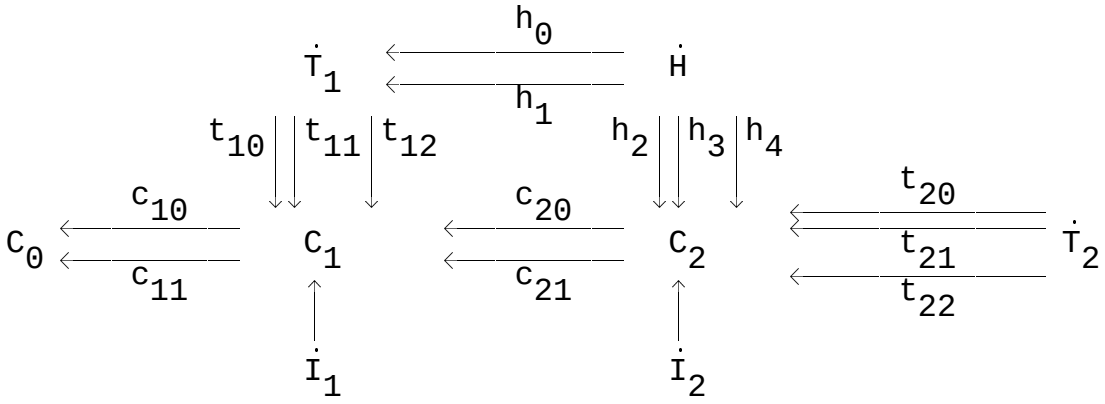
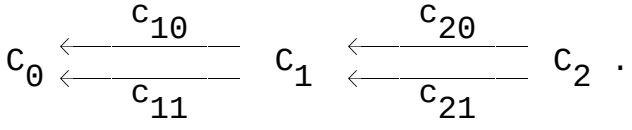
Appendix D: Calculations for §7.

D1. Define the *generalized* DS vocabulary $\mathbb{L}_{2\text{-cat}}$ as the full subcategory of $\mathbb{L}_{\text{anabicat}}$ on the objects of $\mathbb{L}_{2\text{-cat}}$, with relations $\dot{I}_1, \dot{I}_2, \dot{T}_1, \dot{H}, \dot{T}_2$; it is *generalized* since a non-maximal object, T_1, is also made into a relation. Accordingly, an $\mathbb{L}_{2\text{-cat}}$-structure is a functor from $\mathbb{L}_{2\text{-cat}}$ in which the listed relations (including T_1) are interpreted relationally. This is the picture for $\mathbb{L}_{2\text{-cat}}$:



A *2-category-sketch* (2-cat-sketch) is, by definition, a structure of type $\mathbb{L}_{2\text{-cat}}$; maps of 2-cat-sketches are natural transformations of functors. For a 2-cat-sketch S, $|S|$ is its underlying 2-graph, its reduct to



Any bicategory has an underlying 2-cat-sketch. We will look at maps $S \longrightarrow \mathcal{A}$, $S \in 2\text{-catSk}$, $\mathcal{A}$ a bicategory.

Let $S \overset{M}{\underset{N}{\rightrightarrows}} \mathcal{A}$. A *transformation* $\tau : M \to N$ is given by

(i) $\tau_X : MX \to NX$ for each $X \in S(C_0)$;

(ii) for each $(f : X \to Y) \in S(C_1)$, $\tau_f : Nf \circ \tau_X \xrightarrow{\cong} \tau_Y \circ Mf$ as in

$$\begin{array}{ccc} MX & \xrightarrow{\tau_X} & NX \\ {\scriptstyle Mf}\downarrow & \overset{\tau_f}{\swarrow}\cong & \downarrow{\scriptstyle Nf} \\ MY & \xrightarrow[\tau_Y]{} & NY \end{array}$$

such that

(a) for any $X \underset{g}{\overset{f}{\underset{\varphi\downarrow}{\rightrightarrows}}} Y$ in S,

$$\begin{array}{ccc} Nf\circ\tau_X & \xrightarrow{\tau_f} & \tau_Y\circ Mf \\ {\scriptstyle N\varphi\circ\tau_X}\downarrow & \circ & \downarrow{\scriptstyle \tau_Y\circ M\varphi}\ ; \\ Ng\circ\tau_X & \xrightarrow[\tau_g]{} & \tau_Y\circ Mg \end{array}$$

(b) $(f:X\to Y)\in S(I_1) \Longrightarrow \tau_f = 1_{\tau_X}$; and

(c) for every $A \underset{h}{\overset{f}{\rightarrow}} B \overset{g}{\rightarrow} C \in S(\mathsf{T}_1)$ (note that $MgMf=Mh$, $NgNf=Nh$),

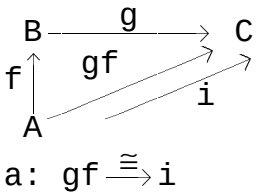
$$\begin{array}{ccccc} MA & \xrightarrow{Mf} & MB & \xrightarrow{Mg} & MC \\ {\scriptstyle \tau_A}\downarrow & \overset{\tau_f}{\swarrow} & {\scriptstyle \tau_B}\downarrow & \overset{\tau_g}{\swarrow} & {\scriptstyle \tau_C}\downarrow \\ NA & \xrightarrow{Nf} & NB & \xrightarrow{Ng} & NC \end{array} \quad = \quad \begin{array}{ccc} MA & \xrightarrow{Mh} & MC \\ {\scriptstyle \tau_A}\downarrow & \overset{\tau_h}{\swarrow} & {\scriptstyle \tau_C}\downarrow \\ NA & \xrightarrow[Nh]{} & NC \end{array},$$

that is,

$$\begin{array}{ccccc} (\tau_C Mg)Mf & \xrightarrow{\tau_g Mf} & (Ng\,\tau_B)Mf & \xleftarrow{\alpha} & Ng(\tau_B Mf) \\ {\scriptstyle \alpha}\uparrow & & \circ & & \downarrow{\scriptstyle \tau_f Ng} \\ \tau_C(MgMf) & \xrightarrow[\tau_h]{} & (NgNf)\,\tau_A & \xleftarrow[\alpha]{} & Ng(Nf\,\tau_A) \end{array}\ .$$

Given $S \underset{N}{\overset{M}{\underset{\tau\downarrow}{\rightrightarrows}}} \mathcal{X}$ and $\Phi: T \longrightarrow S$, we have $T \underset{N\Phi}{\overset{M\Phi}{\underset{\tau\Phi\downarrow}{\rightrightarrows}}} \mathcal{X}$ for which $(\tau\Phi)_f = \tau_{\Phi f}$ for $f\in T(\mathsf{C}_1)$.

D2. Going back to the definition of $\mathcal{R}T_1$ in part (B) of the proof of 7.5, and using the notation there, that definition can be put as follows. Consider the 2-cat-sketch S_0 :

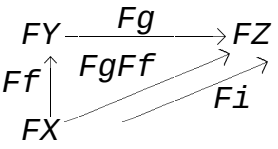


$$(\ S_0(T_1)=\{(f,g,gf)\}\ ,\quad S_0(C_2)=\{a,a^{-1},1_i,1_{gf}\}\ )$$

,

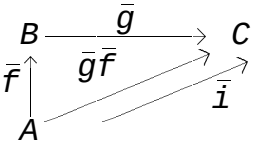
and the two diagrams $S_0 \rightrightarrows \mathcal{A}$ defined as

Φ =



;

Ψ =



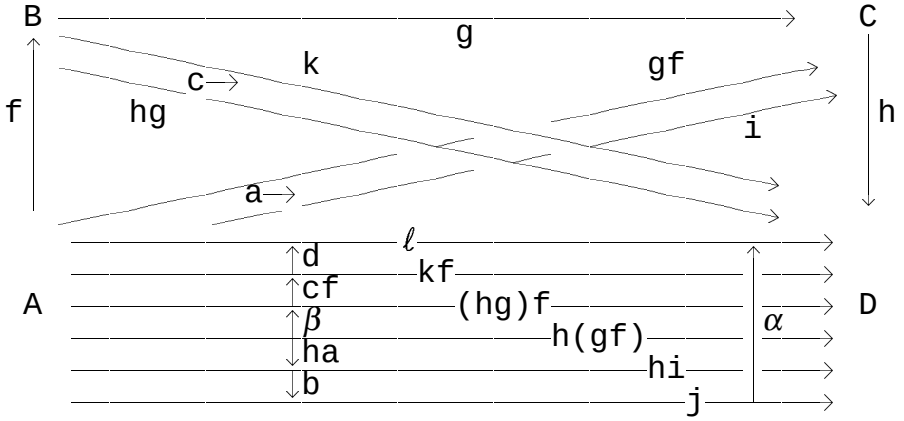
.

$\Phi a = FaF_{f,\,g} : FgFf \xrightarrow{\cong} Fi$ $\qquad$ $\Psi a = \bar{a} : \bar{g}\bar{f} \xrightarrow{\cong} \bar{i}$

Then $\mathcal{R}T_1(\varphi,\ \gamma,\ \eta)[a,\ \bar{a}]$ iff $x,\ y,\ z,\ \varphi,\ \gamma,\ \iota$ are the components of a map $\Phi \to \Psi$.

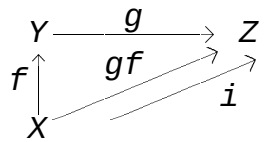
D3. In what follows, we will consider the following 2-cat-sketch S and various of its parts (subsketches):

S =



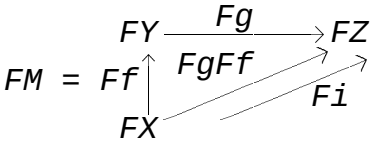
$S(T_1)$ has six elements, (f, g, gf), $(gf, h, h(gf))$, (i, h, hi), (g, h, hg), $(f, hg, h(gf))$, (f, k, kf); the notations showing composition are purely symbolic. The horizontal compositions cf and ha signify the presence of elements "1_f" and "1_h"of $S(I_2)$, and two corresponding elements of $S(H)$. $S(I_1)=\emptyset$. There are further 2-cells and elements of $S(I_2)$ and $S(T_2)$ to the effect that a, b, c, d, α and β are isomorphisms, and α is the composite $d(cf)\beta(ha)^{-1}b^{-1}$.

In case of a general 2-cat-sketch S, for a sketch-map $M:S\to\mathcal{X}$ and a functor $F:\mathcal{X}\to\mathcal{A}$ of bicategories, the composite FM cannot be defined (think of a sketch in which a 1-cell is a composite in two different ways); in the case of our S however, since S is sufficiently "free", a useful sense can be ascribed to FM. First of all, for S_0 from D2, for $M:S_0\to\mathcal{X}$, $F:\mathcal{X}\to\mathcal{A}$, FM is defined as Φ was above: for



$$a:gf\xrightarrow{\cong}i$$

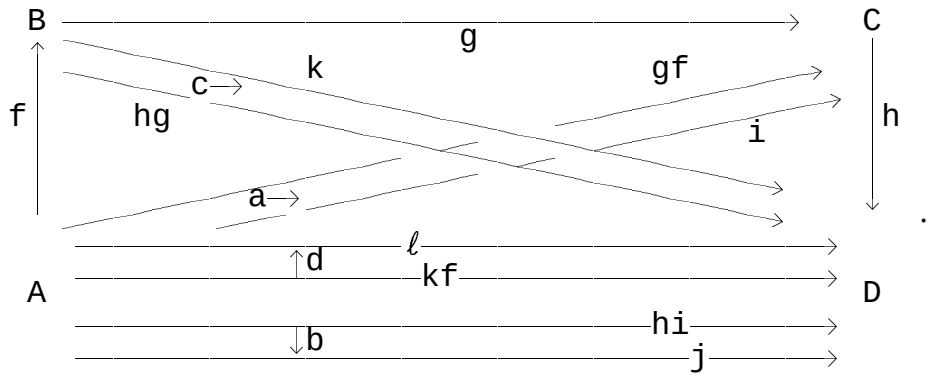
as M, we put FM to be



.

$$Fa=FaF_{f,\,g}:FgFf\xrightarrow{\cong}Fi$$

Now, there are four mappings of the form $S_0\to S$, corresponding to the four items $a:gf\xrightarrow{\cong}i$, $b:hi\xrightarrow{\cong}j$, $c:hg\xrightarrow{\cong}k$, $d:kf\xrightarrow{\cong}\ell$. We define, for any $M:S\to\mathcal{X}$ and $F:\mathcal{X}\to\mathcal{A}$, $FM:S\to\mathcal{A}$ as follows. First, we make sure that for any of the four maps $\sigma:S_0\to S$, $(FM)\sigma=F(M\sigma)$. This requirement determines FM as far as its restriction to the subsketch

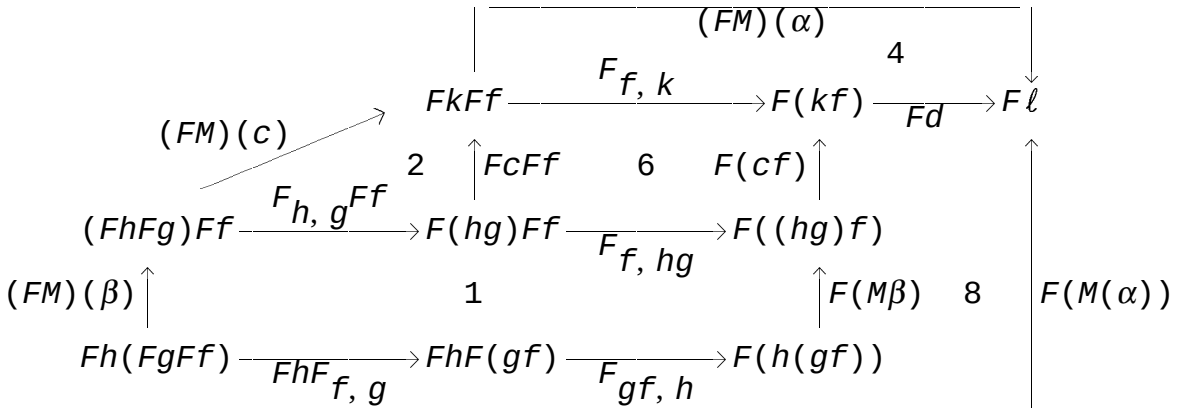


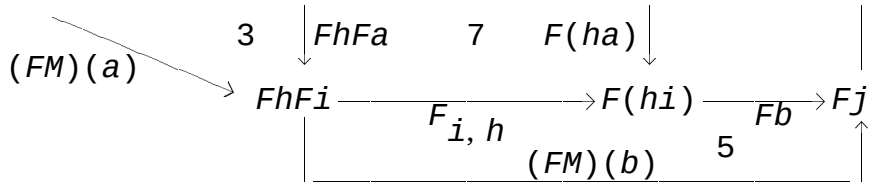
is concerned. But then the effect of FM is uniquely determined on the items $h(gf)$, $(hg)f$, cf, ha. Next, we define $(FM)(\beta)$ so that the following diagram commutes; we wrote f, g, h for Mf, Mg, Mh:

$$\begin{array}{ccccc} (FhFg)Ff & \xrightarrow{F_{h,\,gFf}} & F(hg)Ff & \xrightarrow{F_{f,\,hg}} & F((hg)f) \\ {\scriptstyle (FM)(\beta)}\uparrow & & \circ & & \uparrow{\scriptstyle F(M\beta)} \\ Fh(FgFf) & \xrightarrow[FhF_{f,\,g}]{} & FhF(gf) & \xrightarrow[F_{gf,\,h}]{} & F(h(gf)) \end{array} .$$

Finally, the effect of FM on α in S is now uniquely determined. It is worth noting that if $M\beta = \alpha_{f,\,g,\,h}$, then $(FM)(\beta) = \alpha_{Ff,\,Fg,\,Fh}$ ($f=Mf$, etc.); the reason is that F "preserves" α (see above).

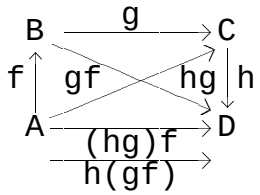
I **claim** that, for $FM:S\to\mathcal{A}$ so defined, $(FM)(\alpha) = F(M(\alpha))$. This is demonstrated by the following commutative diagram:





Here, the cell 1 commutes by the definition of $(FM)(\beta)$; 2, 3, 4, 5 commute by the definition of FM on the 2-cells a, b, c, d; 6 and 7 by the naturality of $F_{-,-}$; and 8 by the fact that $M\alpha$ is the appropriate composite. The assertion is the commutativity of the outside perimeter of the diagram.

D4. Let S_1 be the following subsketch of S:



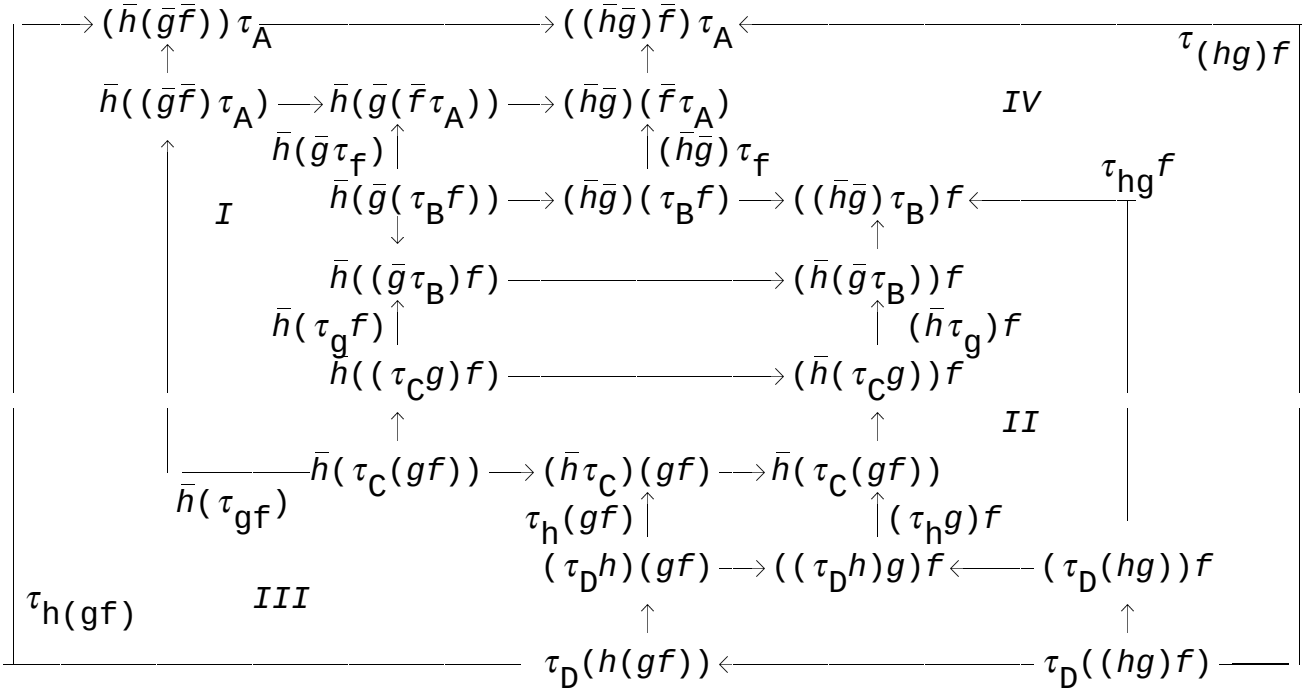
($S_1(t)=\emptyset$ for all $t\in \mathrm{L}_{2\text{-cat}}$, except for $t=\mathrm{C}_0,\mathrm{C}_1,\mathrm{T}_1$), and let S_2 be the sketch (subsketch of S) obtained by adding the 2-cell $\alpha:\mathrm{h(gf)}\longrightarrow\mathrm{(hg)f}$ to S_1. Suppose we have $M, N: S_2 \to \mathcal{A}$ such that $M\alpha = \alpha_{M\mathrm{f},\, M\mathrm{g},\, M\mathrm{h}}$ and $N\alpha = \alpha_{N\mathrm{f},\, N\mathrm{g},\, N\mathrm{h}}$ (associativity isomorphisms), and, also writing M for $M\restriction S_1$, we have

$$S_1 \underset{N}{\overset{M}{\Downarrow \tau}} \mathcal{A} \qquad (1)$$

Then τ is a map with respect to S_2, that is,

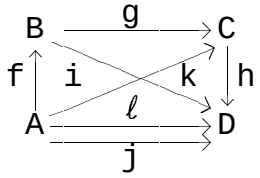
$$S_2 \underset{N}{\overset{M}{\Downarrow \tau}} \mathcal{A} \ .$$

This fact expresses the naturality of the associativity isomorphism in a sense that is considerably stronger than the one required in the definition of bicategory. The proof of the assertion is contained in the diagram



in which t is written for Mt, $\bar{t}$ for Nt, for all relevant values of t, and all unmarked arrows are instances of associativity isomorphisms, possibly horizontally composed with a 1-cell. The issue is the commutativity of the outside quadrangle. The four cells marked I, II, III and IV commute by the definition of τ being a map as in (1). The commutativity of the pentagons are the associativity coherence axioms for bicategory; the commutativity of the small quadrangles are instances of the (ordinary) naturality of the associativity isomorphism. Since all cells commute, the outside commutes as a consequence, and this is what we want.

D5. Now, start with the part (subsketch) S_3



of S ($S_3(t)=\emptyset$ for all $t\in \mathbb{L}_{2\text{-cat}}$, except for $t=C_0, C_1$), and a map

$$S_3 \underset{N}{\overset{M}{\underset{\downarrow\sigma}{\rightrightarrows}}} \mathcal{A} \ . \qquad (2)$$

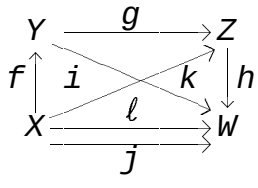
It is clear that if we have any $T \underset{Q}{\overset{P}{\underset{\downarrow\theta}{\rightrightarrows}}} \mathcal{A}$, and T' is the sketch obtained by adding a new element " $gf=h$ " to $T(\mathrm{T}_1)$, where f and g are already in T, but h is new, then P, Q and θ uniquely extend to $T' \underset{Q}{\overset{P}{\underset{\downarrow\theta}{\rightrightarrows}}} \mathcal{A}$. Now, let S_4 be the part of S which is S without the 2-cells ($S_4(t)=S(t)$ for $t=\mathrm{C}_0, \mathrm{C}_1, \mathrm{T}_1$ and $S_4(t)=\emptyset$ otherwise). Applying the above remark four times, we have, a unique extension

$$S_4 \underset{N}{\overset{M}{\underset{\downarrow\sigma}{\rightrightarrows}}} \mathcal{A}$$

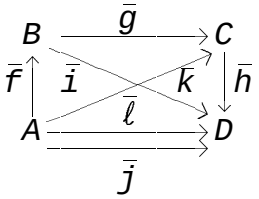
of (2).

D6. Suppose T is a sketch, T' is a subsketch of T missing only some 2-cells and T_2-elements of T, and that T is generated by T' in the sense that T is the least subsketch $T"$ of T such that $T"$ contains T' and every time when $(\rho, \sigma.\theta) \in T(\mathrm{T}_2)$, $\rho, \sigma \in T"(\mathrm{C}_2)$, then $\theta \in T"(\mathrm{C}_2)$, and every time when $(\rho, \sigma.\theta) \in T(\mathrm{H})$, $\rho, \sigma \in T"(\mathrm{C}_2)$, then $\theta \in T"(\mathrm{C}_2)$. Then every transformation $T' \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}$ is also one as in $T \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A}$. This is immediate.

D7. Let us turn to the proof that $\mathcal{R}$ preserves A. What we need to show is this. Assume that we have



in $\mathcal{X}$,



in $\mathcal{A}$,

the items listed under (*) in §7, the further items

$$(w:FW\xrightarrow{\simeq}D)\in\mathcal{R}\mathrm{C}_0[W,D]\ ,$$
$$(b:hi\xrightarrow{\cong}j)\in\mathcal{X}^{\#}\mathrm{T}_1(i,h,j)\ ,\quad(\bar b:\bar h\bar i\xrightarrow{\cong}\bar j)\in\mathcal{A}^{\#}\mathrm{T}_1(\bar i,\bar h,\bar j)\ ,$$
$$(c:hg\xrightarrow{\cong}k)\in\mathcal{X}^{\#}\mathrm{T}_1(g,h,k)\ ,\quad(\bar c:\bar h\bar g\xrightarrow{\cong}\bar k)\in\mathcal{X}^{\#}\mathrm{T}_1(\bar g,\bar h,\bar k)\ ,$$
$$(d:kf\xrightarrow{\cong}\ell)\in\mathcal{X}^{\#}\mathrm{T}_1(f,k,\ell)\ ,\quad(\bar d:\bar k\bar f\xrightarrow{\cong}\bar\ell)\in\mathcal{X}^{\#}\mathrm{T}_1(\bar f,\bar k,\bar\ell)\ ,$$
$$\eta\in\mathcal{R}\mathrm{C}_1(z,w)[h,\bar h]\ ,\quad\psi\in\mathcal{R}\mathrm{C}_1(x,w)[j,\bar j]\ ,\quad\kappa\in\mathcal{R}\mathrm{C}_1(y,w)[k,\bar k]\ ,$$
$$\lambda\in\mathcal{R}\mathrm{C}_1(x,w)[\ell,\bar\ell]\ ;$$

and assume that

$$\mathcal{R}\mathrm{T}_1(\varphi,\gamma,\iota)[a,\bar a]\ ,\quad\mathcal{R}\mathrm{T}_1(\iota,\eta,\psi)[b,\bar b]\ ,$$
$$\mathcal{R}\mathrm{T}_1(\gamma,\eta,\kappa)[c,\bar c]\ ,\quad\mathcal{R}\mathrm{T}_1(\varphi,\psi,\lambda)[d,\bar d]$$

hold. Under these conditions, we want that if $\mathcal{R}\mathrm{C}_2(x,w;\psi,\lambda)[\alpha,\bar\alpha]$, then

$$\mathcal{X}^{\#}\mathrm{A}(a,b,c,d;\alpha)\iff\mathcal{A}^{\#}\mathrm{A}(\bar a,\bar b,\bar c,\bar d;\bar\alpha)\ .$$

I claim that it suffices to show that

$$\mathcal{X}^{\#}\mathrm{A}(a,b,c,d;\alpha)\ \text{ and }\ \mathcal{A}^{\#}\mathrm{A}(\bar a,\bar b,\bar c,\bar d;\bar\alpha)\ \text{ imply }\ \mathcal{R}\mathrm{C}_2(x,w;\psi,\lambda)[\alpha,\bar\alpha]\ .$$

We use that for the given a,b,c,d , there is a unique α such that $\mathcal{X}^{\#}\mathrm{A}(a,b,c,d;\alpha)$ (see (4) in §7), and similarly for $\bar a,\bar b,\bar c,\bar d$; and we use that for the given $x,w;\psi,\lambda$, the relation $\mathcal{R}\mathrm{C}_2(x,w;\psi,\lambda)[\alpha,\bar\alpha]$ of the variables $\alpha,\bar\alpha$ establishes a bijection $\alpha\mapsto\bar\alpha:\mathcal{X}\mathrm{C}_2(j,\ell)\xrightarrow{\cong}\mathcal{A}\mathrm{C}_2(\bar j,\bar\ell)$. The claim now is easily seen.

Thus, we assume $\mathcal{X}^{\#}\mathrm{A}(a,b,c,d;\alpha)$ and $\mathcal{A}^{\#}\mathrm{A}(\bar a,\bar b,\bar c,\bar d;\bar\alpha)$.

Recall the sketch S . The data give us diagrams $M_0:S\longrightarrow\mathcal{X}$, $N:S\longrightarrow\mathcal{A}$; the effect of

M_0 , N are given by the notation, except that $M_0\beta = \alpha_{f,\, g,\, i}$ (associativity iso in $\mathcal{X}$) and $N\beta = \alpha_{\bar{f},\, \bar{g},\, \bar{i}}$ (associativity iso in $\mathcal{A}$). Composing M_0 with F, we get $M{=}FM_0 : S \longrightarrow \mathcal{A}$ (see D3). Consider the restrictions $M : S_3 \to \mathcal{A}$, $N : S_3 \to \mathcal{A}$. The data $x, y, z, w, \varphi, \gamma, \iota, \eta, \psi, \kappa, \lambda$ supply the components of a map

$$S_3 \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ .$$

By D5, we have a unique extension of τ, also denoted by τ, as in

$$S_4 \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ .$$

Let S_5 be the subsketch of S that consists of S_4, and the 2-cells a, b, c, d. The assumptions and D2 (applied to the four maps $S_0 \to S$) tell us that we have

$$S_5 \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ .$$

Now, add also β back to S_5, getting S_6. Since by D3,

$$M\beta = (FM_0)(\beta) = \alpha_{Ff,\, Fg,\, Fh} = \alpha_{M\mathrm{f},\, M\mathrm{g},\, M\mathrm{h}} \ ,$$

D4 says that we have

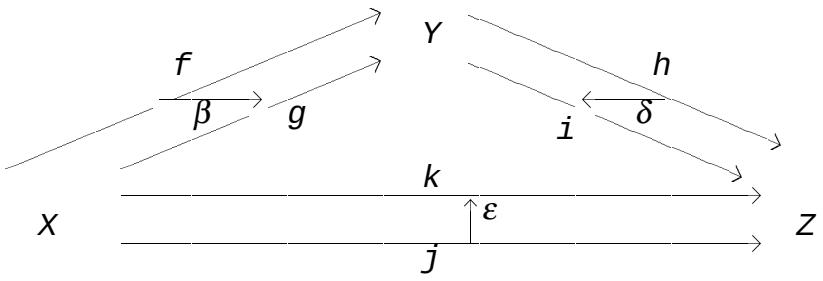
$$S_6 \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ ,$$

and finally D6 says that

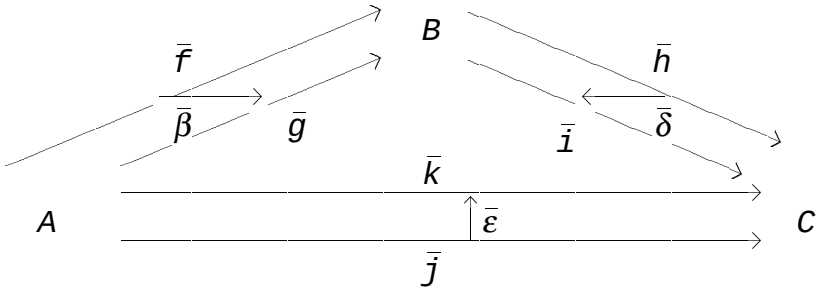
$$S \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ .$$

The fact that τ is natural with respect to α is the desired fact $\mathcal{RC}_2(x, w; \psi, \lambda)[\alpha, \bar{\alpha}]$, since, by D3, $M\alpha = F(M_0\alpha)$.

D8. The proof that $(\mathcal{R}, r_0, r_1)$ preserves H is similar, and simpler. Now, the situation is this. We have



in $\mathcal{X}$, and



in $\mathcal{A}$; we have

$(x{:}FX \xrightarrow{\simeq} A) \in \mathcal{R}\mathsf{C}_0[X, A]$, $(y{:}FY \xrightarrow{\simeq} B) \in \mathcal{R}\mathsf{C}_0[Y, B]$, $(z{:}FZ \xrightarrow{\simeq} C) \in \mathcal{R}\mathsf{C}_0[Z, C]$,
$\varphi \in \mathcal{R}\mathsf{C}_1(x, y)[f, \bar{f}]$, $\eta \in \mathcal{R}\mathsf{C}_1(y, z)[h, \bar{h}]$, $\psi \in \mathcal{R}\mathsf{C}_1(x, z)[j, \bar{j}]$,
$\gamma \in \mathcal{R}\mathsf{C}_1(x, y)[g, \bar{g}]$, $\iota \in \mathcal{R}\mathsf{C}_1(y, z)[i, \bar{i}]$, $\kappa \in \mathcal{R}\mathsf{C}_1(x, z)[k, \bar{k}]$,
$s \in \mathcal{X}^{\#}\mathsf{T}_1(f, h, j)$, $t \in \mathcal{X}^{\#}\mathsf{T}_1(g, i, k)$, $\bar{s} \in \mathcal{A}^{\#}\mathsf{T}_1(\bar{f}, \bar{h}, \bar{j})$, $\bar{t} \in \mathcal{A}^{\#}\mathsf{T}_1(\bar{g}, \bar{i}, \bar{k})$

such that

$\mathcal{R}\mathsf{C}_2(x, y; \varphi, \gamma)[\beta, \bar{\beta}]$, $\mathcal{R}\mathsf{C}_2(y, z; \eta, \iota)[\delta, \bar{\delta}]$, (3)

$\mathcal{R}\mathsf{T}_1(\varphi, \eta, \psi)[s, \bar{s}]$ and $\mathcal{R}\mathsf{T}_1(\gamma, \iota, \kappa)[t, \bar{t}]$. (4)

Under these conditions, we want that

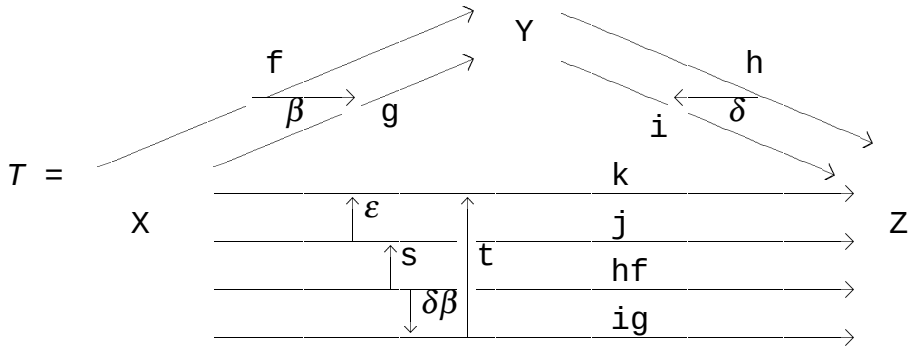
$\mathcal{RC}_2(x, z; \psi, \kappa)[\varepsilon, \bar{\varepsilon}] \Longrightarrow (\mathcal{X}^{\#}\mathrm{H}(s, t; \beta, \delta, \varepsilon) \Longleftrightarrow \mathcal{A}^{\#}\mathrm{H}(\bar{s}, \bar{t}; \bar{\beta}, \bar{\delta}, \bar{\varepsilon}))$.

Again, it suffices to show that

$$\mathcal{X}^{\#}\mathrm{H}(s, t; \beta, \delta, \varepsilon) \text{ and } \mathcal{A}^{\#}\mathrm{H}(\bar{s}, \bar{t}; \bar{\beta}, \bar{\delta}, \bar{\varepsilon}) \tag{5}$$

imply $\mathcal{RC}_2(x, z; \psi, \kappa)[\varepsilon, \bar{\varepsilon}]$. (6)

Assume (5). Consider the 2-cat-sketch

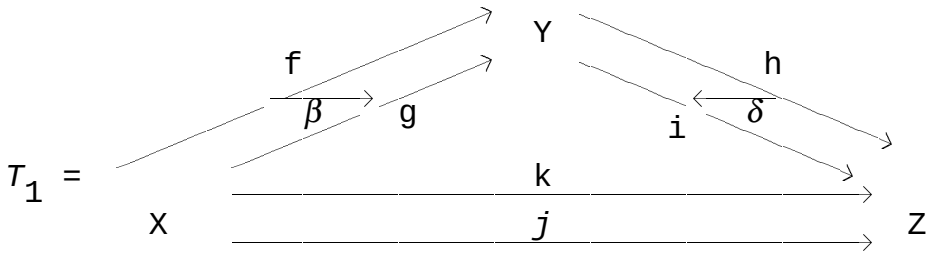


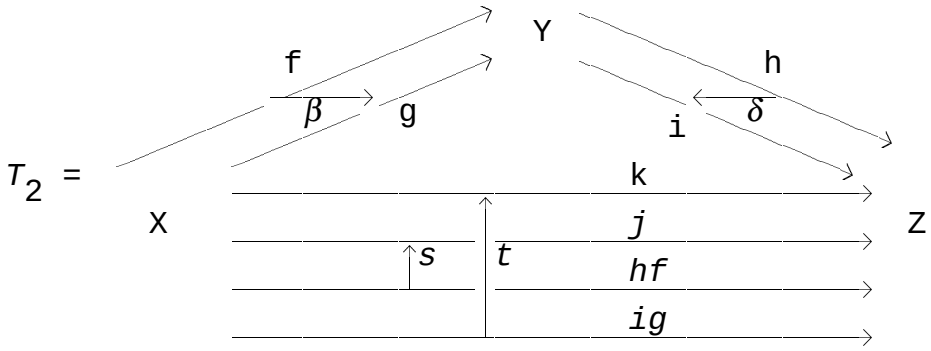
We have (f, h, hf), $(g, i, ig) \in T(\mathrm{T}_1)$, $(\beta, \delta, \delta\beta) \in T(\mathrm{H})$, and $\delta\beta \begin{matrix} hf \xrightarrow{s} j \\ \downarrow \circ \downarrow \varepsilon \\ ig \xrightarrow[t]{} k \end{matrix}$ (the latter by an (unmarked) 2-cell σ , and (s, ε, σ), $(\delta\beta, t, \sigma) \in T(\mathrm{T}_2)$) .

The conditions in (5) ensure that the data we have give rise to morphisms $M_0 : T \to \mathcal{X}$, $N : T \to \mathcal{A}$. As in the case of the sketch S , we can form the composite $M = FM_0 : T \to \mathcal{A}$; we have $M(s) = Fs \circ F_{f, h}$, $M(t) = Ft \circ F_{g, i}$; the commutativity of the diagram

$$\begin{array}{ccc} FhFf & \xrightarrow{F\delta F\beta} & FiFg \\ {\scriptstyle F_{f,h}}\downarrow & \circ & \downarrow{\scriptstyle F_{g,i}} \\ F(hf) & \xrightarrow{F(\delta\beta)} & F(ig) \\ {\scriptstyle Fs}\downarrow & \circ & \downarrow{\scriptstyle Ft} \\ F(j) & \xrightarrow[F\varepsilon]{} & F(k) \end{array}$$

ensures that M is indeed $M:T\to\mathcal{A}$. Consider the following subsketches of T:





The data x, y, z, φ, γ, η, ι, ψ, κ give, via the relation (3), a map

$$T_1 \xrightarrow[N]{M \downarrow \tau} \mathcal{A} \ ,$$

which, by (4) and D2, uniquely extends to

$$T_2 \xrightarrow[N]{M \downarrow \tau} \mathcal{A} \ .$$

By D6, this extends to

$$T \xrightarrow[N]{M \downarrow \tau} \mathcal{A} \ .$$

The naturality of τ with respect to ε is the desired relation (6).