§5. Equivalence

Let $\mathbb{L}$ be a fixed DSV, $\mathbb{K}$ the full subcategory of its kinds.

We have defined what an $\mathbb{L}$-structure is; even, what a $\mathbb{C}$-valued $\mathbb{L}$-structure is, for any $\mathbb{C}$ with finite limits. In what follows, we will make the minimal assumption that $\mathbb{C}$ is a regular category (which is equivalent to saying that $\mathcal{P}(\mathbb{C})$, with "total" $\mathcal{Q}$, is a $\wedge\exists$-fibration: just ignore $\mathbf{f}$ and $\vee$ in the definition of $\wedge\vee\exists$-fibration).

The category of $\mathbb{C}$-valued $\mathbb{L}$-structures, $\mathtt{Str}_{\mathbb{C}}(\mathbb{L})$, has objects the $\mathbb{C}$-valued $\mathbb{L}$-structures, and morphisms natural transformations; $\mathtt{Str}_{\mathbb{C}}(\mathbb{L})$ is a full subcategory of $\mathbb{C}^{\mathbb{L}}$ (with $\mathbb{L}$ in its last occurrence understood as a mere category). We write $\mathtt{Str}(\mathbb{L})$ for $\mathtt{Str}_{\mathtt{Set}}(\mathbb{L})$.

Given $M\in\mathtt{Str}_{\mathbb{C}}(\mathbb{L})$, we have $M\restriction\mathbb{K}:\mathbb{K}\longrightarrow\mathbb{C}$, its $\mathbb{K}$-reduct, the structure of kinds associated to M. For any $R\in\mathtt{Rel}(\mathbb{L})$, we have the canonical monomorphism $\mathtt{m}_R:M(R)\rightarrowtail M[R] = (M\restriction\mathbb{K})[R]$ (see §1). For a natural transformation $(f:U\longrightarrow V)\in\mathbb{C}^{\mathbb{K}}$, we have the canonical arrow $f_{[R]}:U[R]\longrightarrow V[R]$ for which

$$\begin{array}{ccc} U[R] & \xrightarrow{\;f_{[R]}\;} & V[R] \\ {\scriptstyle \pi^U_p}\downarrow & \circlearrowright & \downarrow{\scriptstyle \pi^V_p} \\ U(\mathtt{K}_p) & \xrightarrow[\;h_{\mathtt{K}_p}\;]{} & V(\mathtt{K}_p) \end{array}$$

for all $p\in R\,|\,\mathbb{L}$. If $(h:M\longrightarrow N)\in\mathtt{Str}(\mathbb{L})$, then

$$\begin{array}{ccc} M(R) & \xrightarrow{\;h_R\;} & N(R) \\ {\scriptstyle \mathtt{m}^M_R}\downarrow & \circlearrowright & \downarrow{\scriptstyle \mathtt{m}^N_R} \\ (M\restriction\mathbb{K})[R] & \xrightarrow[\;h_{[R]}\;]{} & (N\restriction\mathbb{K})[R] \end{array}$$

which shows that $h\restriction\mathbb{K}:M\restriction\mathbb{K}\longrightarrow N\restriction\mathbb{K}$ determines h (if any).

We have the forgetful functor $\mathcal{E}_{\mathbf{C},\mathbf{L}}=\mathcal{E}:\mathtt{Str}_{\mathbf{C}}(\mathbf{L})\longrightarrow \mathbf{C}^{\mathbf{K}}$; $\mathcal{E}$ is faithful, by the last remark. $\mathcal{E}$ is a fibration. Indeed, given $f:U\to V$ in $\mathbf{C}^{\mathbf{K}}$, and N over V (that is, $N\restriction\mathbf{K}=V$), then the Cartesian arrow $h:M\to N$ over f is obtained by defining M and h such that $M\restriction\mathbf{K}=U$, $h\restriction\mathbf{K}=f$ and, for all $R\in\mathtt{Rel}(\mathbf{L})$,

$$\begin{array}{ccc} M(R) & \xrightarrow{\quad h_R \quad} & N(R) \\ {\scriptstyle \mathfrak{m}^M_R}\downarrow & \square & \downarrow{\scriptstyle \mathfrak{m}^N_R} \\ U[R] & \xrightarrow[\quad f_{[R]} \quad]{} & V[R] \end{array}$$

is a pullback (it is immediate to see that h so defined is Cartesian). As usual with fibrations, let us denote M so defined by $f^*(N)$, and the Cartesian arrow h by $\theta_f:f^*(N)\to N$.

$\mathcal{E}$ is a fibration with fibers that are preorders.

When in particular $\mathbf{C}=\mathtt{Set}$ (which is the most important case), a functor $U:\mathbf{K}\to\mathtt{Set}$ is called *separated* if $U(K)\cap U(K')=\emptyset$ whenever K, K' are distinct objects of $\mathbf{K}$. For a separated U, we define $|U|=\bigcup_{K\in\mathbf{K}} U(K)$; for a general U, we would put $|U|=\bigsqcup_{K\in\mathbf{K}} U(K)=\{(K,a):k\in\mathbf{K},\ a\in U(K)\}$. Of course, every functor is isomorphic to a separated one. When $f:U\to V$, and U is separated, for $a\in|U|$ we may write $h(a)$ without ambiguity for $h_K(a)$ for which $a\in U(K)$. For notational simplicity, we will restrict attention to separated functors $\mathbf{K}\to\mathtt{Set}$.

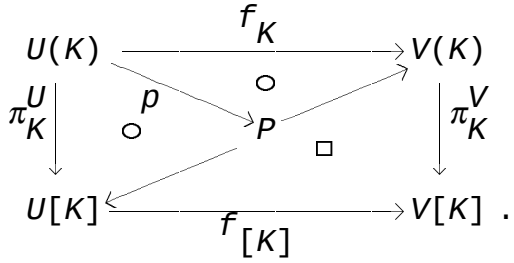
I will now isolate a property of a natural transformation $f:U\to V$ in $\mathbf{C}^{\mathbf{K}}$. Let first $\mathbf{C}=\mathtt{Set}$. We say that f is *very surjective* if whenever $K\in\mathbf{K}$, $\langle a_p\rangle_{p\in K\mid\mathbf{K}}\in U[K]$, the mapping

$$f_{\langle a_p\rangle_{p\in K\mid\mathbf{K}}}:UK(\langle a_p\rangle_{p\in K\mid\mathbf{K}})\longrightarrow VK(\langle fa_p\rangle_{p\in K\mid\mathbf{K}}):a\longmapsto f(a)$$

(see (3) in §1) is surjective.

For a general $\mathbf{C}$ (assumed to be regular), $f:U\to V$ in $\mathbf{C}^{\mathbf{K}}$ is *very surjective* if for every

$K\in\mathbf{K}$, the canonical map $p:U(K)\to P=U[K]\times_{V[K]}V(K)$ from the diagram below is surjective (a regular epimorphism):



It is clear that if f is an isomorphism (in $\mathbf{C}^{\mathbf{K}}$), then it is very surjective. It is easy to see (by induction on the level of $K\in\mathbf{K}$) that very surjective implies surjective (being a regular epimorphism in $\mathbf{C}^{\mathbf{K}}$), but not necessarily conversely.

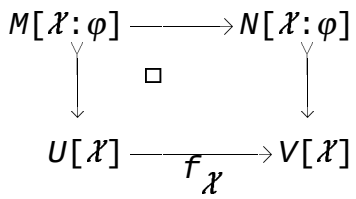
In this section, we consider logic with dependent sorts only without equality; all $\mathbf{L}$-formulas are without equality.

(1) Let $f:U\to V$ in $\mathbf{C}^{\mathbf{K}}$ be very surjective, and any $N\in\mathrm{Str}_{\mathbf{C}}(\mathbf{L})$ over V. Let $h=\theta_f:M=f^*(N)\to N$.

(a) Let first $\mathbf{C}=\mathrm{Set}$. h is *elementary with respect to logic without equality* in the sense that for any context $\mathcal{X}$ and $\mathbf{L}$-formula φ (in logic with dependent sorts and without equality) with $\mathrm{Var}(\varphi)\subset\mathcal{X}$, and any $\langle a_X\rangle_{X\in\mathcal{X}}\in M[\mathcal{X}]$,

$$M\vDash\varphi[\langle a_X\rangle_{X\in\mathcal{X}}]\iff N\vDash\varphi[\langle ha_X\rangle_{X\in\mathcal{X}}]\ .$$

(b) For a general $\mathbf{C}$ which is a Heyting category (to interpret all $\mathbf{L}$-formulas), for any φ and $\mathcal{X}$ as above, there is a pullback



(1b)

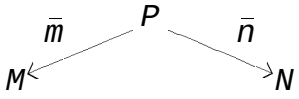
(the vertical monomorphisms are representatives for the subobjects $M[\mathcal{X}:\varphi]\in \mathrm{S}(U[\mathcal{X}])$, $N[\mathcal{X}:\varphi]\in \mathrm{S}(V[\mathcal{X}])$; in other words, (1b) says $M[\mathcal{X}:\varphi] = (f_{\mathcal{X}})^{*} N[\mathcal{X}:\varphi]$).

here, $f_{\mathcal{X}}$ is the canonical map determined through by the definition of $U[\mathcal{X}]$, $V[\mathcal{X}]$ as limits in $\boldsymbol{C}$.

Obviously, (b) generalizes (a).

The proof for (a) can be given as a straightforward induction on the complexity of φ . The clause for atomic formulas is essentially the definition of M . For the propositional connectives, the induction step is automatic. By the duality in Set between $\exists$ and $\forall$, it is enough to handle the inductive step involving $\exists$, which is done using the "very surjective" assumption. In Appendix B, I will take a "fibrational" view of the notion of equivalence, and give a detailed proof of the more general form (b) .

Let M , N be $\boldsymbol{C}$-valued $\boldsymbol{L}$-structures. We say that they are $\boldsymbol{L}$-*equivalent*, and we write $M \sim_{\boldsymbol{L}} N$, if there is a diagram



in $\mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$ such that $\bar{m}\restriction \boldsymbol{K}$, $\bar{n}\restriction \boldsymbol{K}$ are very surjective, and $\bar{m}$ and $\bar{n}$ are Cartesian arrows in the fibration $\mathcal{E}_{\boldsymbol{C},\,\boldsymbol{L}}$. Paraphrased, this means that there exists a functor $W\in \boldsymbol{C}^{\boldsymbol{K}}$ and very surjective maps $m: W\to M\restriction \boldsymbol{K}$, $n: W\to N\restriction \boldsymbol{K}$ such that $m^{*}(M) = n^{*}(N)$, that is, for all $R\in \mathrm{Rel}(\boldsymbol{L})$,

$$\begin{array}{ccccc} M(R) & \longleftarrow & M(R)\times_{M[R]} W[R] = N(R)\times_{N[R]} W[R] & \longrightarrow & N(R) \\ \downarrow & & \square \quad \downarrow \quad \square & & \downarrow \\ M[R] & \xleftarrow[m_{[R]}]{} & W[R] & \xrightarrow[n_{[R]}]{} & N[R] \end{array} \qquad (1')$$

(where the equality means equality of subobjects of $W[R]$). In case $\boldsymbol{C} = \mathrm{Set}$, (1') means that if $R\in \mathrm{Rel}(\boldsymbol{L})$, $\langle c_p\rangle_{p\in R\,|\,\boldsymbol{K}}\in W[R]$, then

$$\langle mc_p \rangle_{p \in R \mid K \in M(R)} \iff \langle nc_p \rangle_{p \in R \mid K \in N(R)} \ . \qquad (1'')$$

The data (W, m, n) are said to form an L-*equivalence* of M and N; in notation, $(W, m, n) : M \overset{L}{\leftrightarrow} N$.

It is easy to see that the relation $\overset{L}{\sim}$ is an equivalence relation (for a proof, see Appendix B). It is also clear that isomorphism of L-structures implies L-equivalence.

Let us write $M \equiv_L N$ for: $M \vDash \sigma \iff N \vDash \sigma$ for all L-sentences in logic with dependent sorts and without equality. We have

(2)(a) $M \overset{L}{\sim} N \implies M \equiv_L N$.

This immediately follows from (1).

The word "equivalence" is used in "L-equivalence" because of the relationship to the various notions of "equivalence" used in category theory; see later.

At this point, the reader may want to look at Appendix C, which may help understand the concept of L-equivalence.

We now will exploit the fact that we have specified variables "with arbitrary parameters". In what follows, a *context* is a, not necessarily finite, set $\mathcal{Y}$ of variables such that $y \in \mathcal{Y}$, $x \in \mathrm{Dep}(y)$ imply that $x \in \mathcal{Y}$. When we want to refer to the previous sense of "context", we will say "finite context". A *specialization* is a map of contexts whose restriction to all finite subcontexts of the domain is a specialization in the original sense. Just as in case of finite contexts, there is a correspondence between contexts and functors $F : K \to \mathrm{Set}$ which is an equivalence of the categories Set^K and $\mathrm{Con}_\infty[K]$, the category of all (small) contexts and specializations.

Given a context $\mathcal{Y}$ and an K-structure M, the set $M[\mathcal{Y}]$ is defined by the formula (1), §1

(which was the definition of $M[\mathcal{Y}]$ for finite $\mathcal{Y}$). Given a formula φ with $\mathrm{Var}(\varphi)\subset\mathcal{Y}$, $M[\mathcal{Y}:\varphi]$ is the subset of $M[\mathcal{Y}]$ for which, for any $\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}]$,

$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\varphi] \iff \langle a_y\rangle_{y\in\mathcal{Y}'}\in M[\mathcal{Y}':\varphi]$$

for any (equivalently, some) finite context $\mathcal{Y}'$ with $\mathrm{Var}(\varphi)\subset\mathcal{Y}'\subset\mathcal{Y}$. As before, we write also $M\models\varphi[\langle a_y\rangle_{y\in\mathcal{Y}}]$ for $\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\varphi]$.

Suppose $\mathcal{X}$ is a context, M, N $\boldsymbol{L}$-structures, $\vec{a}=\langle a_x\rangle_{x\in\mathcal{X}}\in M[\mathcal{X}]$, $\vec{b}=\langle b_x\rangle_{x\in\mathcal{X}}\in N[\mathcal{X}]$. We write

$$(W, m, n):(M,\vec{a})\xleftrightarrow[\boldsymbol{L}]{}(N,\vec{b}) \qquad (3)$$

if $(W, m, n):M\xleftrightarrow[\boldsymbol{L}]{}N$ and there is $\langle s_x\rangle_{x\in\mathcal{X}}\in W[\mathcal{X}]$ such that $ms_x=a_x$ and $ns_x=b_x$ for all $x\in\mathcal{X}$. We write $(M,\vec{a})\sim_{\boldsymbol{L}}(N,\vec{b})$ if there is (W, m, n) such that (3) holds.

With M, N, $\mathcal{X}$, $\vec{a}$, $\vec{b}$ as above, we write $(M,\vec{a})\equiv_{\boldsymbol{L}}(N,\vec{b})$ for: for all $\boldsymbol{L}$-formulas φ with $\mathrm{Var}(\varphi)\subset\mathcal{X}$, we have $M\models\varphi[\langle a_x\rangle_{x\in\mathcal{X}}] \iff N\models\varphi[\langle b_x\rangle_{x\in\mathcal{X}}]$.

We have the following generalization of (2)(a) :

(2)(b) $(M,\vec{a})\sim_{\boldsymbol{L}}(N,\vec{b}) \Longrightarrow (M,\vec{a})\equiv_{\boldsymbol{L}}(N,\vec{b})$;

this also follows immediately from (1) .

Let $\mathcal{Y}$ be a context, x a variable such that $x\notin\mathcal{Y}$ but $\mathcal{Y}\cup\{x\}$ is a context (thus, $\mathrm{x}_{x,\,p}\in\mathcal{Y}$ for all $p\in\mathrm{K}_x\,|\,\boldsymbol{K}$), and let Φ be a set of formulas in logic with dependent sorts over $\boldsymbol{L}$ such that $\mathrm{Var}(\Phi)=\bigcup_{\varphi\in\Phi}\mathrm{Var}(\varphi)\subset\mathcal{Y}\cup\{x\}$; such Φ is called a *$\mathcal{Y}$-set* (of formulas; with x any variable as described with respect to $\mathcal{Y}$). Let M be an $\boldsymbol{L}$-structure, and $\vec{a}=\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}]$. We say that Φ is *satisfiable in* $(M,\vec{a})$ if there is $a\in|M|$ (more

precisely, $a \in M\mathsf{K}_X[\langle a_{\mathsf{X}_{X,\,p}} \rangle_{p \in \mathsf{K}_X} | \boldsymbol{K}]$) such that $M \models \varphi[\vec{a}, a/x]$ (of course, $\vec{a}, a/x$ stands for $\langle a'_y \rangle_{y \in \mathcal{Y} \dot{\cup} \{x\}}$ for which $a'_y = a_y$ for $y \in \mathcal{Y}$, and $a'_x = a$). Φ is *finitely satisfiable in* $(M, \vec{a})$ if every finite subset of Φ is satisfiable in $(M, \vec{a})$. M is said to be $\mathcal{Y}$-$\boldsymbol{L}$*-saturated* if for every $\vec{a} \in M[\mathcal{Y}]$ and every $\mathcal{Y}$-set Φ , if Φ is finitely satisfiable in $(M, \vec{a})$, then Φ is satisfiable in $(M, \vec{a})$.

Let κ be an infinite cardinal. We say that M is κ, $\boldsymbol{L}$*-saturated* if it is $\mathcal{Y}$-$\boldsymbol{L}$-saturated for every context $\mathcal{Y}$ with cardinality smaller than κ .

For saturated models for ordinary first order logic, see [CK]. In [MR2], one can find a detailed introduction to saturated and special models for multisorted logic; the basic facts and their proofs in the multisorted context do not essentially differ from the original one-sorted versions.

κ, $\boldsymbol{L}$-saturation is κ-saturation with respect to $\boldsymbol{L}$-formulas. Since $\boldsymbol{L}$-formulas form a part of the multisorted formulas over $|\boldsymbol{L}|$, it is clear that if M , an $\boldsymbol{L}$-structure, is κ-saturated as a structure for the similarity type $|\boldsymbol{L}|$, then M is κ, $\boldsymbol{L}$-saturated. More generally, suppose that we have "interpreted" $\boldsymbol{L}$ in a theory S in ordinary multisorted first-order logic; that is, we have a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure $I : \boldsymbol{L} \longrightarrow \boldsymbol{C}$, for $\boldsymbol{C}$ the Lindenbaum-Tarski category $[S]$ of S (see [MR]; $[S]$ is a Boolean category). Then if M is a model of S , or equivalently, $M : \boldsymbol{C} \to \mathtt{Set}$ is a coherent functor, and M is κ-saturated in the ordinary sense, then the $\boldsymbol{L}$-structure $M \restriction \boldsymbol{L} = MI : \boldsymbol{L} \to \mathtt{Set}$ is κ, $\boldsymbol{L}$-saturated.

By the *cardinality of* the structure M , $\#M$, we mean the cardinality of its underlying set $|M|$.

(4) Suppose the $\boldsymbol{L}$-structures M , N are κ, $\boldsymbol{L}$-saturated, and both are of cardinality $\leq \kappa$. Then the converses of (2)(a) and (2)(b) hold:

$$M \equiv_{\boldsymbol{L}} N \Longrightarrow M \sim_{\boldsymbol{L}} N \ ;$$

and more generally, if $\mathcal{X}$ is a context of size $< \kappa$, $\vec{a} \in M[\mathcal{X}]$, $\vec{b} \in N[\mathcal{X}]$, then

$$(M, \vec{a}) \equiv_{\boldsymbol{L}} (N, \vec{b}) \implies (M, \vec{a}) \sim_{\boldsymbol{L}} (N, \vec{b}) \ .$$

Proof.

For a given infinite cardinal κ, and a given context $\mathcal{X}$ of cardinality less than κ, let $\mathcal{U}=\mathcal{U}[\kappa, \mathcal{X}]$ be a context such that $\#\mathcal{U} = \kappa$, $\mathcal{X}\subset\mathcal{U}$, and for every sort X with $\mathtt{Var}(X)\subset\mathcal{U}$, the cardinality of the set of variables $x\in\mathcal{U}$ with $x:X$ is equal to κ. It is easy to see that such an $\mathcal{U}$ exists; we define contexts $\mathcal{U}_i$ by recursion on $i\leq k$ for k the height of $\boldsymbol{K}$; let $\mathcal{U}_0=\emptyset$; if $\mathcal{U}_i$ has been defined, pick, for every sort X whose kind is of level i and for which $\mathtt{Var}(X) \subset \mathcal{U}_i$, a set V_X of variables $v:X$ such that $\#V_X=\kappa$, and let $\mathcal{U}_{i+1}$ be the union of $\mathcal{U}_i$ and all the V_X for all such X; if $k=\omega$, let $\mathcal{U}_\omega = \bigcup_{i<\omega} \mathcal{U}_i$; let $\mathcal{U}=\mathcal{U}_k$.

Next, enumerate $\mathcal{U}$ as a sequence $\langle u_\alpha\rangle_{\alpha<\kappa}$ in such a way that for each $\beta<\kappa$, $\langle u_\alpha\rangle_{\alpha<\beta}$ is a context; equivalently, such that for each $\beta<\kappa$, $\mathtt{Dep}(u_\beta)\subset\{u_\alpha:\alpha<\beta\}$. Note first of all that for any finite context $\mathcal{Y}$, there is an enumeration $\mathcal{Y}=\{y_i:i<n\}$ such that $\langle y_i\rangle_{i<j}$ is a context for all $j<n$; enumerate first the level-$\mathtt{0}$ variables, next the level-$\mathtt{1}$ ones, etc. Call such an enumeration of $\mathcal{Y}$ "good". Now, take first an arbitrary enumeration $\langle v_\alpha\rangle_{\alpha<\kappa}$ of $\mathcal{U}$; define the increasing sequence $\langle\beta_\alpha\rangle_{\alpha<\kappa}$ of ordinals and the partial enumeration $\langle u_\gamma\rangle_{\gamma<\beta_\alpha}$ by induction on α as follows. For a limit ordinal α, $\beta_\alpha=\lim_{\delta<\alpha}\beta_\delta$. For $\alpha=\delta+1$, let $\langle u_{\beta_\delta+i}\rangle_{i<n}$ be a good enumeration of $\mathtt{Dep}(v_\delta)\cup\{v_\delta\}$, and let $\beta_\alpha=\beta_\delta+n$.

For every sort X such that $\mathtt{Var}(X)\subset\mathcal{U}$, let $\langle u_{\alpha_{X,\,\nu}}\rangle_{\nu<\kappa}$ be an enumeration in increasing order of all u_α of sort X for which $u_\alpha\notin\mathcal{X}$. Finally, for any $\alpha<\kappa$, let $\nu[\alpha]$ be the ordinal ν for which $\alpha_{X,\,\nu}=\alpha$ where X is the sort of u_α.

Assume $\mathcal{X}$ is a context of size $<\kappa$, $\#M,\#N\leq\kappa$, $\vec{a}=\langle a_X\rangle_{X\in\mathcal{X}}\in M[\mathcal{X}]$, $\vec{b}=\langle b_X\rangle_{X\in\mathcal{X}}\in N[\mathcal{X}]$, and $(M, \vec{a}) \equiv_{\boldsymbol{L}} (N, \vec{b})$. For any M-sort $MK(\langle c_p\rangle_{p\in K\,|\,\boldsymbol{K}})=MK(\vec{c})$, let us fix an enumeration $\langle e_\xi\rangle_{\xi<\lambda} = \langle e_{K,\,\vec{c},\,\xi}\rangle_{\xi<\lambda_{K,\,\vec{c}}}$ of the set $MK(\vec{c})$; here, $\lambda_{K,\,\vec{c}}\leq\kappa$.

Consider $\mathcal{U}=\mathcal{U}[\kappa, \mathcal{X}]$ constructed above.

We define a context $\mathcal{Z}$, a subset of $\mathcal{U}$, by deciding, recursively on $\alpha<\kappa$, whether u_α belongs to $\mathcal{Z}$ or not; furthermore, we also define, for each $u_\alpha\in\mathcal{Z}$, elements $c_\alpha\in |M|$ and $d_\alpha\in |N|$. Let $\mathcal{Z}_\alpha$ denote the set of all u_β with $\beta<\alpha$ for which $u_\beta\in\mathcal{Z}$, and $\vec{c}[\alpha]$ be the sequence $\langle c_z\rangle_{z\in\mathcal{X}\cup\mathcal{Z}_\alpha}\in M[\mathcal{X}\cup\mathcal{Z}_\alpha]$ for which $c_x=a_x$ ($x\in\mathcal{X}$) and $c_{u_\beta}=c_\beta$ ($u_\beta\in\mathcal{Z}_\alpha$). Similarly, we have $\vec{d}[\alpha]\in \nu[\mathcal{X}\cup\mathcal{Z}_\alpha]$. The induction hypothesis of the construction is that

$$(M, \vec{c}[\alpha+1]) \equiv_{\mathbf{L}} (M, \vec{d}[\alpha+1]) . \qquad (5)$$

Suppose $\alpha<\kappa$, and $\mathcal{Z}_\alpha$, $\vec{c}[\alpha]$, $\vec{d}[\alpha]$ have been defined so that, for all $\beta<\alpha$, $(M, \vec{c}[\beta+1]) \equiv_{\mathbf{L}} (M, \vec{d}[\beta+1])$. Since in the definition of "$\equiv_{\mathbf{L}}$", formulas with finitely many free variables are involved, we can conclude that

$$(M, \vec{c}[\alpha]) \equiv_{\mathbf{L}} (M, \vec{d}[\alpha]) . \qquad (6)$$

Look at the variable u_α and its sort X. If $u_\alpha\in\mathcal{X}$, we let $u_\alpha\in\mathcal{Z}$, $c_\alpha=a_{u_\alpha}$, $d_\alpha=b_{u_\alpha}$. (5) is now an automatic consequence of (6).

If not all the variables in X (which are u_β's for $\beta<\alpha$) are in $\mathcal{Z}$, then $u_\alpha\notin\mathcal{Z}$, and we are finished with the stage α.

Assume that $u_\alpha\notin\mathcal{X}$ and all the variables in X are in $\mathcal{Z}$. Look at the ordinal $\nu=\nu[\alpha]$; write ν in the form $\nu=2\cdot\mu$ or $\nu=2\cdot\mu+1$ as the case may be. Let first $\nu=2\cdot\mu$. With $X=K(\langle u_{\beta_p}\rangle_{p\in K\,|\,\mathbf{K}})$, consider the M-sort $MK(\langle c_{\beta_p}\rangle_{p\in K\,|\,\mathbf{K}})=MK(\vec{c})$ and its previously fixed enumeration $\langle e_\xi\rangle_{\xi<\lambda}$ ($=\langle e_{K,\vec{c},\xi}\rangle_{\xi<\lambda_{K,\vec{c}}}$). If $\mu\geq\lambda$, then again $u_\alpha\notin\mathcal{Z}$. If, however, $\mu<\lambda$, then $u_\alpha\in\mathcal{Z}$. Moreover, $c_\alpha \,\bar{\bar{\mathrm{def}}}\, e_\mu$.

Let Φ be the $\mathcal{X}\cup\mathcal{Z}_\alpha$-set of all formulas φ with $\mathrm{Var}(\varphi)\subset\mathcal{X}\cup\mathcal{Z}_\alpha\dot\cup\{u_\alpha\}$ for which $M\models\varphi[\vec{c}[\alpha], e_\mu/u_\alpha]$. I claim that Φ is finitely satisfiable in $(N, \vec{d}[\alpha])$. Let Ψ be a finite subset of Φ. For $\varphi=\bigwedge\Psi$, we have $M\models\varphi[\vec{c}[\alpha], e_\mu/u]$, hence, $M\models(\exists u_\alpha\varphi)[\vec{c}[\alpha]]$ (note that $\exists u_\alpha\varphi$ is well-formed, since for every $z\in\mathrm{Var}(\varphi)$, $z\neq u_\alpha$, we have $z\in\mathcal{X}\cup\mathcal{Z}_\alpha$, hence $\mathrm{Dep}(z)\subset\mathcal{X}\cup\mathcal{Z}_\alpha$, and $u_\alpha\notin\mathrm{Dep}(z)$). As a consequence, by (6),

$N \models (\exists u \varphi)[\vec{d}[\alpha]]$. This means that Ψ is satisfiable in $(N, \vec{d}[\alpha])$ as desired.

Since $\#(\mathcal{X} \cup \mathcal{Z}_\alpha) < \kappa$, and N is κ, $\boldsymbol{L}$-saturated, Φ is satisfiable in $(N, \vec{d}[\alpha])$, by $d_\alpha \in NK(\langle d_{\beta_p} \rangle_{p \in K \mid \boldsymbol{K}})$, say. The choice of Φ ensures that (5) holds.

In case $\nu = 2 \cdot \mu + 1$, we proceed similarly, with the roles of M and N interchanged.

With the construction completed, we put $\mathcal{Z} = \bigcup_{\alpha < \kappa} \mathcal{Z}_\alpha$. We let W be the functor $F_{\mathcal{Z}} : \boldsymbol{K} \to \mathrm{Set}$ associated with the context $\mathcal{Z}$ (see §4). $m : W \to M \restriction \boldsymbol{K}$, $n : W \to N \restriction \boldsymbol{K}$ are defined by $m(u_\alpha) = c_\alpha$, $n(u_\alpha) = d_\alpha$ ($u_\alpha \in \mathcal{Z}$) . The definition ensures that $\mathcal{X} \subset \mathcal{Z}$ and $m(x) = a_x$, $n(x) = b_x$ ($x \in \mathcal{X}$) .

Let us see that m is very surjective. Let $K \in \boldsymbol{K}$. $W[K]$ is the set of all tuples $\langle z_p \rangle_{p \in K \mid \boldsymbol{K}}$ for which each $z_p \in \mathcal{Z}$, and $X = K(\langle z_p \rangle_{p \in K \mid \boldsymbol{K}})$ is a (well-formed) sort; $WK(\langle z_p \rangle_{p \in K \mid \boldsymbol{K}})$ is the set of all $z \in \mathcal{Z}$ such that $z : X$. So, assume that $X = K(\langle z_p \rangle_{p \in K \mid \boldsymbol{K}}) = K(\langle u_{\beta_p} \rangle_{p \in K \mid \boldsymbol{K}})$ is a sort, and

$$a \in MK(\langle mz_p \rangle_{p \in K \mid \boldsymbol{K}}) = MK(\langle c_{\beta_p} \rangle_{p \in K \mid \boldsymbol{K}}) = MK(\vec{c}) \ .$$

Then $a = e_{K, \vec{c}, \mu}$ for some $\mu < \lambda_{K, \vec{c}}$, and for $\alpha = \alpha_{X, 2 \cdot \mu}$, the construction at stage α puts $u_\alpha : X$ into $\mathcal{Z}$; that is, $u_\alpha \in WK(\langle z_p \rangle_{p \in K \mid \boldsymbol{K}})$, with $a = c_\alpha = mu_\alpha$ as desired.

The fact that n is very surjective is seen analogously.

We have that $(W, m, n) : M \xleftarrow{\boldsymbol{L}} N$, since (1") is a consequence of (5) being true for all $\alpha < \kappa$; one has to apply (5) to atomic formulas.

This completes the proof of (4).

Let $\boldsymbol{C}$ be a small Boolean category. By a *model of* $\boldsymbol{C}$ we mean a functor $M : \boldsymbol{C} \to \mathrm{Set}$ preserving the Boolean structure (that is, M is a coherent functor). We write $M \models \boldsymbol{C}$ to say that M is a model of $\boldsymbol{C}$.

There is a theory $\mathsf{T}_{\mathbf{C}} = (\mathsf{L}_{\mathbf{C}}, \Sigma_{\mathbf{C}})$ in multisorted first-order logic, with $\mathsf{L}_{\mathbf{C}}$ the underlying graph of $\mathbf{C}$, such that the models of $\mathbf{C}$ are *the same* as the models of $\mathsf{T}_{\mathbf{C}}$ (note that both the models of $\mathbf{C}$ and the models of $\mathsf{T}_{\mathbf{C}}$ are particular diagrams $\mathsf{L}_{\mathbf{C}} \to \mathsf{Set}$). Moreover, for any subobject $\varphi \in \mathsf{S}_{\mathbf{C}}(A)$, $A \in \mathbf{C}$, there is a (simply defined) $\mathsf{L}_{\mathbf{C}}$-formula $\underline{\varphi}(x)$ with a single free variable $x:A$ such that for every $M \models \mathbf{C}$ and $a \in M(A)$, $M \models \underline{\varphi}[a]$ ($\iff M \models \underline{\varphi}[a/x]$) iff $a \in M(\varphi)$ ($\subset M(A)$). See [MR].

For $\sigma \in \mathsf{S}(1_{\mathbf{C}})$, a subobject of the terminal object in $\mathbf{C}$, we write $M \models \sigma$ for $M(\sigma)=1$ in Set. We will call a subobject of $1_{\mathbf{C}}$ a *sentence* in $\mathbf{C}$.

Let $I:\mathbf{L} \to \mathbf{C}$ a $\mathbf{C}$-valued $\mathbf{L}$-structure (in particular, $I:\mathbf{L} \to \mathbf{C}$ is a functor from $\mathbf{L}$ as a category). When $\mathbf{C}$ is the Lindenbaum-Tarski category $[S]$ of a theory $S=(\mathsf{L}_S, \Sigma_S)$ in ordinary multisorted logic (see [MR] or [M?]), then such an I is what we should consider an *interpretation* of the DS vocabulary $\mathbf{L}$ in the theory S. An example is obtained by taking $S=(|\mathbf{L}|, \Sigma[\mathbf{L}])$ (for $\Sigma[\mathbf{L}]$, see §1), and for $I:\mathbf{L} \to [S]$ the $[S]$-structure defined by $I(A)=[a:\mathbf{t}]$ for $A \in \mathbf{L}$ where $a:A$, and for $f:A \to B$, $I(f)=\langle a \mapsto b : fa=b \rangle : [a:\mathbf{t}] \to [b:\mathbf{t}]$. $I:\mathbf{L} \to [S]$ is the *canonical* interpretation of logic with dependent types in multisorted logic. In this case, for any formula φ of FOLDS over $\mathbf{L}$, with $\mathsf{Var}(\varphi) \subset \mathcal{X}$, we have $I[\mathcal{X}:\varphi] = m^*[\mathcal{X}:\varphi^*]$; here, $m:I[\mathcal{X}:\varphi] \rightarrowtail \{\mathcal{X}\}_{\mathsf{def}} \prod_{X \in \mathcal{X}} \mathsf{K}_X$ is the canonical monomorphism, m^* denotes pulling back along m; φ^* was defined in §1.

For a general $I:\mathbf{L} \to \mathbf{C}$, and for an $\mathbf{L}$-sentence θ, let us write $I(\theta)$ for the sentence $I[\emptyset:\theta]$ of $\mathbf{C}$. In case $\mathbf{C}=[S]$, $I(\theta)$ also stands for any one of the S-equivalent L_S-sentences which are the representatives of the $\mathbf{C}$-subobject $I(\theta)$.

When $M \models \mathbf{C}$, the composite $MI:\mathbf{L} \to \mathsf{Set}$ is an $\mathbf{L}$-structure. We also write $M \restriction \mathbf{L}$ for MI; $M \restriction \mathbf{L}$ is the $\mathbf{L}$-*reduct of* M (via I).

Let $\mathbf{C}$ and $\mathbf{D}$ be small Boolean categories, $I:\mathbf{L} \to \mathbf{C}$ and $J:\mathbf{L} \to \mathbf{D}$. Notational conventions introduced above for $I:\mathbf{L} \to \mathbf{C}$ are valid for $J:\mathbf{L} \to \mathbf{D}$, *mutatis mutandis.*

(7)(a) Assume that σ is a sentence of $\mathbf{C}$, τ a sentence of $\mathbf{D}$, and for all $M \models \mathbf{C}$, $N \models \mathbf{D}$,

$$M \models \sigma \;\&\; M\restriction \mathbf{L} \sim_{\mathbf{L}} N\restriction \mathbf{L} \;\Longrightarrow\; N \models \tau\ .$$

Then there is an $\mathbf{L}$-sentence θ in logic with dependent sorts without equality such that for all $M \models \mathbf{C}$, $N \models \mathbf{D}$, we have

$$M \models \sigma \Longrightarrow M\restriction \mathbf{L} \models \theta \qquad \text{and} \qquad N\restriction \mathbf{L} \models \theta \Longrightarrow N \models \tau\ .$$

For a more general formulation, consider a finite $\mathbf{L}$-context $\mathcal{X}$, and the object $I[\mathcal{X}] \in \mathbf{C}$. $I[\mathcal{X}]$ is defined as a finite limit in $\mathbf{C}$; see the end of §1; let $\pi_{[x]} : I[\mathcal{X}] \to I(\mathsf{K}_x)$ be the limit projections ($x \in \mathcal{X}$). Given any $M \models \mathbf{C}$, we have similar projections $\rho_{[x]} : (M\restriction \mathbf{L})[\mathcal{X}] \to MI(\mathsf{K}_x)$ in $\mathtt{Set}$, and a canonical isomorphism $\mu : (M\restriction \mathbf{L})[\mathcal{X}] \xrightarrow{\cong} M(I[\mathcal{X}])$ making each diagram

$$\begin{array}{ccc} (M\restriction \mathbf{L})[\mathcal{X}] & \xrightarrow{\quad\mu\quad} & M(I[\mathcal{X}]) \\ & {\scriptstyle\rho_{[x]}}\searrow \quad \cong \quad \swarrow{\scriptstyle M(\pi_{[x]})} & \\ & MI(\mathsf{K}_x) & \end{array} \qquad (7')$$

commute. If $\vec{a} = \langle a_x \rangle_{x \in \mathcal{X}} \in (M\restriction \mathbf{L})[\mathcal{X}]$, we write $\langle \vec{a} \rangle$ for $\mu(\vec{a}) \in M(I[\mathcal{X}])$. Once again, similar conventions apply in the context of $J : \mathbf{L} \to \mathbf{D}$.

(7)(b) Assume that $\mathcal{X}$ is a finite $\mathbf{L}$-context, $\sigma \in \mathsf{S}_{\mathbf{C}}(I[\mathcal{X}])$, $\tau \in \mathsf{S}_{\mathbf{D}}(J[\mathcal{X}])$, and for all $M \models \mathbf{C}$, $N \models \mathbf{D}$, $\vec{a} \in (M\restriction \mathbf{L})[\mathcal{X}]$, $\vec{b} \in (N\restriction \mathbf{L})[\mathcal{X}]$,

$$\langle \vec{a} \rangle \in M(\sigma) \;\&\; (M\restriction \mathbf{L}, \vec{a}) \sim_{\mathbf{L}} (N\restriction \mathbf{L}, \vec{b}) \;\Longrightarrow\; \langle \vec{b} \rangle \in N(\tau)\ . \qquad (8)$$

Then there is an $\mathbf{L}$-formula θ in logic with dependent sorts without equality with $\mathtt{Var}(\varphi) \subset \mathcal{X}$ such that

$$\sigma \leq_{I[\mathcal{X}]} I[\mathcal{X} : \theta]\ , \qquad J[\mathcal{X} : \theta] \leq_{J[\mathcal{X}]} \tau\ . \qquad (8')$$

Note that (8') may be written equivalently as

for all $M \models \boldsymbol{C}$, $N \models \boldsymbol{D}$, $\vec{a} \in (M \restriction \boldsymbol{L})[\mathcal{X}]$ and $\vec{b} \in (N \restriction \boldsymbol{L})[\mathcal{X}]$,

$$\langle \vec{a} \rangle \in M(\sigma) \Longrightarrow M \restriction I \models \theta[\vec{a}] \text{ and } N \restriction J \models \theta[\vec{b}] \Longrightarrow \langle \vec{b} \rangle \in N(\tau) .$$

Proof. Let us extend the vocabulary $\mathrm{L}_{\boldsymbol{C}}$ to $\mathrm{L}_{\boldsymbol{C}}(c)$ by adding a single new individual constant c of sort $A_{\mathrm{def}} I[\mathcal{X}]$. For any $\varphi \in \mathrm{S}_{\boldsymbol{C}}(A)$, let $\varphi(c)$ denote $\underline{\varphi}(c/x)$, the result of substituting c for x in $\underline{\varphi}(x)$. For an $\boldsymbol{L}$-formula θ with $\mathrm{Var}(\theta) \subset \mathcal{X}$, let $\theta(c)$ stand for $(I[\mathcal{X}:\theta])(c)$. Similarly, we introduce $d : B_{\mathrm{def}} J[\mathcal{X}]$; for $\psi \in \mathrm{S}_{\boldsymbol{D}}(B)$, $\psi(d)$ and for θ as before, $\theta(d)$.

Let Θ be the set of all $\boldsymbol{L}$-formulas θ with $\mathrm{Var}(\theta) \subset \mathcal{X}$ such that $\sigma \leq_A I[\mathcal{X}:\theta]$. Consider the set $\Sigma_{\mathrm{def}} \Sigma_{\boldsymbol{D}} \cup \{\theta(d) : \theta \in \Theta\}$ of $\mathrm{L}_{\boldsymbol{D}}(d)$-sentences. I claim that

$$(\mathrm{L}_{\boldsymbol{D}}(d), \Sigma) \models \tau(d) . \qquad (9)$$

Once the claim is proved, by compactness there are finitely many $\theta_i \in \Theta$ ($i<n$) such that $(\mathrm{L}_{\boldsymbol{D}}(d),\ \Sigma_{\boldsymbol{D}} \cup \{\theta_i(d) : i<n\}) \models \tau(d)$, which means, for $\theta = \bigwedge_{i<n} \theta_i \in \Theta$ that $(\mathrm{L}_{\boldsymbol{D}}(d), \Sigma_{\boldsymbol{D}}) \models \theta(d) \to \tau(d)$, that is, $(\mathrm{L}_{\boldsymbol{D}}(d), \Sigma_{\boldsymbol{D}}) \models \forall x : B.(\underline{\theta}(x) \to \underline{\tau}(x))$, which means $J[\mathcal{X}:\theta] \leq_B \tau$; thus, it is enough to see the claim.

Assume that there is an infinite cardinal $\lambda \geq \#\mathrm{L}_{\boldsymbol{C}}$ such that $\lambda^+ = 2^\lambda$ (see below for the legitimacy of this assumption). Let $\kappa = \lambda^+$. According to the existence theorem for saturated models (see [CK], [MR2]), any $\mathrm{L}_{\boldsymbol{D}}(d)$-structure is elementarily equivalent to a κ-saturated structure of cardinality $\leq \kappa$. Therefore, to show (9), take $(N, b/d)$, a κ-saturated model of cardinality $\leq \kappa$ of $(\mathrm{L}_{\boldsymbol{D}}(d), \Sigma)$, to show $(N, b/d) \models \tau(d)$.

Let Φ be the set of $\boldsymbol{L}$-formulas φ with $\mathrm{Var}(\varphi) \subset \mathcal{X}$ such that $b \in N(I[\mathcal{X}:\varphi]) \subset NB$; for every $\boldsymbol{L}$-formula φ with $\mathrm{Var}(\varphi) \subset \mathcal{X}$, exactly one of φ, $\neg\varphi$ belongs to Φ. Since $(N, b/d)$ is a model of $(\mathrm{L}_{\boldsymbol{D}}(d), \Sigma)$, with Σ defined as it is, we have $\Theta \subset \Phi$. I make the subclaim that the theory

$$(\mathrm{L}_{\boldsymbol{C}}(c),\ \Sigma_{\boldsymbol{C}} \cup \{\sigma(c)\} \cup \{\varphi(c) : \varphi \in \Phi\}) \qquad (10)$$

is consistent. Consider a finite subset $\{\varphi_i : i<n\}$ of Φ. If

$(L_C(c),\ \Sigma_C \cup \{\sigma(c)\} \cup \{\varphi_i(c) : i<n\})$ were not consistent, then we would have, for $\varphi = \bigwedge_{i<n} \varphi_i \in \Phi$, that $\sigma \leq_A I[\mathcal{X} : \neg\varphi]$, which would mean that $\neg\varphi \in \Theta \subset \Phi$, contradicting $\varphi \in \Phi$. This shows the subclaim.

Now, let $(M, a/c)$ be a κ-saturated model of (10) of cardinality $\leq \kappa$. Let $\vec{a} \in (M \restriction \boldsymbol{L})[\mathcal{X}]$ such that $a = \langle \vec{a} \rangle$ (see (7')) and $\vec{b} \in (N \restriction \boldsymbol{L})[\mathcal{X}]$ such that $b = \langle \vec{b} \rangle$. Then, for any $\boldsymbol{L}$-formula θ with $\mathrm{Var}(\theta) \subset \mathcal{X}$ such that $M \restriction \boldsymbol{L} \models \theta[\vec{a}]$, we have $\neg\theta \notin \Phi$, hence $\theta \in \Phi$, hence $N \restriction \boldsymbol{L} \models \theta[\vec{b}]$. This says that $(M \restriction \boldsymbol{L}, \vec{a}) \equiv_{\boldsymbol{L}} (N \restriction \boldsymbol{L}, \vec{b})$. By (4), $(M \restriction \boldsymbol{L}, \vec{a}) \sim_{\boldsymbol{L}} (N \restriction \boldsymbol{L}, \vec{b})$, and by the (8), the assumption of the proposition, $\langle \vec{b} \rangle \in N(\tau)$, that is, $N \models \underline{\tau}[\langle \vec{b} \rangle / x]$, that is, $(N, b/d) \models \tau(d)$ as promised.

The set-theoretic assumption used in the proof is redundant, by a general absoluteness theorem (arithmetic statements are absolute with respect to the constructible universe, in which the Generalized Continuum Hypothesis (GCH) holds; see [J]). On the other hand, one may use "special" models in place of saturated ones, and avoid the use of GCH; see [CK], [MR2].

(11)(a) Assume that S is a theory in multisorted logic, and $I : \boldsymbol{L} \to [S]$ is an interpretation of the DSV $\boldsymbol{L}$ in S. Suppose that the class $\mathrm{Mod}(S)$ of models of S is invariant under $\boldsymbol{L}$-equivalence in the sense that for any L_S-structures M and N, $M \in \mathrm{Mod}(S)$ and $M \restriction \boldsymbol{L} \sim_{\boldsymbol{L}} N \restriction \boldsymbol{L}$ imply that $N \in \mathrm{Mod}(S)$. Then S is $\boldsymbol{L}$-axiomatizable; that is, for a set Θ of $\boldsymbol{L}$-sentences, $\mathrm{Con}_{L_S}(\{I(\theta) : \theta \in \Theta\}) = \mathrm{Con}_{L_S}(\Sigma_S)$; here, $\mathrm{Con}_L(\Phi)$ is the set of L-sentences that are consequences of the theory (L, Φ).

Note that the conclusion can also be expressed by saying that for any L_S-structure M, $M \models \Sigma_S$ iff $M \restriction \boldsymbol{L} \models \Theta$.

(11)(b) More generally, assume, in addition to S and $I : \boldsymbol{L} \to [S]$, a theory T in a language extending that of S ($L_S \subset L_T$) such that

for any $M, N \in \mathrm{Mod}(T)$, $M \restriction L_S \in \mathrm{Mod}(S)$ and $M \restriction \boldsymbol{L} \sim_{\boldsymbol{L}} N \restriction \boldsymbol{L}$ imply that

$N\restriction \mathrm{L}_S \in \mathrm{Mod}(S)$.

Then, there is a set Θ of $\boldsymbol{L}$-sentences such that, for any $M\models T$, $M\models \Sigma_S$ iff $M\restriction \boldsymbol{L}\models\Theta$.

(11)(a) is the special case when $T=(\mathrm{L}_S, \emptyset)$.

Proof of (11)(b). For any $\tau\in\Sigma_S$, $M\models T$ and $N\models T$, we have

$$M\models\Sigma_S \;\&\; M\restriction \boldsymbol{L} \sim_{\boldsymbol{L}} N\restriction \boldsymbol{L} \implies N\models\tau .$$

By appropriately coding the condition $M\restriction \boldsymbol{L} \sim_{\boldsymbol{L}} N\restriction \boldsymbol{L}$ in first order logic with suitable additional primitives, and by applying compactness, we can find $\sigma[\tau]$, a finite conjunction of elements of Σ_S, such that for any $M\models T$ and $N\models T$,

$$M\models\sigma[\tau] \;\&\; M\restriction \boldsymbol{L} \sim_{\boldsymbol{L}} N\restriction \boldsymbol{L} \implies N\models\tau .$$

Then by (7)(a), applied to $\boldsymbol{C}=\boldsymbol{D}=[T]$, and $I=J:\boldsymbol{L}\xrightarrow{I}[S]\xrightarrow{\mathrm{incl}}[T]$, we can find $\theta[\tau]$, an $\boldsymbol{L}$-sentence, such that $T\models\sigma[\tau]\longrightarrow I(\theta[\tau])$, $T\models I(\theta[\tau])\longrightarrow\tau$. Clearly, $\Theta=\{\theta[\tau]:\ \tau\in\Sigma_S\}$ is then appropriate for the assertion.

We leave it to the reader to formulate a version of (11) with formulas in a given context $\mathcal{X}$ instead of sentences.

The following, which is a special case of (7)(b), says that a first-order property invariant under $\boldsymbol{L}$-equivalence is expressible in logic with dependent types over $\boldsymbol{L}$.

(12) Let $I:\boldsymbol{L}\to\boldsymbol{C}$ be as before. Assume that $\mathcal{X}$ is a finite $\boldsymbol{L}$-context, $\sigma\in\mathrm{S}(I[\mathcal{X}])$, and for all $M, N\models\boldsymbol{C}$ and $\vec{a}\in(M\restriction\boldsymbol{L})[\mathcal{X}]$, $\vec{b}\in(N\restriction\boldsymbol{L})[\mathcal{X}]$,

$$\langle\vec{a}\rangle\in M(\sigma) \;\;\&\;\; (M\restriction\boldsymbol{L},\vec{a}) \sim_{\boldsymbol{L}} (N\restriction\boldsymbol{L},\vec{b}) \;\;\Longrightarrow\;\; \langle\vec{b}\rangle\in N(\sigma) \;\;.$$

Then there is an $\boldsymbol{L}$-formula θ in logic with dependent sorts without equality with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that $\sigma=_{I[\mathcal{X}]} I[\mathcal{X}:\theta]$.

The notion of $\mathbf{L}$-equivalence as defined is relevant to FOLDS without equality. However, frequently we deal with FOLDS with restricted equality. As explained in §1, when M is an $\mathbf{L}$-structure, it can be considered as an $\mathbf{L}^{\mathrm{eq}}$-structure, with the additional relations E_K interpreted as true equality; let us write M for the resulting "standard" $\mathbf{L}^{\mathrm{eq}}$-structure as well. What does it mean to have an equivalence $(W, m, n): M \xleftarrow[\mathbf{L}^{\mathrm{eq}}]{} N$ for $\mathbf{L}$-structures M, N? Clearly, this is to say that $(W, m, n): M \xleftarrow[\mathbf{L}]{} N$ and, for any maximal kind K, and $\vec{c} \in W[K]$, $c_1, c_2 \in WK(m\vec{c})$, we have that $mc_1 = mc_2$ iff $nc_1 = nc_2$. Let us write $(W, m, n): M \xleftarrow[\mathbf{L}]{\approx} N$ for $(W, m, n): M \xleftarrow[\mathbf{L}^{\mathrm{eq}}]{} N$, and let us call such (W, m, n) an $\mathbf{L}, \approx$-equivalence; also, write $M \approx_{\mathbf{L}} N$ for $M \sim_{\mathbf{L}^{\mathrm{eq}}} N$; note that throughout, M and N are $\mathbf{L}$-structures.

Let us define $M \equiv_{\mathbf{L}^{=}} N$ as we did $M \equiv_{\mathbf{L}} N$ above, except that we refer to logic with equality. Then, using the translation $\varphi \mapsto \hat{\varphi}$ mentioned in §1, we obviously have $M \equiv_{\mathbf{L}^{=}} N \iff M \equiv_{\mathbf{L}^{\mathrm{eq}}} N$. Thus, by (2)(a) we have

(13) For $\mathbf{L}$-structures M and N, $M \approx_{\mathbf{L}} N \implies M \equiv_{\mathbf{L}^{=}} N$.

$\mathbf{L}, \approx$-equivalences can be "normalized" in a certain way, which will be useful for us later.

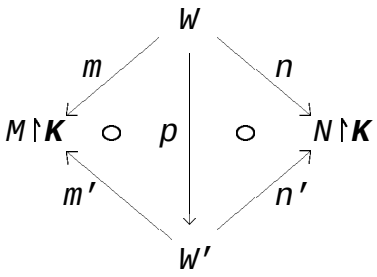
Let $U, V \in \mathrm{Set}^{\mathbf{K}}$. A very surjective morphism $f: U \to V$ is *normal* if for any maximal kind K, and any $\vec{a} \in U[K]$, "f is 1-1 in the fiber over $\vec{a}$", that is, if $b, c \in UK(\vec{a})$, then $f(b) = f(c)$ implies $b = c$. Together with the very surjective condition, this says that f induces a bijection $UK(\vec{a}) \xrightarrow{\cong} VK(f\vec{a})$.

Let M, N be $\mathbf{L}$-structures. A *normal* $\mathbf{L}, \approx$-equivalence $(W, m, n): M \xleftarrow[\mathbf{L}]{\approx} N$ is an $\mathbf{L}, \approx$-equivalence in which both m and n are normal. We have the fact

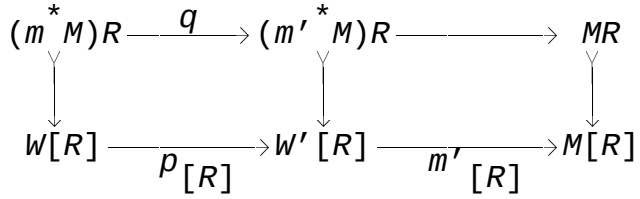
(14) For any $\mathbf{L}$-structures M, N, if $M \approx_{\mathbf{L}} N$, then there is a normal $\mathbf{L}, \approx$-equivalence

$(W, m, n) : M \overset{\approx}{\underset{\boldsymbol{L}}{\leftrightarrow}} N$.

The argument is as follows. Start with any $\boldsymbol{L}$,≈-equivalence $(W, m, n) : M \overset{\approx}{\underset{\boldsymbol{L}}{\leftrightarrow}} N$. Define $W' \in \mathsf{Set}^{\boldsymbol{K}}$ by setting $W'K = WK$ for all $K \in \boldsymbol{K}$ except the maximal ones; for a maximal K, $W'K \underset{\mathrm{def}}{=} WK/\sim$, where $\sim$ is the equivalence relation on WK for which $b \sim c$ iff b and c are over the same $\vec{a} \in W[K]$, and $m(b) = m(c)$. When in this definition, we replace m by n , the result is the same; this is because (W, m, n) being an $\boldsymbol{L}$,≈-equivalence, $m(b) = m(c)$ iff $n(b) = n(c)$ for b , c over the same element in $W[K]$. For an arrow $p : K \to \mathsf{K}_p$, $W'(p) = W(p)$ when K is not maximal (in which case K_p is not maximal either); and for K maximal, $(W'p)(b/\sim) = (Wp)(b)$; the latter is well-defined, since by the definition of $\sim$, if $b \sim c$, then $(Wp)(b) = (Wp)(c)$. Clearly, $W' : \boldsymbol{K} \to \mathsf{Set}$ is well-defined, and we have obvious maps $p : W \to W'$, $m' : W' \to M \restriction \boldsymbol{K}$, $n' : W' \to N \restriction \boldsymbol{K}$ such that



I claim that $(W', m', n') : M \overset{\approx}{\underset{\boldsymbol{L}}{\leftrightarrow}} N$; the normality condition is clearly satisfied. Consider a relation R in $\boldsymbol{L}$. In the commutative diagram



the outside rectangle and the right-hand square are pullbacks. It follows that the left-hand square is a pullback too. Obviously, $p_{[R]}$ is surjective. It follows that q is surjective. This determines the subobject $(m'^{*}M)R \rightarrowtail W'[R]$ as the image of $(m^{*}M)R \rightarrowtail W[R]$ under $p_{[R]}$. Switching to N from M , $(n'^{*}N)R \rightarrowtail W'[R]$ is the image of $(n^{*}N)R \rightarrowtail W[R]$

under $p_{[R]}$. Since $(m^*M)R =_{W[R]} (n^*N)R$, it follows that $(m'^*M)R =_{W'[R]} (n'^*N)R$ as desired. The additional condition concerning equality is clearly satisfied.

Notice that the above proof works for an essentially arbitrary $\mathbf{C}$ in place of $\mathtt{Set}$.

Note that if $m:W\to M\restriction \mathbf{K}$ is normal, then m^*M formed from M as a standard $\mathbf{L}^{\mathrm{eq}}$-structure is a standard $\mathbf{L}^{\mathrm{eq}}$-structure too. Put in another way, the standard fiberwise equality relations on the maximal kinds in m^*M are formed by the same pullback operation from the corresponding relation on M as any primitive $\mathbf{L}$-relation.

We have the following variant of (12).

(15) Let $\mathbf{C}$ be a small Boolean category, $I:\mathbf{L}\to\mathbf{C}$. Assume that $\mathcal{X}$ is a finite $\mathbf{L}$-context, $\sigma\in \mathtt{S}(I[\mathcal{X}])$, and for all $M, N\models\mathbf{C}$ and $\vec{a}\in(M\restriction\mathbf{L})[\mathcal{X}]$, $\vec{b}\in(N\restriction\mathbf{L})[\mathcal{X}]$,

$$\langle\vec{a}\rangle\in M(\sigma)\ \ \&\ \ (M\restriction\mathbf{L},\vec{a})\approx_{\mathbf{L}}(N\restriction\mathbf{L},\vec{b})\ \ \Longrightarrow\ \ \langle\vec{b}\rangle\in N(\sigma)\ \ .$$

Then there is an $\mathbf{L}$-formula θ in logic with dependent sorts *with* equality with $\mathtt{Var}(\theta)\subset\mathcal{X}$ such that $\sigma =_{I[\mathcal{X}]} I[\mathcal{X}:\theta]$.

Proof. By definition, for each maximal K, $I[\mathtt{E}_K]=I(K)\times_{I[K]}I(K)$. Let us form $I^{\mathrm{eq}}:\mathbf{L}^{\mathrm{eq}}\longrightarrow\mathbf{C}$ extending $I:\mathbf{L}\to\mathbf{C}$ by specifying that, $I^{\mathrm{eq}}(\mathtt{E}_K) = I[E_K]$, with $I^{\mathrm{eq}}(\mathtt{e}_{K0})=I^{\mathrm{eq}}(\mathtt{e}_{K1})=1_{I[\mathtt{E}_K]}$. We apply (12) to $I^{\mathrm{eq}}:\mathbf{L}^{\mathrm{eq}}\to\mathbf{C}$. For $M\models\mathbf{C}$, $M\restriction\mathbf{L}^{\mathrm{eq}}=M\circ I^{\mathrm{eq}}$ is, clearly, the same as $M\restriction\mathbf{L}$ as a standard $\mathbf{L}^{\mathrm{eq}}$-structure. Thus,

$$(M\restriction\mathbf{L}^{\mathrm{eq}},\vec{a})\ \sim_{\mathbf{L}^{\mathrm{eq}}}\ (N\restriction\mathbf{L}^{\mathrm{eq}},\vec{b})\ \ \Longleftrightarrow\ \ (M\restriction\mathbf{L},\vec{a})\approx_{\mathbf{L}}(N\restriction\mathbf{L},\vec{b})\ \ .$$

Thus, from the hypothesis of (15), that of (12) follows. By (12), we have some θ in FOLDS without equality over $\mathbf{L}^{\mathrm{eq}}$ such that $\sigma =_{I[\mathcal{X}]} I^{\mathrm{eq}}[\mathcal{X}:\theta]$; but clearly, for θ' in FOLDS with equality over $\mathbf{L}$ such that $\hat{\theta}' = \theta$, we have $I[\mathcal{X}:\theta'] = I^{\mathrm{eq}}[\mathcal{X}:\theta]$; thus $\sigma =_{I[\mathcal{X}]} I[\mathcal{X}:\theta']$ as required.