First Order Logic with Dependent Sorts,
with Applications to Category Theory

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Contents:

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Introduction

1. This work introduces First-Order Logic with Dependent Sorts (FOLDS). FOLDS is inspired by Martin-Löf's Theory of Dependent Types (TDT) [M-L]; in fact, FOLDS may be regarded a proper part of TDT, similarly to ordinary first-order logic being a proper part of higher-order logic. At the same time, FOLDS is of a much simpler nature than the theory of dependent types. First of all, the expressive power of FOLDS is no more than that of ordinary first-order logic; in fact, FOLDS may be regarded as a constrained form of Multi-Sorted First-Order Logic (MSFOL). Secondly, the syntax of FOLDS is quite simple, only slightly more complicated than that of MSFOL.

In general terms, the significance of FOLDS is analogous to that of ordinary first-order logic (FOL). On the one hand, FOL has a simple and powerful semantic metatheory; on the other hand, FOL is the basis of a multitude of specific foundational theories. Correspondingly, FOLDS has a simple semantic metatheory, not essentially more complicated than that for FOL. It is one of the aims of this work to develop the basic semantic theory of FOLDS. On the other hand, I make a start on showing that FOLDS is good, and better than FOL, for the purposes of formal systems dealing with sets, categories, and more general categorical concepts.

FOLDS is very simple; for the understanding of the motivation for, and the basic mechanics of, FOLDS there is no need for any prior knowledge of the, by now, extensive literature of dependent types. I find the idea of FOLDS so simple and natural (we will also see that FOLDS is *useful*, which is another issue) that I am thoroughly surprised by the apparent fact that, in the literature, it has not so far been singled out for study. (Nevertheless, there are important pointers to FOLDS in the literature that I will point out below.) Incidentally, I decided to use the word "sort", instead of "type", in "first-order logic with dependent sorts", to emphasize the closeness of FOLDS to MSFOL, and because of the strongly-felt connotation, in phrases like "type-theory", of the word "type", that implies the presence of a higher-order structure; you would not say "multi-typed first-order logic", would you?

J. Cartmell [C] introduced a syntax of variable types for the purposes of a novel presentation of generalized algebraic theories; Cartmell's syntax was also "abstracted from ... Martin-Löf type theory". FOLDS differs in two ways from Cartmell's syntax. Firstly, in Cartmell's syntax, there are no logical operators in the usual sense; there are no propositional connectives, or quantifiers; FOLDS has them, with quantification constrained in the natural way already given in TDT. Secondly, the type-structure of FOLDS is much simpler than that of Cartmell's syntax.

Cartmell's syntax may be characterized as the result of abstracting the structure of *contexts, types, terms* and *equality* out of TDT. FOLDS has the first two of these, contexts and types (although the latter are called "sorts"), but it does not have the third, terms (except in the rudimentary form of mere variables), and it has equality in a greatly restricted form only.

The restriction on the use of equality in FOLDS is a fundamental feature. FOLDS is to be used in formulating categorical situations in which, for example, equality of objects of a category is not an admissible primitive. The absence of term-forming operators, to be interpreted as functions, is a consequence of the absence of equality; it seems to me that the notion of "function" is incoherent without equality.

It is convenient to regard FOLDS a logic without equality entirely, and deal with equality, as much as is needed of it , as extralogical primitives.

It is worth-while for the reader at this point to make a quick comparison of the way [C] formulates the theory of categories (pp. 212, 213 in [C]), and the way FOLDS formulates the same (see §1, p.11). Let me emphasize that essentially this particular instance of FOLDS have been introduced early on by G. Blanc [B], in his characterization (mathematically equivalent to P. Freyd's earlier characterization) of first-order properties of categories invariant under equivalence of categories. A. Preller [P] makes the specification of the specific instance of FOLDS clearer. The theme of invariance under equivalence is in fact the main theme for this work; see below.

The FOLDS formulation of the theory of categories is, admittedly, longer than the Cartmell formulation. It consists in writing out the axioms of "category" in essentially the usual first-order terms, with a special regard for the typing of variables. The main points to observe are that (1) no equality on objects is used; (2) equality of arrows is used only when the arrows already are assumed to be parallel; and (3) quantification on arrows is restricted to one hom-set at a time.

The formulation in [C] is more "mathematical"; in particular, the essential algebraic nature of the concept of category is clear on it, whereas, because of the presence of the usual first-order operators that in general do not yield essentially algebraic concepts, in the FOLDS formulation the essential algebraic quality of the concept of category is obscured.

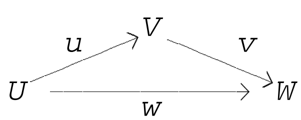
In the case of the theory of categories, the notions of *context* in the two formulations coincide; in fact, now a context is a finite diagram of objects and arrows represented by variables. Below, we will take a look at the formulations of the concept of a category with finite limits in the two frameworks, when the differences become greater.

The most obvious difference of the two formulations is that the one in FOLDS is *purely relational*, in Cartmell's syntax, purely operationalI. In FOLDS, the concepts of identity and composition are represented by relations, rather than operations as in [C]. The arity of a relation is the type of a particular *context*; the places of a relation are to be filled by variables forming a context of a given type. To give an example, in case of composition as a relation, the variables filling the places of the relation T (for (commutative) triangle) form a system consisting of variables U, V, W, u, v, w (not necessarily all distinct), related to each other by sorting data

$$U, V, W{:}O;\ \ u{:}\mathrm{A}(U, V),\ \ v\in\mathrm{A}(V, W),\ \ w\in\mathrm{A}(U, W)$$

(O for "object", A for "arrow"),

or more pictorially,



;

T then *says* of this diagram that it is commutative.

A general context in the FOLDS language for categories is a finite graph of object and arrow variables, with sorting data specifying which object variable is the domain of each arrow variable, and the same for codomain (when we say "graph", we mean to imply that there is no arrow-variable without a corresponding object-variable designated as its domain, or codomain).

An immediate consequence of the absence of operations in FOLDS is the simplification of the notions of *context* and *type* (*sort*) in FOLDS with respect to the Cartmell syntax. To see the effect of this, we take the example of the theory of categories with finite limits. Although this example is not discussed in [C], it is highly relevant to the subject of [C] as acknowledged by the title of section 6: "Essentially algebraic theories and categories with finite limits".

In the Cartmell syntax, pullbacks would be introduced by the following introductory rules:

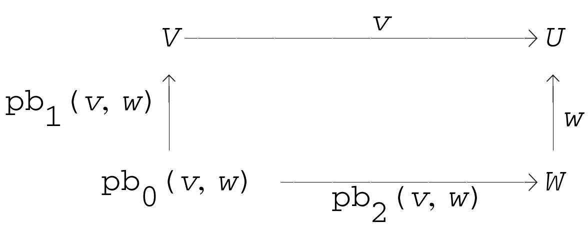
$U, V, W \in \mathrm{Ob}\ ,\ \ v \in \mathrm{Hom}(V, U)\ ,\ \ w \in \mathrm{Hom}(W, U)\ :\ \mathrm{pb}_0(U, V, W, v, w) \in \mathrm{Ob}$

(here, the "informal syntax" allows writing $\mathrm{pb}_0(v, w)$ in place of the longer term) ;

$U, V, W \in \mathrm{Ob}\ ,\ \ v \in \mathrm{Hom}(V, U)\ ,\ \ w \in \mathrm{Hom}(W, U)\ :$

$\mathrm{pb}_1(v, w) \in \mathrm{Hom}(\mathrm{pb}_0(v, w), V)\ ,\qquad \mathrm{pb}_2(v, w) \in \mathrm{Hom}(\mathrm{pb}_0(v, w), W)\ .$

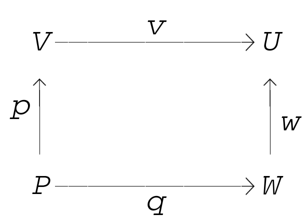
(Of course, one has in mind the pullback diagram



.)

There are further terms and rules expressing the universal property of the pullback.

Now, in FOLDS, we have two possibilities. One is simply adopting the same language of categories as before; after all, pullbacks are first-order definable in the language of categories; in fact, pullbacks are definable in FOLDS over the language of categories. Another possibility would be adopting an additional primitive relation of arity the diagram



;

we would do this if we wanted (as we may) to keep down the quantifier complexity of the axioms of the resulting theory. In either case, appropriate first-order axioms, formulated in FOLDS, are adopted.

Now, compare the notions of context and type (sort) in the Cartmell formulation, to those in the FOLDS formulations, in this example. In either of the FOLDS formulations, the notions of context and type remain the same as in the previous example of the theory of categories; in particular, contexts are finite graphs of variables. However, in the Cartmell formulation, because of the presence of terms of arbitrarily high complexity, both of the type of an object and of an arrow, contexts and types of arbitrarily high complexity will come up. In particular, the second rule above features a type with a place filled by a term which is not a variable.

This example explains the reason for the complexity of the definition the general concept of theory in Cartmell's syntax; see section 6, *loc.cit*. In particular, the definition of "type" cannot be made independent of the axioms of the theory in question; what counts as a well-formed type depends on what axioms are present. This is not at all unexpected; M. Coste's earlier syntax for essentially algebraic theories [Co] (not referred to in [C]) also had this feature. In contrast, in FOLDS, there is no such complication in the definition of "type" ("sort").

Let me point out another aspect in which FOLDS is simpler than Cartmell's syntax. In FOLDS, one never substitutes in a sort expression; in the formal system, there is a substitution rule, but it does not effect sorts. Related to this is the circumstance that the sorting of variables can be given rigidly; that is, when we say that the variable x is sort X, where the sort X may contain further variables, we mean a *formal*, once-for-all specification concerning x. In FOLDS, in contrast to Cartmell's syntax, it is impossible to have the same variable x to be declared of types X and Y unless X and Y are *literally* the same.

I consider the just-described feature of FOLDS to be of foundational importance. The view underlying FOLDS is that sort-declarations are not subject to logical manipulation; they are not propositions; one cannot negate a sort-declaration. One cannot ask *whether* x is of sort X *within* logic; the variable x being of sort X is purely notational, or conventional, matter. More pointedly, *membership in a set* is not a matter for logic; what *is* the matter for logic is whether certain elements, declared to belong to various sets, do or do not satisfy certain *predicates*. One should compare simple type theory (higher-order logic), in which typing of variables is also absolute. The difference in FOLDS is only that the type of a variable may also contain variables; however, the latter variables are uniquely determined from the variable being typed.

There is an important difference between the FOLDS-formulation and the Cartmell formulation, indicated above, of the notion of category with finite limits; in fact, the very notions formulated, not just their formulation, differ. Cartmell's syntax formalizes the notion of category with *specified* finite limits; FOLDS (in our application) formalizes the notion of category with finite limits, with the latter defined *only up to isomorphism*. Moreover, Cartmell's syntax *cannot* formalize the latter notion, for the simple reason that that notion is not an essentially algebraic one. Conversely, FOLDS, with the restriction that no equality on objects is allowed, cannot formalize the notion of category with *specified* finite limits.

It is possible to recapture the full expressive power, and more, of Cartmell's syntax within the framework of FOLDS. This will essentially be shown in Appendix C, when discussing "global equality". However, FOLDS with global equality captures more than Cartmell's syntax; because of this, it fails to represent that syntax faithfully. Thus, Cartmell's syntax is not rendered superfluous, or redundant, in any sense by what we do here. There is a similar situation with Coste's syntax for essentially algebraic theories mentioned above. Coste's syntax is one using the unique existential quantifier; it can be easily subsumed under *the simpler* regular logic which uses the ordinary existential quantifier. The point of Coste's syntax, and of Cartmell's, is that they capture *exactly* the essentially algebraic doctrine. In addition, I want to stress the great practical value of Cartmell's syntax. It is, in my opinion, the most practical specification language for structures such as (possibly) higher dimensional categories, with (possible) additional structure.

In this work, I present two ways of introducing FOLDS, which, however, are ultimately equivalent; one in §1, the other in Appendix A. The one in Appendix A is the more direct one. It starts with a simultaneous inductive definition of the concepts of kind, context, sort and variable, together with some other auxiliary concepts. Kinds are the heads (names) of sorts; each sort is obtained by *appropriately* filling out the places of a kind by variables. After defining the syntax in a global manner, one isolates specific *vocabularies*, or similarity types, for the purposes of formulating specific *theories* in FOLDS.

On the other hand, the treatment in §1 starts with the idea of a vocabulary for FOLDS (DSV). It is interesting that the data for a DSV can be naturally and succinctly captured by a, usually finite, *one-way category*. One-way categories were isolated by F. W. Lawvere in [L]; a category is one-way if its endomorphism monoids are trivial; in the skeletal case, this means that there are no non-trivial circuits of arrows. Subsequently, Lawvere observed that one-way categories are intimately related to the sketch-based syntax of [M1]. Their appearance in this paper is related to their role in [M1], although this fact is not worked out here. The DSV as a one-way category has objects the kinds and the relation-symbols; the latter are "top"-level objects in the category; the arrows between kinds represent the dependencies built into the syntax.

The formulation of FOLDS based on one-way categories is simpler than the "direct" approach. In fact, it can be put into a succinct algebraic form, in the form of certain hyperdoctrine-type structures. We will exploit this possibility for the presentation of the Gödel and the Kripke completeness theorems for FOLDS.

2. Let me indicate the foundational motivation behind this work.

P. Benaceraff, in a well-known paper [Ben] entitled "What numbers could not be", expressed a criticism of the set-theoretical reconstruction of mathematical concepts such as that of "natural number". Benaceraff's point is that any one set-theoretical definition of "natural number" gives rise to truths, such as "17 has exactly seventeen members", that become false under an alternative, but equally legitimate set-theoretical definition of "natural number" (his illustration compares the von Neumann definition ($0=\emptyset$, $n+1=n\cup\{n\}$) and the Zermelo definition ($0=\emptyset$, $n+1=\{n\}$). Thus, the set-theoretical reconstruction of mathematics is inevitably cluttered with irrelevant and arbitrary truths.

The way out of this requires a language of mathematics in which one talks about the system $(\mathbb{N}, 0, S)$ of the natural numbers in such a way *that any property of* $(\mathbb{N},0,S)$ *that can be expressed in the language is necessarily invariant under isomorphism* of structures of the form $(A;\ a\in A,\ f:A\to A)$. We quickly realize, as did Benaceraff, that in such a language, we cannot allow an equality predicate relating things belonging to various sets; we may contemplate equality $a=_A a'$ of elements a, a' of a *fixed*, but arbitrary, set A only. As a consequence, we cannot allow an equality predicate whose arguments are sets; for if A and B are sets, $A=B$ should imply that $\forall a\in A.\exists b\in B.a=b$, but the last use of the equality predicate is not restricted to elements of a fixed set!

Doing mathematics under such restrictions is not as absurd as it may sound first. In fact, considering sets to be objects of a category, with functions as arrows, and using the FOLDS language of category theory mentioned above, one may do, specifically in the Lawvere-Tierney theory of elementary toposes with a natural numbers object, a significantly large part of mathematics, without violating the said exclusions, and in fact, fully observing the above-italicized requirement.

One may contemplate a comprehensive language of abstract mathematics, with the property that in it, only "relevant", that is, *suitably invariant*, predicates can be expressed. In the case of properties of sets, "suitably invariant" means "invariant under isomorphism (bijection)". In the contemplated foundational framework, sets are singled out among arbitrary totalities by the quality of a set that an equality predicate on its elements as arguments is present as part of the "structure" of the set. The totality of all sets is not a set, since there is no equality predicate on sets as arguments.

But then, what kind of structure does the totality of all sets form? Answer: a *category*. The isomorphisms will be particular arrows. We quickly realize that, to do set-theory, we need more general arrows than isomorphisms. In category theory, equality of *parallel* arrows is fundamental; we stipulate that the arrows from a fixed object to another fixed object form a *set*. We find that there are other categories, such as that of groups and homomorphisms, which in many ways are similar to that of sets and functions. For instance, we do not want to have equality of groups as a primitive. Categories appear as *generalizations* of sets; every set is a category, a discrete category. There is, in general, no such thing as the "underlying *set* of the objects of a category", not because of size considerations, but rather because, in general, there is no equality predicate whose arguments are the objects of the category.

We find that the idea of an *isomorphism* of categories, let alone equality of categories, is incoherent; it is obvious that the notion of an isomorphism of two categories must involve reference to equality of objects in each of the categories. This entails that a totality of categories cannot be, in general, a category; in any category, the notion of isomorphism is well-defined. For totalities of categories, we must have a new type of structure, some kind of 2-dimensional category.

However, in our quest for the "perfectly invariant" language we quickly get into conflict with standard category theory. The trouble is that we must conclude that the notion of functor, surely a mainstay of the subject, is not acceptable. The problem with it is that it implicitly refers to equality of objects in the codomain category, in the requirement that its value at any given object in the domain category be *uniquely* determined. Is there a way out of this?

In an old paper ([Kel]), G. M. Kelly described a common situation one finds oneself when one wants to define a functor. It appears that all data are there to define the functor, still, it is not possible to canonically single out the value of the functor at an argument-object; one needs to make an arbitrary choice of a value, while it is also clear that it is immaterial what choice one makes. Frequently, the choice cannot be made without the Axiom of Choice. Kelly described in precise terms what the data are like *before* one makes the arbitrary choices. Relatively recently, without knowing about Kelly's paper, I also went through a similar consideration, and made a formal definition of the notion of anafunctor (a term suggested by D. Pavlovic), anticipated by Kelly some thirty years ago (he did not give a name to the concept). (Related ideas occurred to R. Paré some time ago.) I have found that one can live, quite well actually, with anafunctors, without converting them into functors by making non-canonical choices. There is a basic category theory that, in its main outline, does not deviate too much from the standard one, and which uses anafunctors in place of functors; this theory gets by to a large extent without the Axiom of Choice. The beginnings of anafunctor theory is presented in [M2].

Let me emphasize that the work in [M2] is done in a traditional set-theoretic framework. The "perfectly invariant" foundation is not yet available for use; the mathematical work in [M2] is intended to help formulate such a foundation.

I envisage a foundational set-up, a *universe* of abstract concepts, in which we have sets, functions, categories and *anafunctors* as specific distinct kinds of entities. It is clear that we cannot stop here. We will have *natural transformations* of anafunctors. But the totality of all categories, anafunctors and natural transformations of the latter will form a new kind of entity, an *anabicategory*. This differs from a bicategory in that each composition operation of 1-cells, one for each triple of objects (0-cells), instead of being a functor, is an anafunctor. [M2] treats the afore-mentioned concepts.

The concepts of anafunctor and anabicategory mentioned above are "non-radical" revisions of established notions of category theory. As Kelly explained in [Kel], using a global version of the Axiom of Choice, anafunctors can be "converted" into functors. Technically, this amounts to saying that, under an appropriate Axiom of Choice, every anafunctor is isomorphic to a functor (this makes sense since a functor is canonically an anafunctor; "anafunctor" is a generalization of "functor"). Thus, under the full force of the usual set-theoretic foundations, anafunctors are of no importance. (Let me mention in this context that the global Axiom of Choice we have in mind is in fact *meaningless* in the Invariant Foundation, since it talks about a function with values which are sets, the very idea of which is inexpressible because of the lack of equality on sets. In fact, Kelly already in *loc.cit.* considered the global type of choice involved here more suspect than ordinary choice.)

The universe of the Invariant Foundation is not clearly defined as yet. It should contain ana-n-categories for all natural n's; the totality of ana-n-categories, with their morphisms, etc., will form an ana-$n+1$-category. The task of formulating these concepts is closely related with the task of defining the general notion of "weak n-category", mentioned in [BD].

3. In the previous subsection, I gave an incomplete outline of the universe of the Invariant Foundation. The contribution of the present work is to the language of that foundation. The proposal is to use FOLDS as the basic language.

For any vocabulary $\mathcal{L}$ for FOLDS, taken (for convenience) completely without equality, I introduce the notion of $\mathcal{L}$-*equivalence* of $\mathcal{L}$-structures; this is the replacement for the notion of isomorphism for ordinary kinds of structure. An $\mathcal{L}$-structure M is at the same time an ordinary structure for an ordinary language $|\mathcal{L}|$; the properties of M expressible in FOLDS are particular ordinary first-order properties of M as an $|\mathcal{L}|$ -structure, but not vice versa. It turns out (General Invariance Theorem, GIT) that the first-order properties that are invariant under $\mathcal{L}$-equivalence are precisely the ones that are expressible in FOLDS over $\mathcal{L}$. This indicates that $\mathcal{L}$-equivalence is the right notion of "isomorphism" for structures for FOLDS.

As was mentioned above, anafunctors are a generalization of functors. But, upon closer look, we see that the requirements of the "logic of (generalized) equality" impose an additional condition on anafunctors. Whereas an anafunctor determines its value at a given argument up to isomorphism, meaning that any two possible values are isomorphic, in the case of a *saturated anafunctor*, the value is determined also *no more* than up to isomorphism, meaning that any object isomorphic to a possible value is also one. (The precise definition also relates to the given isomorphism between a possible value and a new object.) The requirement of saturation is an extension of the principle of substitutability of equal for equal, transferred to isomorphism from equality. Now, it turns out that every anafunctor, in particular every functor, has a canonically defined *saturation*, a parallel saturated anafunctor, to which it is isomorphic. The right notion of "functor" is "saturated anafunctor".

On the one hand, we have traditional types of categorical structures, examples which are (1) categories, (2) diagrams of categories, functors and natural transformations, and (3) bicategories, *etc.*. We have notions of equivalence for each of these kinds; e.g., the one for bicategories is usually called "biequivalence".

On the other hand, we have anaversions of each of the above kinds of structure. In particular, we have a canonical saturation of any structure of each of the above kinds; in case of the first (category), the saturation is identical to the original. Each kind of anastructure has a vocabulary $\mathcal{L}$ for FOLDS as its similarity type; as a result, we have the notion of $\mathcal{L}$-equivalence for these anastructures. The chief point of the work here is that the concept of equivalence for traditional structures of a given kind, and the concept of $\mathcal{L}$-equivalence for their saturations correspond to each other. E.g., two bicategories are biequivalent iff their saturations are $\boldsymbol{L}$-equivalent, where $\boldsymbol{L}$ is the FOLDS vocabulary for anabicategories.

The saturation $\mathcal{A}^{\#}$ of a categorical structure (e.g., bicategory) $\mathcal{A}$ is quite simply defined in terms of $\mathcal{A}$; in particular, the definition is a first-order interpretation. As a result, any first-order property, and in particular, any FOLDS property, of $\mathcal{A}^{\#}$ is also, by a direct translation, a first-order property of $\mathcal{A}$. Hence, it is meaningful to ask of a first-order property P of $\mathcal{A}$ whether it is expressible as a FOLDS property of $\mathcal{A}^{\#}$. We have the conclusion that this holds iff P is invariant under equivalence of the appropriate kind. E.g., a first-order property of a variable bicategory $\mathcal{A}$ is invariant under biequivalence iff it is expressible in FOLDS as a property of the saturation of $\mathcal{A}$. This theorem is a result of a combination of the relation of the two kinds of equivalence mentioned above, and an appropriate generalization of the GIT.

The last result for categories is due to P. Freyd [F], and G. Blanc [B]; Blanc's formulation is closer to the spirit of this work. A detailed proof is available in [FS]. The methods of the present work are entirely different from Freyd's. Restricted to the case of categories, the former give stronger results, although the additional strength that I cannot reproduce by Freyd's methods seems of minor importance. More important is the fact that Freyd's methods employ the axiom of choice, through the use of the skeleton of a category, and thus do not generalize to "constructive category theory". In Appendix E, I give a proof of the GIT for intuitionistic FOLDS. This gives rise to an intuitionistic version of the Freyd-Blanc characterization theorem for properties of categories invariant under equivalence. This does not seem to be accessible by the methods of [FS].

The main mathematical results of the present work are thus syntactic characterizations of formulas that are invariant under equivalence, in various senses of "equivalence". For the statement of these results, there is no need to understand the anaconcepts. In fact, for the case of bicategories, I organized the presentation in a way that does not refer to anabicategories explicitly, although, in this way, I missed the proof of the full strength of the main result. By contrast, in the case of diagrams of categories, functors and natural transformations, the anaconcepts are displayed.

From the foundational point of view, the results give confirmation to the idea that FOLDS employed in the context of anastructures is a suitable foundational language. I expect that the analysis started here will extend with similar results to higher dimensions. This is a concrete matter in the case of tricategories [GPS]; but I believe the case of general n-dimensional structures will soon be accessible too. I find it an interesting proposition, verified up to dimension 2 here, and conjectured to hold in all dimensions, that the appropriate notion of equivalence, "weak n-equivalence" in the terminology of [BD], has a form, namely $\mathcal{L}$-equivalence for the saturations of the structures involved, which is of a general "logical nature"; the original notion of "weak n-equivalence" looks *a priori* to be a rather involved idea.

4. Let me give an overview of the contents. I have organized the material into seven sections and five appendices, with the obvious implication as to what parts I felt to be the more important ones. §1 is the basic introduction to the syntax and semantics of FOLDS. The reader may immediately look at Appendix A, which contains the alternative, "more logical", introduction of FOLDS. §2 contains the formal systems for the classical, intuitionistic and coherent versions of FOLDS. §3 is a purely algebraic (categorical) study of "fibrations with quantification". I deal with hyperdoctrine-like structures; specifically, fibrations in which the base category has finite limits, but there is a distinguished class of arrows along which quantification is allowed. The applications to FOLDS is given in §4. I was surprised at the appropriateness of this simple idea for the purposes of FOLDS. The (Gödel, Kripke) completeness of the systems of §2 are thus seen to be a special case of something much more general.

§5 introduces the concept of $\mathcal{L}$-equivalence, the main new concept of the work, and proves, in a suitably general form, the General Invariance Theorem (GIT). Appendices B and C are elaborations on the theme of $\mathcal{L}$-equivalence. In Appendix C, I give, among others, proofs that follow the spirit of the treatment in [FS]. §§6 and 7 work out the conclusions concerning the three kinds of categorical structure we discussed above. In §6, the example of a single functor between two categories as a categorical structure is considered in some detail. In particular, fibrations are such structures. Appendix D contains some of calculations for §7.

Finally, Appendix E does two main things. One is the extension of the theory of $\mathcal{L}$-equivalence to intuitionistic logic and Kripke models. The other is ordinary Craig interpolation and Beth definability for FOLDS.

I would like to thank George Janelidze and Dusko Pavlovic for valuable conversations on the

subject of this work.

§1. Logic with dependent sorts

First, we describe the kinds of structure which the assertions of logic with dependent sorts are about.

It is well-known from categorical logic that the similarity types that are graphs (having sorts the objects, and unary sorted operation symbols only) are sufficient for all purposes. The simplest consideration here replaces a relation-symbol sorted as $R\subset A_1\times...\times A_n$ by a new sort R, and operations $R\xrightarrow{p_i}A_i$, $i=1, ..., n$. Our first move is to restrict attention to *one-way graphs*; in fact, more conveniently, to *one-way categories*.

The concept of one-way category is due to F. W. Lawvere [L]. In [M1], I reproduce Lawvere's observation to the effect that categories of finite sketches obtained by the repeated use of the second construction of [M1] starting from Set are exactly the ones of the form $\mathrm{Set}^{\boldsymbol{C}}$, with $\boldsymbol{C}$ a finite one-way category.

A one-way category is one in which all endomorphisms are trivial (identities). In a skeletal one-way category, for any objects A and B, it is not possible that there are *proper* (non-identity) arrows in both direction $A\to B$ and $B\to A$. As a consequence, there are no cycles (positive-length paths $A_0\to A_1\to ...\to A_n$ of proper arrows with $A_0=A_n$).

We are mainly interested in finite, skeletal, one-way categories. However, for certain purposes, we need to relax the finiteness condition.

A category $\boldsymbol{C}$ has *finite fan-out* (I owe this concept to Jim Otto) if for every object A, there are altogether finitely many arrows with domain A; the set $\bigsqcup\{\boldsymbol{C}(A, C) : C\in\mathrm{Ob}(\boldsymbol{C})\}$ is finite. A *simple* category is one which is one-way, skeletal, and has finite fan-out.

A simple category is reverse-well-founded; in other words, it satisfies the ascending chain condition: there are no infinite paths $A_0\to A_1\to ...\to A_n\to A_{n+1}\to ...$ ($n<\omega$) consisting of proper arrows. (Namely, any such would have to have the objects A_n pairwise distinct, by the above, and that would mean, *a fortiori*, infinitely many arrows out of A_0 .)

If $\boldsymbol{L}$ is a simple category, the set $\mathrm{Ob}(\boldsymbol{L})$ of objects is partitioned as in

$$\mathrm{Ob}(\boldsymbol{L}) \;=\; \dot{\bigcup_{i<\ell}} \boldsymbol{L}_i$$

into non-empty *levels* $\boldsymbol{L}_i$, for $i<\ell$, ℓ the *height* of $\boldsymbol{L}$, $\ell\leq\omega$, such that $\boldsymbol{L}_0$ consist of the objects A for which there is no proper arrow with domain A, and such that, for $i>0$, $\boldsymbol{L}_i$ consists of those objects A for which all proper arrows $A\to B$ have $B\in\boldsymbol{L}_{<i} = \bigcup_{j<i}\boldsymbol{L}_j$, and there is at least one arrow $A\to B$ with $B\in\boldsymbol{L}_{i-1}$. (If $A\in\mathrm{Ob}(\boldsymbol{L})$, and for all proper $f:A\to B$, $B\in\dot{\bigcup_{i<\omega}}\boldsymbol{L}_i$, then $A\in\dot{\bigcup_{i<\omega}}\boldsymbol{L}_i$; in fact $A\in\boldsymbol{L}_i$ for some i not greater than the maximum of the levels of the codomains of the finitely many proper arrows with domain A plus one. Therefore, if $A\in\mathrm{Ob}(\boldsymbol{L})-\dot{\bigcup_{i<\omega}}\boldsymbol{L}_i$, then there is a proper $A\to B$ with $B\in\mathrm{Ob}(\boldsymbol{L})-\dot{\bigcup_{i<\omega}}\boldsymbol{L}_i$, and thus there is an infinite proper path out of A.) All proper arrows go from a level to a lower level. Of course, the height of a finite simple category is finite.

A *maximal* object in a simple category is one which is not the codomain of a proper arrow. Every object of the maximal level (if any) is maximal, but not necessarily conversely.

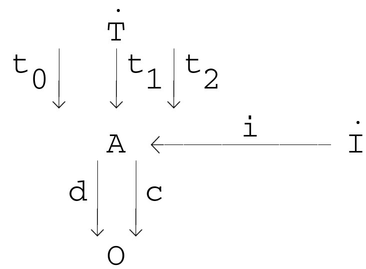
By a *vocabulary for logic with dependent sorts*, or *DS vocabulary*, or even DSV, we mean a simple category given with a distinguished, but otherwise arbitrary (possibly empty) set of maximal objects. The distinguished maximal objects of the DSV are its *relation symbols* (or *relations*); the rest of its objects are its *kinds*. We write $\mathrm{Rel}(\boldsymbol{L})$ and $\mathrm{Kind}(\boldsymbol{L})$ for the sets of relations and of kinds of $\boldsymbol{L}$, respectively.

DS vocabularies are our similarity types for structures for logic with dependent sorts; concomitantly, they figure as vocabularies for the syntax of logic with dependent sorts. Unlike in multisorted logic, the arrows of a DSV do not enter the syntax of FOLDS as operation-symbols; the role of the arrows in a DSV and their composition will serve to determine the "dependence structure" of the variables.

Here are some examples for DSV's .

$$\boldsymbol{L}_{\mathrm{graph}}: \qquad \begin{array}{c} A \\ d\downarrow\;\downarrow c \\ O \end{array}$$

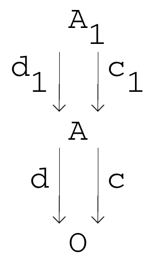
$\mathbf{L}_{\text{cat}}$:



$dt_1=ct_0$, $dt_2=ct_1$,
$dt_2=dt_0$, $di=ci$.

relations: I , T

$\mathbf{L}_{\text{2-graph}}$:



$dd_1=dc_1$, $cd_1=cc_1$.

Only non-identity arrows are shown. The proper arrows are those shown and their composites, among which we have the equalities shown, and no more. *E.g.*, there are three distinct arrows $T\to O$. $\mathbf{L}_{\text{graph}}$ and $\mathbf{L}_{\text{2-graph}}$ have no relations. The dots in $\mathbf{L}_{\text{cat}}$ signify that I and T are relations.

For a DSV $\mathbf{L}$, and an object A in it, we write $A|\mathbf{L}$ for the set of *proper* arrows with domain A (the notation resembles the notation $A{\downarrow}\mathbf{L}$ for the comma category). For an arrow p , K_p denotes its codomain.

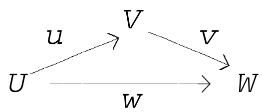
Given a DSV $\mathbf{L}$, the intended structures for $\mathbf{L}$, the $\mathbf{L}$-*structures*, are the functors $M:\mathbf{L}\to\mathrm{Set}$ in which for each relation $R\in\mathrm{Rel}(\mathbf{L})$ the following holds: the family $\langle M(p):M(R)\to M(K_p)\rangle_{p\in R|\mathbf{L}}$ of functions, indexed by the proper arrows in $\mathbf{L}$ with domain R, is jointly monomorphic: for $a, b\in M(R)$, if $M(p)(a)=M(p)(b)$ for all $p\in R|\mathbf{L}$, then $a=b$. The condition means that $M(R)$ is essentially a subset of the set $\prod_{p\in R|\mathbf{L}} M(K_p)$, actually a subset of $M[R]$; here, for any $A\in\mathrm{Ob}(\mathbf{L})$,

$$M[A] \stackrel{\text{def}}{=} \{\langle a_p\rangle_p\in \prod_{p\in A|\mathbf{L}} M(K_p) \ :\ M(q)(a_p)=a_{p'}, \text{ whenever } qp=p'\} \qquad (1)$$

($M[A]$ is the limit (joint pullback) of the diagram $A{\downarrow}(\mathbf{L}-\{1_A\})\xrightarrow{\Phi}\mathbf{L}\xrightarrow{M}\mathrm{Set}$ (with Φ the forgetful functor) mapping $(A\to K)$ to $M(K)$). We will usually (and without loss of generality) assume that in case $R\in\mathrm{Rel}(\mathbf{L})$, the canonical monomorphism

$m^M_R: M(R) \twoheadrightarrow M[R]$ taking a to $\langle (Mp)(a) \rangle_{p \in R|K_p}$ is an *inclusion* of sets.

We recognize that the L_{graph}-structures are the graphs, the $L_{\text{2-graph}}$-structures are the 2-graphs. Categories are particular L_{cat}-structures. If M is a category, for M as an L_{cat}-structure, $M(\mathrm{O})$, $M(\mathrm{A})$ are the sets of objects and of arrows, $M(\mathrm{d})$ and $M(\mathrm{c})$ are the domain and codomain functions; as a consequence, $M[\mathrm{T}]$ is the set of triangles



in M; by definition, $M(\mathrm{T})$ is the set of *commutative* triangles, a subset of $M[\mathrm{T}]$; $M(\mathrm{I})$ is the set of identity arrows. In fact, we realize that the L_{cat}-structures are exactly the *category-sketches* of [M1] .

For $\boldsymbol{L}$ a DSV , $|\boldsymbol{L}|$ denotes its underlying graph. Any (small) graph L can be used as a similarity type for multisorted logic; the L-structures are the graph-maps (diagrams) $L \to \mathrm{Set}$; $\boldsymbol{C}$-valued L-structures are the diagrams $L \to \boldsymbol{C}$. Multisorted first-order logic with L as vocabulary uses the objects of L as sorts and the arrows of L as sorted unary operation symbols; we always allow equality (to be interpreted in the *standard* way) when we refer to multisorted logic. For these matters, see [MR1]. First-order logic with dependent sorts over $\boldsymbol{L}$ will be a *proper part* of multisorted first order logic over $|\boldsymbol{L}|$.

To be sure, the $|\boldsymbol{L}|$ -structures are not exactly the $\boldsymbol{L}$-structures; the latter are those among the former that satisfy a certain set $\Sigma[\boldsymbol{L}]$ of axioms over $|\boldsymbol{L}|$, to be described next. $\Sigma[\boldsymbol{L}]$ consists of the following sentences:

$$\forall x \in A.(\bigwedge \{q(p(x))=p'(x) \ : p, p' \in A|\boldsymbol{L}, \ q \in \mathrm{Arr}(\boldsymbol{L}), qp=p'\} ,$$

one for each $A \in \mathrm{Ob}(\boldsymbol{L})$ ($= \mathrm{Kind}(\boldsymbol{L}) \cup \mathrm{Rel}(\boldsymbol{L})$) ; and

$$\forall x \in R. \forall y \in R.[(\bigwedge_{p \in R|\boldsymbol{L}} p(x)=p(y)) \longrightarrow x=y] ,$$

one for each $R \in \mathrm{Rel}(\boldsymbol{L})$.

One feature of logic with dependent sorts is that there will not be any operation symbols (explicitly) used in it; thus, the just-listed sentences are definitely not in logic with dependent sorts over $\mathbf{L}$.

Let us explain the intuition behind logic with dependent sorts for the case when the vocabulary is $\mathbf{L}_{\mathrm{cat}}$. First of all, logic with dependent sorts is a (proper) *part* of what we know as ordinary multisorted logic over $|\mathbf{L}_{\mathrm{cat}}|$, the (a) language of categories. In logic with dependent sorts over $\mathbf{L}_{\mathrm{cat}}$, we have variables ranging over O; we can quantify these variables. However, instead of variables ranging over A, we will have ones that range over $\mathrm{A}(U, V)$, where U and V are variables of sort O. $\mathrm{A}(U, V)$ is a "dependent sort", one depending on the variables U and V. A variable u ranging over $\mathrm{A}(U, V)$ is *of* sort $\mathrm{A}(U, V)$, and we write $u{:}\mathrm{A}(U, V)$. Of course, we should think of $\mathrm{A}(U, V)$ as $\mathrm{hom}(U, V)$, and of $u{:}\mathrm{A}(U, V)$ as $u{:}U{\to}V$. In terms of the semantics of $\mathrm{L}_{\mathrm{cat}}$-structures, the interpretation of $\mathrm{A}(U, V)$ in M is $\{a{\in}M\mathrm{A}{:}(M\mathrm{d})(a){=}(M\mathrm{c})(a)\}$. Thus, we have no variables ranging over all arrows at once; only ones ranging over arrows with a *fixed* domain and codomain.

An immediate consequence of this is that if a formula has the free variables U and V, and also $u{:}U{\to}V$ (that is, $u{:}\mathrm{A}(U, V)$), then forming $\forall U\varphi$ should and will be illegal; the free variable u in $\forall U\varphi$ has lost its fixed reference to a domain.

In FOLDS in general, and in particular over $\mathbf{L}_{\mathrm{cat}}$, we will have a restricted use of equality only. The reason for this is our main aim, which is to formulate languages for categorical structures in which all statements are invariant under the equivalence appropriate for the kind of categorical structure at hand. Typically, equivalences does not respect equality of certain kinds of entities; in the case of categories, equality of objects, in the case of bicategories, equality of objects (0-cells) and equality of 1-cells. In FOLDS with restricted equality, we will allow "fiberwise equality" over maximal kinds; in the case of $\mathbf{L}_{\mathrm{cat}}$, this means fiberwise equality over A. The restrictions on equality in FOLDS over $\mathbf{L}_{\mathrm{cat}}$ will correspond to the intuition that in category theory, one should not refer to equality of objects, and equality on arrows should be mentioned only with reference to arrows which have the same domain and the same codomain.

The above remarks, made for the case $\mathbf{L} = \mathbf{L}_{\mathrm{cat}}$, on how logic with dependent sorts over $\mathbf{L}$ is constrained with respect to ordinary first-order multi-sorted logic over $|\mathbf{L}|$ have natural extensions to the case of a general vocabulary $\mathbf{L}$. The constraints will be built into the general definition of the syntax.

Before giving the general definitions, to illustrate FOLDS (first-order logic with dependent sorts), we write down the axioms for category in this logic.

$$\forall U{:}\mathrm{O}.\exists i{:}U\to U.\mathrm{I}(i)\ ;$$
$$\forall U{:}\mathrm{O}.\forall i{:}U\to U.\forall j{:}U\to U.(\mathrm{I}(i)\wedge\mathrm{I}(j)\to i=j)\ ;$$
$$\forall U{:}\mathrm{O}.\forall V{:}\mathrm{O}.\forall W{:}\mathrm{O}.\forall u{:}U\to V.\forall v{:}V\to W.\exists w{:}U\to W.\mathrm{T}(u,\ v,\ w)\ ;$$
$$\forall U{:}\mathrm{O}.\forall V{:}\mathrm{O}.\forall W{:}\mathrm{O}.\forall u{:}U\to V.\forall v{:}V\to W.\forall w{:}U\to W.\forall w'{:}U\to W$$
$$(\mathrm{T}(u,\ v,\ w)\wedge\mathrm{T}(u,\ v,\ w')\longrightarrow w=w')\ ;$$
$$\forall U{:}\mathrm{O}.\forall i{:}U\to U.\forall u{:}U\to V.\mathrm{T}(i,\ u,\ u)\ ;$$
$$\forall U{:}\mathrm{O}.\forall i{:}U\to U.\forall u{:}V\to U.\mathrm{T}(u,\ i,\ u)\ ;$$
$$\forall U{:}\mathrm{O}.\forall V{:}\mathrm{O}.\forall W{:}\mathrm{O}.\forall X{:}\mathrm{O}.$$
$$\forall u{:}U\to V.\forall v{:}V\to W.\forall w{:}U\to W.\forall x{:}W\to X.\forall y{:}V\to X.\forall z{:}U\to X$$
$$((\mathrm{T}(u,\ v,\ w)\wedge\mathrm{T}(v,\ x,\ y)\wedge\mathrm{T}(w,\ x,\ z))\longrightarrow\mathrm{T}(u,\ y,\ z))\ .$$

We have applied certain abbreviations in writing these formulas. The atomic formula $\mathrm{I}(i)$ should be really $\mathrm{I}(U,\ i)$; U is also a variable in it; in fact, $i{:}U\to U$ cannot appear anywhere without U. Similarly, $\mathrm{T}(u,\ v,\ w)$ is really $\mathrm{T}(U,\ V,\ W,\ u,\ v,\ w)$. However, the abbreviations used are systematic, and can be made into a formal feature. Also, $w=w'$ is an atomic formula depending on all of the variables $U,\ W,\ w,\ w'$; it is written, more fully, as $w=_{\mathrm{A}(U,\ W)}w'$.

Many of the usual properties of categories, and of diagrams of objects and arrows in categories, can be expressed in FOLDS over $\mathbf{L}_{\mathrm{cat}}$. For instance, the definition of elementary topos (with operations defined by universal properties up to isomorphism, not specified as univalued operations) can be given as a finite set of sentences in FOLDS over $\mathbf{L}_{\mathrm{cat}}$; the reader will find it easy to write down the axioms for elementary topos in the style of the above axioms for category. As Freyd [F] and Blanc [B] have shown, and as we will see below, this is closely related to the fact that the usual properties of categories, and of diagrams in categories, are invariant under equivalence of categories.

Let us turn to the formal specification of the syntax of logic with dependent sorts. We fix a DSV $\boldsymbol{L}$. For a while, only the kinds in $\boldsymbol{L}$ will be used; let $\boldsymbol{K}$ be the full subcategory of $\boldsymbol{L}$ on the objects the kinds; $\boldsymbol{K}$ is a simple category, the *category of kinds* of $\boldsymbol{L}$; it may regarded as a DSV without relations.

Note that kinds have been assigned a level in $\mathbf{K}$; levels range over the natural numbers less than k, where k is the height of $\mathbf{K}$. Recall that for any $K\in\mathbf{K}$, we use the notation $K|\mathbf{K}$ for the set of all *proper* arrows $p:K\to K_p$ with domain K. The set $K|\mathbf{K}$ will figure as the *arity* of the symbol K. In particular, the ones with empty arity are exactly the level-0 kinds.

We are going to define what sorts are, and what variables of a given sort are. These notions are relative to a given $\mathbf{L}$ (actually, to the category $\mathbf{K}$ of kinds of $\mathbf{L}$), which is considered fixed now.

When X is anything, we write $x:X$ to mean that $x=\langle 2, X, a\rangle$ for some (any) a. When we have defined sorts, and X is a sort, $x:X$ is to be read as " x is a variable of sort X ".

By definition, a *sort* is an entity of the form

$$\langle 1, K, \langle x_p\rangle_{p\in K|\mathbf{K}}\rangle$$

such that K is a kind, and for each $p\in K|\mathbf{K}$, and for

$$X_p \overset{=}{\text{def}} \langle 1, K_p, \langle x_{qp}\rangle_{q\in K_p|\mathbf{K}}\rangle ,$$

we have $x_p:X_p$.

We will also write $K(\langle x_p\rangle_{p\in K|\mathbf{K}})$ for $\langle 1, K, \langle x_p\rangle_{p\in K|\mathbf{K}}\rangle$; thus, a sort is obtained by filling in the " pth " place of a kind K, for any p in the arity $K|\mathbf{K}$ of K, by a *suitable* variable x_p. The sort $K(\langle x_p\rangle_{p\in K|\mathbf{K}})$ is said to be *of the kind* K.

When X is a sort, and $x:X$, that is, $x=\langle 2, X, a\rangle$ for some a, x is called a *variable of sort* X; a is called the *parameter* of the variable x. Usually, the notation $x:X$ will imply that X is a sort.

Note that every variable "carries" its own sort with it. This is in contrast with the practice of most of the relevant literature (see e.g. [C]), where variables are "locally" declared to be of certain definite sorts, but by themselves, they do not carry sort-information. For a sort $X=K(\langle x_p\rangle_{p\in K|\mathbf{K}})$, $\mathrm{Var}(X) \overset{=}{\text{def}} \{x_p:p\in K|\mathbf{K}\}$; and if $x:X$, $\mathrm{Dep}(x) \overset{=}{\text{def}} \mathrm{Var}(X)$; x *depends on* the variables in $\mathrm{Dep}(x)$.

Note also that *any* parameter gives rise to a variable of a given sort; for any sort X, and for any a whatever, $\langle 2, X, a\rangle$ is a variable of sort X. In the "purely syntactic" contexts, it suffices to restrict the parameters to be natural numbers (thereby ensuring a countable infinite recursive set of variables of each sort). However, for the purposes of model-theory, it is convenient to have a proper class of variables of each sort (as a consequence, we have a proper class of sorts). Let us call a variable *natural* when its parameter, as well as that of each variable it depends on, *etc.*, is a natural number.

For a variable y, let's write X_y for the sort of y ($y:\mathrm{X}_y$), and let's use the notation

$$y:\mathrm{X}_y=\mathrm{K}_y(\langle \mathrm{x}_{y,\,p}\rangle_{p\in \mathrm{K}_y|\boldsymbol{L}}) \tag{1'}$$

displaying the ingredients of the sort X_y in dependence on y. Also, let's write $\mathrm{a}(y)$ for the parameter of y.

The first question arising concerning the definition of "sort" is whether the constituent entities X_p are also sorts; the answer is "yes". Assume $X=\langle 1, K, \langle x_p\rangle_{p\in K|\boldsymbol{K}}\rangle$ is a sort. Applying the definition of "sort" to X_p, for $q\in \mathrm{K}_p|\boldsymbol{K}$, we want that for

$$(X_p)_q=\langle 1, \mathrm{K}_q, \langle x_{(rq)p}\rangle_{r\in \mathrm{K}_q|\boldsymbol{K}}\rangle \ ,$$

we have $x_{qp}:(X_p)_q$. But since $\mathrm{K}_q=\mathrm{K}_{qp}$ and $(rq)p=r(qp)$, we have $(X_p)_q=X_{qp}$; and $x_{qp}:X_{qp}$, by X being a sort.

Although the definition unambiguously defines what sorts and variables are, it is not (quite) clear, for instance, that for every $K\in\boldsymbol{K}$, *there are* sorts of the kind K. We show that the sorts of the kind K are in a bijective correspondence with families $\langle a_p\rangle_{p\in K|\boldsymbol{K}}$ of arbitrary entities a_p; the correspondence maps $X=\langle 1, K, \langle x_p\rangle_{p\in K|\boldsymbol{K}}\rangle$ to $\mathrm{a}(X)\overset{=}{\mathrm{def}}\langle \mathrm{a}(x_p)\rangle_{p\in K|\boldsymbol{K}}$ for which $x_p=\langle 2, X_p, a_p\rangle$ for a suitable X_p.

We want to prove that for any $\langle a_p\rangle_{p\in K|\boldsymbol{K}}$, there is a unique sort X of the kind K with $\mathrm{a}(X)=\langle a_p\rangle_{p\in K|\boldsymbol{K}}$.

Let $K\in\boldsymbol{K}$ and $\langle a_p\rangle_{p\in K|\boldsymbol{K}}$ be given. By recursion on the level of K_p, for each $p\in K|\boldsymbol{K}$, we define X_p (a sort, as it turns out), and the variable $x_p:X_p$. Let $p\in K|\boldsymbol{K}$. We put

$$X_p \overset{=}{\mathrm{def}} \langle 1, \mathrm{K}_p, \langle x_{qp}\rangle_{q\in \mathrm{K}_p|\boldsymbol{K}}\rangle \text{ , and } x_p \overset{=}{\mathrm{def}} \langle 2, X_p, a_p\rangle \text{ .}$$

Since for each $q\in \mathrm{K}_p|\boldsymbol{K}$, K_{qp} is of lower level than K_p, the entity x_{qp} has been defined; thus, X_p and x_p are defined for p as well. This defines X_p and x_p for all p.

Put $X \overset{=}{\mathrm{def}} \langle 1, K, \langle x_p\rangle_{p\in K|\boldsymbol{K}}\rangle$. Then X_p formed for X as in the definition of "sort" is the same as the X_p we just defined. Since $x_p : X_p$, X is a sort. Clearly, $\mathrm{a}(X) = \langle a_p\rangle_{p\in K|\boldsymbol{K}}$.

The uniqueness of X with this property is (also) easily seen.

Let us remark that for kinds K of level 0, there is exactly one sort of the kind K, namely $K(\emptyset)$; this can safely be identified with K itself.

Let us consider the case $\boldsymbol{K} = \mathrm{L}_{\mathrm{graph}}$. We have the level-$0$ sort O ; let us use the letters U, V, W, ... for denoting variables of sort O ; $U{:}\mathrm{O}$, *etc*. The level-1 sorts are of the form $\mathrm{A}(\langle x_p\rangle)_{p\in\{\mathrm{d},\mathrm{c}\}}$ with $x_{\mathrm{d}}, x_{\mathrm{c}}{:}\mathrm{O}$, for which we write $\mathrm{A}(x_{\mathrm{d}}, x_{\mathrm{c}})$. Thus, we have sorts $\mathrm{A}(U, V)$, $\mathrm{A}(V, U)$, $\mathrm{A}(U, U)$, ... Let us use u, v, for variables of level 1 ; we may have $u{:}\mathrm{A}(U, V)$, which we paraphrase as $u{:}U\to V$.

In the case of $\mathrm{L}_{\mathrm{graph}}$, the ones listed are all the sorts and variables.

For $\boldsymbol{K} = \mathrm{L}_{2\text{-}\mathrm{graph}}$, we have the additional sorts of the kind A_1. Let us write d_{10} for the arrow $\mathrm{dd}_1 = \mathrm{dc}_1$, and c_{10} for $\mathrm{cd}_1 = \mathrm{cc}_1$. $\mathrm{A}_1|\boldsymbol{K} = \{\mathrm{d}_{10}, \mathrm{c}_{10}, \mathrm{d}_1, \mathrm{c}_1\}$. Writing $\mathrm{A}_1(x_{\mathrm{d}_{10}}, x_{\mathrm{c}_{10}}, x_{\mathrm{d}_1}, x_{\mathrm{c}_1})$ for $\mathrm{A}(\langle x_p\rangle_{p\in\{\mathrm{d}_{10},\mathrm{c}_{10},\mathrm{d}_1,\mathrm{c}_1\}})$, we see that the sorts of level-2 are those

$$\mathrm{A}(U, V, u, v)$$

for which $U{:}\mathrm{O}$, $V{:}\mathrm{O}$, $u{:}U\to V$ and $v{:}U\to V$. Here, U and V, as well as u and v, may coincide. We would like to paraphrase $\mathrm{A}(U, V, u, v)$ as

$$\mathrm{A}(U \underset{v}{\overset{u}{\rightrightarrows}} V) \text{ .}$$

Before we complete the definition of the syntax of logic with dependent sorts, let us discuss the semantics of sorts. To begin with an example, let us take $\boldsymbol{K} = \mathrm{L}_{\mathtt{2-graph}}$. Now, we know that the intended $\boldsymbol{K}$-structures are the functors $\boldsymbol{K}\to\mathtt{Set}$. But we may take a different view. We may say that a $\boldsymbol{K}$-structure M consists of

a set $M\mathrm{O}$,

for each $A, B\in M(\mathrm{O})$, a set $M\mathrm{A}(A, B)$,

and

for each $A, B\in M(\mathrm{O})$ and $f, g\in M\mathrm{A}(A, B)$, a set $M\mathrm{A}_1(A, B; f, g)$.

This way of thinking of a $\boldsymbol{K}$-structure emphasizes that an "arrow" f cannot be conceived of *before* its "domain and codomain" A, B, which have to be elements of $M\mathrm{O}$, have been given; there is a similar consideration for "2-cells". Also note that this kind of $\boldsymbol{K}$-structure is not literally the same as a functor $\boldsymbol{K}\to\mathtt{Set}$. The main difference is that, in the new version of the concept, we are not saying anything about the sets $M\mathrm{A}(A, B)$ being disjoint from each other for distinct pairs (A, B). Recall the two different styles of definition of "category" (or "2-category"). The one in which arrows determine their domain and codomain is in the spirit of our notion of structure in the original sense; the other in which we talk about a function $A, B\mapsto\mathrm{hom}(A, B)$ assigning a hom-set to pairs of objects is related to the new concept.

The second version of the concept of $\boldsymbol{K}$-structure has the following general form. A $\boldsymbol{K}$-structure M is given by specifying when, for $K\in\boldsymbol{K}$, the entities $MK(\langle a_p\rangle_{p\in K|\boldsymbol{K}})$ *are defined*, and when they are, what *sets* they are; such data are subject to the following condition:

(2) $MK(\langle a_p\rangle_{p\in K|\boldsymbol{K}})$ is defined iff for each $p\in K|\boldsymbol{K}$, $MK_p(\langle a_{qp}\rangle_{q\in K_p|\boldsymbol{K}})$ is defined and $a_p\in MK_p(\langle a_{qp}\rangle_{q\in K_p|\boldsymbol{K}})$.

This formulation hides the recursive character of the concept. Once it is clarified, for all K of level less than i, when $MK(\langle a_p\rangle_{p\in K|\boldsymbol{K}})$ is defined, and if so, what set it is, then for any K of level i, $MK(\langle a_p\rangle_{p\in K|\boldsymbol{K}})$ is defined iff for all $p\in K|\boldsymbol{K}$, $MK_p(\langle a_{qp}\rangle_{q\in K_p|\boldsymbol{K}})$ is defined, and $a_p\in MK_p(\langle a_{qp}\rangle_{q\in K_p|\boldsymbol{K}})$ (note that each K_p is of level $<i$), and *in that case*, $MK(\langle a_p\rangle_{p\in K|\boldsymbol{K}})$ is *any* set.

Any functor $M:\boldsymbol{K}\to\mathtt{Set}$ gives rise to a $\boldsymbol{K}$-structure in the new sense. For any $K\in\boldsymbol{K}$, define $M[K]$ as in (1); declare that $MK(\langle a_p\rangle_{p\in K|\boldsymbol{K}})$ is defined iff $\langle a_p\rangle_{p\in K|\boldsymbol{K}}\in M[K]$, and in that case, put

$$MK(\langle a_p\rangle_{p\in K|\mathbf{K}}) \overset{\text{def}}{=} \{a\in MK: \bigwedge_{p\in K|\mathbf{K}} (Mp)(a)=a_p\} . \tag{3}$$

It is clear (using that M is a functor) that now, (2) is satisfied.

But conversely, "essentially all" $\mathbf{K}$-structures in the second sense are obtained as functors $\mathbf{K}\to\mathrm{Set}$. The passage from a structure in the new sense to one in the old sense is as follows. Given a $\mathbf{K}$-structure M in the new sense, for any $K\in\mathbf{K}$, $M(K)$ is defined as the disjoint union of all *defined* sets $MK(\langle a_p\rangle_{p\in K|\mathbf{K}})$, indexed by the tuples $\langle a_p\rangle_{p\in K|\mathbf{K}}$, and, for $p:K\to K_p$, $M(p)$ is given by $M(p)(\langle\langle a_p\rangle_{p\in K|\mathbf{K}}, a\rangle)=a_p$; this defines a functor $\mathbf{K}\to\mathrm{Set}$.

Making the statement that the two notions of $\mathbf{K}$-structure are "essentially equivalent" precise would require defining what we mean by an isomorphism of two $\mathbf{K}$-structures M and N in the new sense, and showing that the above two passages represent an equivalence of the category of functors $\mathbf{K}\to\mathrm{Set}$ with natural isomorphisms as arrows on the one hand, and the category of $\mathbf{K}$-structures in the new sense, with isomorphisms in the new sense between them as arrows on the other. We will not go through this exercise, and return to our original concept of " $\mathbf{K}$-structure" (" $\mathbf{L}$-structure"). However, the concept of an M-*sort* as a set of the form (3) will be used.

Let us now return to the full DS vocabulary $\mathbf{L}$, and define what $\mathbf{L}$-formulas in logic with dependent sorts are. We will have two versions: logic with dependent sorts *with* (*restricted*) *equality*, and logic with dependent sorts *without equality*. FOLDS *with unrestricted equality* also makes sense; however, it turns out to be essentially the same as full multisorted logic with equality over $|\mathbf{L}|$ (see Appendix C), hence, it is of no real interest.

Let us fix $\mathbf{L}$.

Atomic formulas are defined very similarly to sorts. An *atomic formula* in logic with dependent sorts *without equality* is an entity of the form

$$\langle 3, R, \langle x_p\rangle_{p\in R|\mathbf{L}}\rangle$$

such that R is a relation in $\boldsymbol{L}$, and for each $p \in R|\boldsymbol{L}$, and

$$X_p \overset{=}{\text{def}} \langle 1, \mathrm{K}_p, \langle x_{qp} \rangle_{q \in \mathrm{K}_p | \boldsymbol{K}} \rangle \ ,$$

we have $x_p : X_p$. Under these conditions, the X_p are sorts (just as with the definition of "sort"). We write $R(\langle x_p \rangle_{p \in R|\boldsymbol{L}})$ for $\langle 3, R, \langle x_p \rangle_{p \in R|\boldsymbol{L}} \rangle$.

In logic with (restricted) equality, we also have additional atomic formulas as follows. For any *maximal* kind K (maximal object of $\boldsymbol{K}$), sort $X = K(\langle x_p \rangle_{p \in K|\boldsymbol{K}})$, and variables x, y, both of sort X, we have that $\langle 4, X, x, y \rangle$, written as

$$x =_X y \ ,$$

is an atomic (*equality*) formula.

We define *formulas* φ and the set $\mathrm{Var}(\varphi)$ of the *free variables of* φ by a simultaneous induction. Any atomic formula is a formula; if $\varphi = R(\langle x_p \rangle_{p \in R|\boldsymbol{L}})$, $\mathrm{Var}(\varphi) = \{x_p : p \in R|\boldsymbol{L}\}$; if $\varphi :=: x =_X y$, $\mathrm{Var}(\varphi) = \mathrm{Var}(X) \cup \{x, y\}$.

The sentential connectives $\mathbf{t}$, $\mathbf{f}$, $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\longleftrightarrow$ can be applied in an unlimited manner; $\mathrm{Var}(\ )$ for the compound formulas formed using connectives is defined in the expected way; e.g., $\mathrm{Var}(\varphi \wedge \psi) = \mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi)$.

Suppose φ is a formula, x is a variable *such that there is no* $y \in \mathrm{Var}(\varphi)$ *with* $x \in \mathrm{Dep}(y)$. Then $\forall x\varphi$, $\exists x\varphi$ are (well-formed) formulas;

$$\mathrm{Var}(\forall x\varphi) \ \overset{=}{\text{def}} \ \mathrm{Var}(\exists x\varphi) \ \overset{=}{\text{def}} \ (\mathrm{Var}(\varphi) - \{x\}) \cup \mathrm{Dep}(x) \ .$$

All formulas are obtained as described. (Of course, we have some determinations such as $\forall x\varphi \ \overset{=}{\text{def}} \ \langle \forall, x, \varphi \rangle$, where $\forall = 7$ (?), *etc*.)

Let us make some remark on logic with (restricted) equality. Just as in ordinary first-order logic, the syntax of logic with equality is the same as that of logic with equality, with the equality-symbol understood as another relation symbol; it is only the semantics that makes the difference.

Formally, for each maximal kind K, add to $\boldsymbol{L}$ an additional relation $\dot{\mathrm{E}}_K$, with morphisms $\mathrm{E}_K \underset{e_{K1}}{\overset{e_{K0}}{\rightrightarrows}} K$ subject to $pe_{K0}=pe_{K1}$, for all $p\in K|\boldsymbol{L}$; let us denote by $\boldsymbol{L}^{\mathrm{eq}}$ the extension of $\boldsymbol{L}$ by these additions. The equality formula $x =_X y$ corresponds to $\mathrm{E}_K(\langle z_r\rangle_{r\in \mathrm{E}_K|\boldsymbol{L}})$ where $z_{e_{K0}}=x$, $z_{e_{K1}}=y$, $z_{pe_{K0}}=z_{pe_{K1}}=x_p$. Up to the exchange of these two formulas, for each maximal K, the syntax of FOLDS with (restricted) equality over $\boldsymbol{L}$, and the syntax of FOLDS without equality over $\boldsymbol{L}^{\mathrm{eq}}$ coincide.

A *context* is a finite set $\mathcal{Y}$ of variables such that if $y\in\mathcal{Y}$, then $\mathrm{Dep}(y)\subset\mathcal{Y}$. It is easy to see that for any formula φ, $\mathrm{Var}(\varphi)$ is a context.

We explain the semantics of logic with dependent sorts. Let M be any $\boldsymbol{L}$-structure. Let $\mathcal{Y}$ be a context. We define

$$M[\mathcal{Y}] \underset{\mathrm{def}}{=} \{\langle a_y\rangle_{y\in\mathcal{Y}} : a_y \in MK(\langle a_{x_{y,p}}\rangle_{p\in K_y|\boldsymbol{L}}) \text{ for all } y\in\mathcal{Y}\} \quad (4)$$

(recall the notations (1') and (3)).

By recursion on the complexity of the formula φ, we define $M[\mathcal{Y}:\varphi]$, *the interpretation of* φ *in* M *in the context* $\mathcal{Y}$, whenever $\mathcal{Y}$ is a context such that $\mathrm{Var}(\varphi)\subset\mathcal{Y}$; we will have that $M[\mathcal{Y}:\varphi]\subset M[\mathcal{Y}]$. For an atomic formula $R(\langle x_p\rangle_{p\in R|\boldsymbol{K}})$, we stipulate, for any $\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}]$,

$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:R(\langle x_p\rangle_{p\in R|\boldsymbol{K}})] \underset{\mathrm{def}}{\Longleftrightarrow} \langle a_{x_p}\rangle_{p\in R|\boldsymbol{K}}\in M(R)$$

(recall that $M(R)\subset M[R]$; clearly, $\langle a_{x_p}\rangle_{p\in R|\boldsymbol{K}}\in M[R]$ automatically).

In case of logic with equality,

$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:u=_X v] \underset{\mathrm{def}}{\Longleftrightarrow} a_u = a_v .$$

For the propositional connectives, the clauses are the expected ones; *e.g.*,

$$\langle a_y\rangle_{y\in\mathcal{Y}} \in M[\mathcal{Y}:\psi\wedge\theta] \quad \underset{\text{def}}{\Longleftrightarrow} \quad \langle a_y\rangle_{y\in\mathcal{Y}} \in M[\mathcal{Y}:\psi] \text{ and } \langle a_y\rangle_{y\in\mathcal{Y}} \in M[\mathcal{Y}:\theta] \ .$$

Let us consider $\forall x\psi$, and a context $\mathcal{Y}$ such that $\mathrm{Var}(\forall x\psi)\subset\mathcal{Y}$. Let $x{:}K(\langle x_p\rangle_{p\in K|\boldsymbol{L}})$. First, assume that $x\notin\mathcal{Y}$; this is the case in particular when

$$\mathcal{Y}=\mathrm{Var}(\forall x\psi)=(\mathrm{Var}(\varphi)-\{x\})\cup\mathrm{Dep}(x) \ .$$

Let $\mathcal{Y}'=\mathcal{Y}\dot{\cup}\{x\}$; $\mathcal{Y}'$ is a context. When $\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}]$ and $a\in MK(\langle a_{x_p}\rangle_{p\in K|\boldsymbol{L}})$, let $\langle a_y\rangle_{y\in\mathcal{Y}}(a/x)$ denote $\langle a'_y\rangle_{y\in\mathcal{Y}'}$ for which $a'_y=a_y$ for $y\in\mathcal{Y}$, and $a'_x=a$. We see that $\langle a_y\rangle_{y\in\mathcal{Y}}(a/x)\in M[\mathcal{Y}']$ as follows. Note that for $y\in\mathcal{Y}$, we have $x\notin\mathrm{Dep}(y)$ (since $x\in\mathrm{Dep}(y)$ would imply that $x\in\mathcal{Y}$); as a consequence, $a'_y\in MK_y(\langle a'_{x_{y,p}}\rangle_{p\in K_y|\boldsymbol{L}})$ is equivalent to $a_y\in MK_y(\langle a_{x_{y,p}}\rangle_{p\in K_y|\boldsymbol{L}})$ for $y\in\mathcal{Y}$; while for $y=x$, the same holds by the assumption on a. We define

$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\forall x\psi] \quad \underset{\text{def}}{\Longleftrightarrow}$$

$$\text{for all } a\in MK(\langle a_{x_p}\rangle_{p\in K|\boldsymbol{L}}), \text{ we have } \langle a_y\rangle_{y\in\mathcal{Y}}(a/x)\in M[\mathcal{Y}':\psi] \ ;$$

and

$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\exists x\psi] \quad \underset{\text{def}}{\Longleftrightarrow}$$

$$\text{there is } a\in MK(\langle a_{x_p}\rangle_{p\in K|\boldsymbol{L}}) \text{ such that } \langle a_y\rangle_{y\in\mathcal{Y}}(a/x)\in M[\mathcal{Y}':\psi] \ .$$

In the general case for $\mathcal{Y}\supset\mathrm{Var}(\forall x\psi)$, define

$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\forall x\psi] \iff \langle a_y\rangle_{y\in\hat{\mathcal{Y}}}\in M[\hat{\mathcal{Y}}:\forall x\psi] \ ,$$
$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\exists x\psi] \iff \langle a_y\rangle_{y\in\hat{\mathcal{Y}}}\in M[\hat{\mathcal{Y}}:\exists x\psi] \ ,$$

where $\hat{\mathcal{Y}}=\mathrm{Var}(\forall x\psi)=\mathrm{Var}(\exists x\psi)$. It is clear that when $x\notin\mathcal{Y}$, the second definition gives the same answer as the first one.

As usual, we also write $M \models \varphi[\langle a_y \rangle_{y\in\mathcal{Y}}]$ for $\langle a_y \rangle_{y\in\mathcal{Y}} \in M[\mathcal{Y}:\varphi]$.

This completes the definition of the standard, Set-valued semantics of FOLDS.

Let us note that when φ is a formula in logic with equality over $\boldsymbol{L}$, and $\hat{\varphi}$ is the corresponding formula in logic without equality over $\boldsymbol{L}^{\mathrm{eq}}$, obtained by exchanging the equality subformulas for E_K-formulas, and M is an $\boldsymbol{L}$-structure, then $M[\mathcal{Y}:\varphi] = M[\mathcal{Y}:\hat{\varphi}]$; in the latter instance, M denotes the *standard* $\boldsymbol{L}^{\mathrm{eq}}$-structure in which each E_K is interpreted as true equality. In short, the semantics of logic with equality over $\boldsymbol{L}$ coincides with the semantics of logic without equality over $\boldsymbol{L}^{\mathrm{eq}}$ when the latter is restricted to standard structures.

Let us formulate a simple translation of logic with dependent sorts into ordinary multisorted logic. This amounts to a mapping $\varphi \mapsto \varphi^*$ of $\boldsymbol{L}$-formulas φ of FOLDS to $|\boldsymbol{L}|$ -formulas of multisorted logic. Let us agree that every variable $x:X$, with X a sort of kind K , will be regarded, in multisorted logic over $\boldsymbol{L}$, a variable of sort K .
The mapping $\varphi \mapsto \varphi^*$ will be so defined that the free variables of φ^* are exactly the same as those of φ . Moreover, the essential property of the translation is that, for any $\boldsymbol{L}$-structure M ,

$$M \models \varphi[\langle a_y \rangle_{y\in\mathcal{Y}}] \iff M \models \varphi^*[\langle a_y \rangle_{y\in\mathcal{Y}}] \;;$$

here, in the second instance, we referred to the usual notion of truth for multisorted logic. The definition is this:

for an atomic formula $\varphi :=: R(\langle x_p \rangle_{p\in R|\boldsymbol{K}})$,

$$\varphi^* \overset{=}{\mathrm{def}} \dot{R}(\langle x_p \rangle_{p\in R|\boldsymbol{K}}) \overset{=}{\mathrm{def}} \exists y\in R. \bigwedge_{p\in R|\boldsymbol{L}} p(y) =_{K_p} x_p \;; \tag{5}$$

for an equality formula $\varphi :=: x =_X y$,

$$\varphi^* \overset{=}{\mathrm{def}} x =_K y$$

(here, X is a sort of the kind K) ;

$(\ )^*$ commutes with propositional connectives;

$$(\forall x\varphi)^* \;\overset{=}{\text{def}}\; \forall x(\bigwedge_{p\in K|\boldsymbol{L}} p(x)=_{\mathrm{K}_p} x_p \longrightarrow \varphi^*)\ ;$$

$$(\exists x\varphi)^* \;\overset{=}{\text{def}}\; \exists x(\bigwedge_{p\in K|\boldsymbol{L}} p(x)=_{\mathrm{K}_p} x_p \wedge \varphi^*)$$

(in the last two clauses, $x:K(\langle x_p\rangle_{p\in K|\boldsymbol{L}})$).

We have a straightforward extension of the semantics of logic with dependent sorts to interpretations in categories so that the standard semantics will appear as the special when the target category is Set . First of all notice that the notion of $\boldsymbol{C}$*-valued* $\boldsymbol{L}$*-structure* makes sense for *any* category $\boldsymbol{C}$; it is that of a functor $M:\boldsymbol{L}\to\boldsymbol{C}$ such that for any $R\in\mathrm{Rel}(\boldsymbol{L})$, the family $\langle M(p)\rangle_{p\in R|\boldsymbol{L}}$ of morphisms in $\boldsymbol{C}$ is jointly monomorphic. The use of the notation $M:\boldsymbol{L}\to\boldsymbol{C}$ will imply that M is a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure. From now on, let us assume, at least, that $\boldsymbol{C}$ has finite limits.

Let $M:\boldsymbol{L}\to\boldsymbol{C}$. For any object A of $\boldsymbol{L}$ (kind or relation), we define $M[A]$ as the limit (joint pullback) of the diagram $A\downarrow(\boldsymbol{L}-\{1_A\})\xrightarrow{\Phi}\boldsymbol{L}\xrightarrow{M}\boldsymbol{C}$, with Φ the forgetful functor; $M\circ\Phi$ maps $p:A\to\mathrm{K}_p$ to $M(\mathrm{K}_p)$. Let us write π_p , or π^M_p , for the limit projection $M[A]\to M(\mathrm{K}_p)$, and let $\pi_A=\pi^M_A:M(A)\to M[A]$ be the canonical arrow for which $\pi_p\circ\pi_A=M(p)$. When A is a relation R , then π_R is a monomorphism; we also write m^M_R for π^M_R . When A is a kind K , then $U[K]$ and $\pi^U_p:U(K)\to U[K]$ are defined for any $U:\boldsymbol{K}\to\boldsymbol{C}$ (formally, by using the above definition for $\boldsymbol{K}$ in place of $\boldsymbol{L}$); of course, when $U=M\restriction\boldsymbol{K}$, then $U[K]=M[K]$, $\pi^U_p=\pi^M_p$.

Continuing, let $\mathcal{Y}$ be a context; we will define $M[\mathcal{Y}]$. We construct a graph $\langle\mathcal{Y}\rangle$ and a diagram $\Phi_{\mathcal{Y}}:\langle\mathcal{Y}\rangle\longrightarrow\boldsymbol{L}$ as follows. The objects of $\langle\mathcal{Y}\rangle$ are the elements of $\mathcal{Y}$, $\mathrm{Ob}\langle\mathcal{Y}\rangle=\mathcal{Y}$. The arrows of $\langle\mathcal{Y}\rangle$ are $\langle y,z;p\rangle:y\to z$, one for each $p\in\mathrm{K}_y|\boldsymbol{L}$ such that $z=x_{y,p}$. $\Phi_{\mathcal{Y}}$ maps y to K_y , $\langle y,z;p\rangle:y\to z$ to $p:\mathrm{K}_y\to\mathrm{K}_p(=\mathrm{K}_z)$. $M[\mathcal{Y}]$ is defined as the limit of the composite $M\Phi_{\mathcal{Y}}:\langle\mathcal{Y}\rangle\longrightarrow\boldsymbol{C}$; let us denote the projections for this limit by $\pi_y=\pi^M_y=\pi^M_{\mathcal{Y},y}:M[\mathcal{Y}]\to M(\mathrm{K}_y)\quad(y\in\mathcal{Y})$.

We define $M[\mathcal{Y}:\varphi]$ as a certain subobject of $M[\mathcal{Y}]$, by recursion on the complexity of φ.

Let φ be the atomic formula $R(\langle x_p\rangle_{p\in R|\boldsymbol{L}})$, and let $\mathcal{Y}$ be a context such that $\mathrm{Var}(\varphi)=\{x_p : p\in R|\boldsymbol{L}\}\subset\mathcal{Y}$. Let $f:M[\mathcal{Y}]\to M[R]$ be the arrow, given by the universal property of the limit defining $M[R]$, for which $\pi_p\circ f=\pi_{x_p}$ ($p\in R|\boldsymbol{L}$). $M[\mathcal{Y}:\varphi]$ is defined by the pullback

$$\begin{array}{ccc}
M[\mathcal{Y}:\varphi] & \longrightarrow & M(R) \\
\mathrm{m}^M_{\mathcal{Y},\varphi}\big\downarrow & \square & \big\downarrow \mathrm{m}^M_R \\
M[\mathcal{Y}] & \xrightarrow[f]{} & M[R]
\end{array}$$

(that is, $M[\mathcal{Y}:\varphi]$ as a subobject of $M[\mathcal{Y}]$ is represented by the monomorphism $\mathrm{m}^M_{\mathcal{Y},\varphi}$).

For formulas built by a propositional connective from simpler formulas, the definition is the expected one. *E.g.*,

$$M[\mathcal{Y}:\varphi\to\psi] = M[\mathcal{Y}:\varphi]\longrightarrow M[\mathcal{Y}:\psi] ,$$

where on the right-hand-side, reference is made to the Heyting implication $\longrightarrow$ in the subobject lattice $\mathrm{S}(M[\mathcal{Y}])$; of course, $M[\mathcal{Y}:\varphi\to\psi]$ is defined if and only if the corresponding instance of Heyting implication is defined in $\mathrm{S}(M[\mathcal{Y}])$.

Let $x\notin\mathcal{Y}$, and $\mathrm{Var}(\forall x\varphi)\subset\mathcal{Y}$. We have $f:M[\mathcal{Y}\dot\cup\{x\}]\to M[\mathcal{Y}]$ for which $\pi_y\circ f=\pi'_y$ ($y\in\mathcal{Y}$; $\pi_y=\pi^M_{\mathcal{Y},y}$, $\pi'_y=\pi^M_{\mathcal{Y}\dot\cup\{x\},y}$). Let $\exists_f$, $\forall_f : \mathrm{S}(M[\mathcal{Y}\dot\cup\{x\}])\supset\!\!\to S(M[\mathcal{Y}])$ be the partial left and right adjoints to $f^*:M[\mathcal{Y}]\to M[\mathcal{Y}\dot\cup\{x\}]$, the latter defined by pulling back along f. We define

$$M[\mathcal{Y}:\exists x\varphi] = \exists_f(M[\mathcal{Y}\dot\cup\{x\}:\varphi]) ,$$

$$M[\mathcal{Y}:\forall x\varphi] = \forall_f(M[\mathcal{Y}\dot\cup\{x\}:\varphi]) .$$

For $M[\mathcal{Y}:\exists x\varphi]$ or $M[\mathcal{Y}:\forall x\varphi]$ to be defined, it is necessary and sufficient that the corresponding instance of $\exists_f$, respectively $\forall_f$ be defined.

For the coherent part of the language (atomic formulas, **t** , **f** , $\wedge$, $\vee$, $\exists$) to be interpretable in the category, it suffices that ***C*** is a coherent category (see e.g. [MR1]). For the interpretation of the full language, it is suffices to have that ***C*** is a Heyting category (see *e.g.* [MR2]).

§2. Formal systems

In this section, a vocabulary $\boldsymbol{L}$ for logic with dependent sorts is assumed fixed. Relations, formulas, *etc.*, are all from/over $\boldsymbol{L}$.

For a formula φ, $\mathtt{Var}^*(\varphi)$ is "the set of all variables in φ, free or bound". More precisely, $\mathtt{Var}^*(\varphi) = \mathtt{Var}(\varphi)$ for atomic φ; $\mathtt{Var}^*(\varphi\wedge\psi) = \mathtt{Var}^*(\varphi)\cup\mathtt{Var}^*(\psi)$, and similarly for the other connectives;

$$\mathtt{Var}^*(\forall x\varphi) = \mathtt{Var}^*(\exists x\varphi) = \{x\}\cup\mathtt{Dep}(x)\cup\mathtt{Var}^*(\varphi) .$$

Let $\mathcal{X}$, $\mathcal{Y}$ be contexts. A map $s:\mathcal{X}\to\mathcal{Y}$ is called a *specialization* if whenever $x\in\mathcal{X}$, $x:K(\langle x_p\rangle_{p\in K|\boldsymbol{L}})$, we have $X=K(\langle s(x_p)\rangle_{p\in K|\boldsymbol{L}})$ is a sort, and $s(x):X$. The identity map $\mathcal{X}\to\mathcal{X}$ is a specialization, the composite of specializations is a specialization. Moreover, if a specialization is a bijection, then its inverse is also a specialization, and the restriction of a specialization to a subset of its domain which is a context is also a specialization. A notation such as $s:\mathcal{X}\to\mathcal{Y}$ will refer to a specialization.

For a sort X, resp. a formula φ, and a specialization $s:\mathcal{X}\to\mathcal{Y}$ such that $\mathtt{Var}(X)\subset\mathcal{X}$, resp. $\mathtt{Var}(\varphi)\subset\mathcal{X}$, we define $X|s$, resp. $\varphi|s$, "the result of substituting $s(x)$ for all free occurrences of x in X, resp. in φ, simultaneously for all $x\in\mathcal{X}$".

If X is the sort $K(\langle x_p\rangle_{p\in K|\boldsymbol{L}})$, and if φ is the atomic formula $R(\langle x_p\rangle_{p\in R|\boldsymbol{L}})$, we put

$$X|s \underset{\text{def}}{=} K(\langle s(x_p)\rangle_{p\in K|\boldsymbol{L}}) , \quad \varphi|s \underset{\text{def}}{=} R(\langle s(x_p)\rangle_{p\in R|\boldsymbol{L}}) .$$

For the equality formula $\varphi :=: x=_X y$, $\varphi|s :=: s(x)=_{X|s}s(y)$. The property of s being a specialization ensures that $X|s$ is a sort, and $\varphi|s$ is a(n atomic) formula in both cases.

$$(\varphi\wedge\psi)|s \underset{\text{def}}{=} (\varphi|s)\wedge(\psi|s) ,$$

and similarly for the other connectives.

Suppose $\varphi = \forall x\psi$. Let us first assume that $\mathcal{X}=\mathrm{Var}(\forall x\varphi)$. Consider the sort X of x, $x:X$; let y be a variable of sort $X|s$ which is *new* in the sense that $y\notin\mathrm{Var}^*(\psi)\cup\mathcal{X}\cup\mathcal{Y}$. Define t to be the function $t:\mathcal{X}\cup\{x\}\longrightarrow\mathcal{Y}\cup\{y\}$ for which $t\restriction\mathcal{X} = s$, and $t(x) = y$ (note that $x\notin\mathcal{X}$). Notice that $\mathrm{Var}(X)\subset\mathcal{X}$, $\mathrm{Var}(X|s)\subset\mathcal{Y}$, thus $\mathcal{X}\cup\{x\}$ and $\mathcal{Y}\cup\{y\}$ are contexts, and t is a specialization. We put $(\forall x\psi)|s \underset{\mathrm{def}}{=} \forall y(\psi|t)$. For a general $s:\mathcal{X}\to\mathcal{Y}$, $(\forall x\psi)|s$ is defined as $(\forall x\psi)|s'$, with $s' = s\restriction\mathrm{Var}(\forall x\psi)$. We make a similar definition for $\exists$ in place of $\forall$.

Since in the above description, y was not uniquely determined by the conditions given, substitution is not quite well-defined. We may correct this by making a particular, but artificial, choice of y. A better procedure is to *identify* the formulas obtained by different choices of y; this we do by defining the equivalence relation on formulas of one being an *alphabetic variant* of the other. However, for defining "alphabetic variant" it is convenient to use substitution. As long as substitution is not "well-defined", what we have is a relation " $\varphi|s = \theta$ " of three variables φ, s, θ rather than an operation $(\varphi, s)\longmapsto\varphi|s$.

Let φ be a formula, x and u variables of the same sort (for which we write $x\simeq u$), and assume that for all $v\in\mathrm{Var}(\varphi)$, $x\notin\mathrm{Dep}(v)$ (that is, either $x\notin\mathrm{Var}(\varphi)$ or it is a "top" element in $\mathrm{Var}(\varphi)$). Then the mapping

$$s:\mathrm{Var}(\varphi)\cup\{x\}\to\mathrm{Var}(\varphi)\cup\{u\},$$

defined by $s(v)=v$ for $v\in\mathrm{Var}(\varphi)-\{x\}$ and $s(x)=u$, is a specialization. Under these conditions, we put $\varphi|(x\mapsto u) \underset{\mathrm{def}}{=} \varphi|s$ [more precisely, " $\varphi|(x\mapsto u) = \theta$ " iff " $\varphi|s = \theta$ "].

The relation $\varphi\sim\psi$, " φ is an alphabetic variant of ψ ", is defined as follows.

If φ is atomic, then $\varphi\sim\psi$ iff $\varphi = \psi$.

$\varphi_1\wedge\varphi_2 \sim \psi$ iff $\psi = \psi_1\wedge\psi_2$ for some ψ_i with $\varphi_i\sim\psi_i$ $(i=1, 2)$; and similarly for the other connectives.

$\forall x\varphi \sim \psi$ iff $\psi = \forall x'\varphi'$ for some $x'\simeq x$ and φ' such that, for some u for which $u\simeq x\simeq x'$ and $u\notin\mathrm{Var}^*(\varphi)\cup\mathrm{Var}^*(\varphi')$, we have that $\varphi|(x\mapsto u) \sim \varphi'|(x'\mapsto u)$. Similarly for $\exists$ in place of $\forall$. [More precisely, we should say, in place of $\varphi|(x\mapsto u) \sim \varphi'|(x'\mapsto u)$, that for some σ and τ such that " $\varphi|(x\mapsto u) = \sigma$ " and

" $\varphi|(x'\mapsto u) = \tau$ ", we have $\sigma\sim\tau$.]

One shows in a routine manner that $\sim$ is an equivalence relation, $\varphi\sim\psi$ implies that $\mathtt{Var}(\varphi) = \mathtt{Var}(\psi)$, and $\sim$ is compatible with substitution: if $\varphi\sim\psi$, " $\varphi|s = \varphi'$ ", and " $\psi|s = \psi'$ " imply that $\varphi'\sim\psi'$. In particular, substitution $(\ )|s$ is an operation on equivalence classes of $\sim$. Note that the logical operations are compatible with $\sim$; $\varphi\sim\psi$ implies that $\forall x\varphi\sim\forall x\psi$, etc. Also, the semantics of alphabetic variants are identical. Henceforth, we *identify* alphabetic variants. In other words, a formula is, strictly speaking, an equivalence class of the "alphabetic variant" relation $\sim$.

When $s:\mathcal{X}\to\mathcal{Y}$, $\mathtt{Var}(\varphi)\subset\mathcal{X}$, we have $\mathtt{Var}(\varphi|s)\subset\mathcal{Y}$. If, in addition, $t:\mathcal{Y}\to\mathcal{Z}$, then $(\varphi|s)|t = \varphi|(ts)$. Also, $\varphi|1_{\mathcal{X}} = \varphi$.

An *entailment* is an entity of the form $\varphi\underset{\mathcal{X}}{\Longrightarrow}\psi$, where φ , ψ are formulas, $\mathcal{X}$ is a context, and $\mathtt{Var}(\varphi)$, $\mathtt{Var}(\psi)\subset\mathcal{X}$. We formulate *rules of inference* involving entailments. Each rule is a relation $\mathcal{R}(\varepsilon_0, \ldots, \varepsilon_{n-1}; \varepsilon_n)$ between entailments $\varepsilon_0, \ldots, \varepsilon_{n-1}, \varepsilon_n$; $\varepsilon_0, \ldots, \varepsilon_{n-1}$ are the *premises*, ε_n is the *conclusion* of the respective instance of $\mathcal{R}$. We display instances of $\mathcal{R}$ in the form

$$\mathcal{R}\quad\frac{\varepsilon_0\quad\varepsilon_1\qquad\qquad\varepsilon_{n-1}}{\varepsilon_n}\ .$$

n may be 0 , in which case we have a rule with no premises, an *axiom schema*.

I. Structural rules:

$$(\mathtt{Taut})\qquad\frac{}{\varphi\underset{\mathcal{X}}{\Longrightarrow}\varphi}$$

$$(\mathtt{Cut})\qquad\frac{\varphi\underset{\mathcal{X}}{\Longrightarrow}\psi\qquad\psi\underset{\mathcal{X}}{\Longrightarrow}\sigma}{\varphi\underset{\mathcal{X}}{\Longrightarrow}\sigma}$$

(Subst)
$$\frac{\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi}{\varphi|s \underset{\mathcal{Y}}{\Longrightarrow} \psi|s} \qquad (s:\mathcal{X}\longrightarrow\mathcal{Y})$$

II. Rules for the connectives

(**t**)
$$\frac{}{\psi \underset{\mathcal{X}}{\Longrightarrow} \mathbf{t}}$$

(**f**)
$$\frac{}{\mathbf{f} \underset{\mathcal{X}}{\Longrightarrow} \psi}$$

($\wedge$)
$$\frac{\theta \underset{\mathcal{X}}{\Longrightarrow} \varphi \qquad \theta \underset{\mathcal{X}}{\Longrightarrow} \psi}{\theta \underset{\mathcal{X}}{\Longrightarrow} \varphi\wedge\psi}$$

($\vee$)
$$\frac{\varphi \underset{\mathcal{X}}{\Longrightarrow} \theta \qquad \psi \underset{\mathcal{X}}{\Longrightarrow} \theta}{\varphi\vee\psi \underset{\mathcal{X}}{\Longrightarrow} \theta}$$

($\rightarrow$)
$$\frac{\theta\wedge\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi}{\theta \underset{\mathcal{X}}{\Longrightarrow} \varphi\rightarrow\psi}$$

($\neg$)
$$\frac{}{\mathbf{t} \underset{\mathcal{X}}{\Longrightarrow} \theta\vee\neg\theta} \qquad (\neg\theta \text{ abbreviates } \theta\rightarrow\mathbf{f})$$

($\wedge\vee$)
$$\frac{}{(\varphi\vee\psi)\wedge\theta \underset{\mathcal{X}}{\Longrightarrow} (\varphi\wedge\theta)\vee(\psi\wedge\theta)}$$

III. Quantifier rules

($\forall$)
$$\frac{\theta \underset{\mathcal{X}\dot{\cup}\{x\}}{\Longrightarrow} \varphi}{\theta \underset{\mathcal{X}}{\Longrightarrow} \forall x\varphi} \qquad (x\notin\mathcal{X})$$

$$(\exists)\qquad \frac{\varphi \underset{\mathcal{X}\dot{\cup}\{x\}}{\Longrightarrow} \theta}{\exists x\varphi \underset{\mathcal{X}}{\Longrightarrow} \theta}\ \text{(double line)}\qquad (x\notin\mathcal{X})$$

$$(\wedge\exists)\qquad \frac{}{\theta\wedge\exists x\varphi \underset{\mathcal{X}}{\Longrightarrow} \exists x(\theta\wedge\varphi)}\qquad (x\notin\mathtt{Var}(\theta))$$

IV. Equality axioms

$$(\mathrm{E}_1)\qquad \frac{}{\mathbf{t} \underset{\mathcal{X}}{\Longrightarrow} x=_X x}$$

$$(\mathrm{E}_2)\qquad \frac{}{x=_X y \underset{\mathcal{X}}{\Longrightarrow} y=_X x}$$

$$(\mathrm{E}_3)\qquad \frac{}{x=_X y\ \wedge\ \varphi \underset{\mathcal{X}}{\Longrightarrow} \varphi|(x\mapsto y)}$$

In the rules, φ, ψ, θ and σ ranges over formulas, x over variables, $\mathcal{X}$ and $\mathcal{Y}$ over finite contexts. An implicit condition is that each entailment shown has to be well-formed. *E.g.*, in $(\mathbf{t})$ and $(\mathbf{f})$, $\mathtt{Var}(\sigma)\subset\mathcal{X}$. In $(\forall)$ and $(\exists)$, $\mathtt{Var}(\theta)\subset\mathcal{X}$; since $x\notin\mathcal{X}$ is explicitly assumed, it follows that $x\notin\mathtt{Var}(\theta)$. Note that, in the same rules, the condition for the well-formedness of $\forall x\varphi$, $\exists x\varphi$ is satisfied as a consequence of the other provisos. More precisely, if $\mathcal{X}$ is a context, $\mathtt{Var}(\varphi)\subset\mathcal{X}\dot{\cup}\{x\}$ (in particular $x\notin\mathcal{X}$), then $\forall x\varphi$, $\exists x\varphi$ are well-formed. Namely, for $y\in\mathtt{Var}(\varphi)$, if $y\neq x$, then $y\in\mathcal{X}$, hence $\mathtt{Dep}(y)\subset\mathcal{X}$, and thus $x\notin\mathtt{Dep}(y)$; and if $y=x$, then $x\notin\mathtt{Dep}(x)$ anyway.

For (E_3), note that since X is a sort of a maximal kind, $\varphi|(x\mapsto y)$ is well-defined.

The double-lined "rules" contain more than one rule. The double line indicates that inference can proceed in both directions. E.g., in $(\vee)$, three rules are contained: the one that infers the entailment below $=\!=$ from the two above $=\!=$, and the ones allowing to infer either of the two entailments above $=\!=$ from the one below $=\!=$.

We have coherent, classical and intuitionistic logic with dependent sorts, each with or without equality. *Coherent* logic involves the (*coherent*) operators $\mathbf{t}$, $\mathbf{f}$, $\wedge$, $\vee$, $\exists$; classical and intuitionistic logics involve the remaining two, $\to$ and $\forall$. Coherent logic without equality has the rules all those not mentioning $\to$ and $\forall$ in their names; intuitionistic logic also has the additional rules $(\to)$ and $(\forall)$ (and then $(\wedge\vee)$, $(\wedge\exists)$ become superfluous); classical logic has also the remaining rule $(\neg)$. The versions with equality also have the rules (E_1) , (E_2) and (E_3) .

A *coherent formula* is one built up by the coherent operators starting with the atomic formulas; an entailment $\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$ is *coherent* if both φ , ψ are coherent formulas. A *coherent theory in logic with dependent logic* is a pair $T=(\boldsymbol{L}, \Sigma)$ of a DS vocabulary $\boldsymbol{L}$ and a set Σ of coherent entailments over $\boldsymbol{L}$. $\mathrm{Cons}_{\mathrm{coh}}(T)$ is the least set of coherent $\boldsymbol{L}$-entailments that contains Σ as a subset, and is closed under the rules for coherent logic; we write $T\vdash \varepsilon$, or $T\vdash_{\mathrm{coh}} \varepsilon$, and say that ε is *deducible from* T *in coherent logic with* dependent sorts, for $\varepsilon\in\mathrm{Cons}_{\mathrm{coh}}(T)$. Again, we have the versions with or without equality.

A theory in intuitionistic logic , or in classical logic (with dependent sorts) is defined similarly, *mutatis mutandis*. Again, we have logic with or without equality. Aside the exclusion of equality in the logics without equality, all formulas are used, in contrast to coherent logic. We have the concept $T\vdash \varepsilon$ of deducibility for each of these logics with dependent sorts.

We have completeness theorems for the various logics (coherent, intuitionistic, classical) with dependent sorts. What these completeness theorems show is that logic with dependent sorts is "self-contained". The initial view of logic with dependent sorts is that it is a fragment of ordinary multi-sorted logic. The fact that truths in the fragment can be deduced by deductions using only formulas also in the fragment is a sign, indeed, a necessary sign, that the fragment deserves the designation "logic".

To formulate completeness, let us fix a semantic category $\boldsymbol{C}$ (in the first instance, $\boldsymbol{C} = \mathrm{Set}$). Let M be a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure. Let us write $M\vDash \varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$ for $M[\mathcal{X}:\varphi] \leq_{M[\mathcal{X}]} M[\mathcal{X}:\psi]$, and say that M *satisfies* the entailment $\varphi \underset{\mathcal{X}}{\Longrightarrow} \psi$. A *model* of a theory $T=(\boldsymbol{L}, \Sigma)$ is a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure that satisfies all entailments in Σ . For a theory T , and an entailment ε , let us write $T\vDash_{\boldsymbol{C}} \varepsilon$, and say that the entailment ε is a $\boldsymbol{C}$-*consequence* of T , to mean that all $\boldsymbol{C}$-valued models M of T satisfy ε . For a class $\mathcal{C}$ of categories, $T\vDash_{\mathcal{C}} \varepsilon$ means that $T\vDash_{\boldsymbol{C}} \varepsilon$ for all $\boldsymbol{C}\in\mathcal{C}$.

$\mathrm{Mod}_{\boldsymbol{C}}(T)$ is the category of all $\boldsymbol{C}$-valued models of T; it is a full subcategory of $\mathrm{Fun}(\boldsymbol{L},\boldsymbol{C})$. We write $\mathrm{Mod}(T)$ when $\boldsymbol{C} = \mathrm{Set}$.

The completeness theorem for coherent logic, as well as for classical logic, with dependent sorts, with or without equality, is expressed by the equivalence

$$T \vdash \varepsilon \iff T \models_{\mathrm{Set}} \varepsilon$$

(of course, the symbol $\vdash$ is to be taken in any one of the four distinct senses corresponding to the four logics listed; ε accordingly ranges over the entailments of the corresponding logic). The completeness theorem for intuitionistic logic with dependent sorts, with or without equality, is

$$T \vdash \varepsilon \iff T \models_{\mathrm{Kr}} \varepsilon \ ,$$

where Kr (for Kripke) denotes the class of categories of the form Set^{P}, with P any poset.

As usual (see *e.g.* [MR2]), the completeness theorem for intuitionistic logic with dependent sorts may be formulated in the style of Kripke's semantics.

We will prove of the completeness theorems in §4.

§3. Quantificational fibrations

The notation and terminology of [M3] is used. The particular kinds of fibrations introduced here do not appear in *loc.cit.*, but most of the needed ingredients do.

Let $\overset{\boldsymbol{E}}{\underset{\boldsymbol{B}}{\mathcal{C}\downarrow}} = \overset{\mathbf{E}_{\mathcal{C}}}{\underset{\mathbf{B}_{\mathcal{C}}}{\mathcal{C}\downarrow}}$ be a fibration; let $\mathcal{Q}$ be a class of arrows in $\boldsymbol{B}$. Assume:

$\boldsymbol{B}$ has a terminal object, and pullbacks ($\boldsymbol{B}$ is left exact) .

$\mathcal{Q}$ is closed under pullbacks: when

$$\begin{array}{ccc} A & \xrightarrow{\quad q \quad} & B \\ \uparrow & \Box & \uparrow \\ A' & \xrightarrow{\quad q' \quad} & B' \end{array} \qquad\qquad (1)$$

is a pullback, then $q\in\mathcal{Q}$ implies $q'\in\mathcal{Q}$.

Each fiber $\mathcal{C}^A$ ($A\in\boldsymbol{B}$) is a poset; in fact, it is a lattice (with top and bottom elements, denoted $\mathbf{t}_A$, $\mathbf{f}_A$; the meet and join operations are written as $\wedge_A$, $\vee_A$, or more simply as $\wedge$, $\vee$ if no confusion may arise).

For each $(q\colon A\to B)\in\mathcal{Q}$, $q^*\colon\mathcal{C}^B\longrightarrow\mathcal{C}^A$ has a left adjoint $\exists_q\colon\mathcal{C}^A\longrightarrow\mathcal{C}^B$, which satisfies the Beck-Chevalley condition with respect to *all* pullback squares (1), and which satisfies Frobenius reciprocity (see pp. 342 and 343 in [M3]).

(Note that a fibration with posetal fibers (the only ones we are interested in here) is the same as a functor

$$\boldsymbol{B}^{\mathrm{op}}\longrightarrow\mathtt{Poset} : A\xrightarrow{f}B \longmapsto \mathcal{C}^B\xrightarrow{f^*}\mathcal{C}^A$$

to the category $\mathtt{Poset}$ of posets and order-preserving maps.)

The data $\mathcal{C}$, $\mathcal{Q}$ as described make the pair $(\mathcal{C},\mathcal{Q})$ a $\wedge\vee\exists$-*fibration*. We may denote $(\mathcal{C},\mathcal{Q})$ by $\mathcal{C}$; we may write $\mathcal{Q}_{\mathcal{C}}$ for $\mathcal{Q}$. Dropping the references to $\mathbf{f}_A$ and $\vee_A$ results in the notion of $\wedge\exists$-*fibration*.

A *morphism* $M:\mathcal{C}\to\mathcal{D}$ *of* $\wedge\vee\exists$-*fibrations* is a morphism of fibrations (among others, $M=(M_1, M_2)$, $M_1:\mathbf{B}_{\mathcal{C}}\to\mathbf{B}_{\mathcal{D}}$, $M_2:\mathbf{E}_{\mathcal{C}}\to\mathbf{E}_{\mathcal{C}}$, $\begin{array}{ccc}\mathbf{E}_{\mathcal{C}} & \longrightarrow & \mathbf{E}_{\mathcal{D}}\\ \downarrow & \circ & \downarrow\\ \mathbf{B}_{\mathcal{C}} & \longrightarrow & \mathbf{B}_{\mathcal{C}}\end{array}$; in practice, we omit the subscripts 1 and 2, and write $M(A)$ for $M_1(A)$, *etc.*) that takes $\mathcal{Q}_{\mathcal{C}}$-arrows to $\mathcal{Q}_{\mathcal{D}}$-arrows, induces lattice homomorphisms on the fibers, and preserves all instances of each $\exists_q$ ($q\in\mathcal{Q}_{\mathcal{C}}$). M is *conservative with respect to* a pair (X, Y) of predicates over the same base-object A if $MX\leq_{MA}MY$ implies $X\leq_A Y$; M is conservative if it is *conservative* for all such (X, Y) .

The $\wedge\vee\exists$-fibrations and their morphisms form a category $\wedge\vee\exists$. In fact, we can make $\wedge\vee\exists$ into a 2-category, by making $\wedge\vee\exists(\mathcal{C},\mathcal{D})$ into a category; the latter is a full subcategory of $[\mathcal{C}, \mathcal{D}]$ (see p. 348 in [M3]). An arrow

$$\mathcal{C}\ \underset{N}{\overset{M}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow h}}}}\ \mathcal{D}$$

is a natural transformation $h:M_1\longrightarrow N_1$ satisfying $MP\leq_{h_A}NP$ for all $A\in\mathbf{B}_{\mathcal{C}}$, $P\in\mathcal{C}^A$ (for the notation $X\leq_f Y$, see p. 349 in [M3]; $X\leq_f Y \Longleftrightarrow X\leq f^* Y$).

For a category $\boldsymbol{C}$ with pullbacks, $\mathcal{P}(\boldsymbol{C})$, the *fibration of predicates of* $\boldsymbol{C}$, is the fibration $\mathcal{C}$ with base-category $\boldsymbol{C}$ for which $\mathcal{C}^A=\mathrm{S}(A)$, the $\wedge$-semi-lattice of subobjects of A, and for $f:A\to B$, $f^*:\mathrm{S}(B)\to\mathrm{S}(A)$ is the usual pullback-mapping. To say that $\mathcal{P}(\boldsymbol{C})$ is a $\wedge\vee\exists$-fibration, with $\mathcal{Q}$ the class of all arrows in $\boldsymbol{C}$, is the same as to say that $\boldsymbol{C}$ is a coherent category (see, *e.g.*, [MR2]).

Consider $\mathcal{P}(\mathrm{Set})$ as a $\wedge\vee\exists$-fibration, with $\mathcal{Q}$ the class of all arrows in Set . A *model* of $\mathcal{C}$ is a morphism $\mathcal{C}\to\mathcal{P}(\mathrm{Set})$. $\mathrm{Mod}(\mathcal{C})$ is the *category of models* of $\mathcal{C}$; $\mathrm{Mod}(\mathcal{C})$ = $\wedge\vee\exists(\mathcal{C},\mathcal{P}(\mathrm{Set}))$. More generally, let us write $\mathrm{Mod}_{\mathcal{D}}(\mathcal{C})$ for $\wedge\vee\exists(\mathcal{C},\mathcal{D})$.

Until further notice, fix $\mathcal{C}$ = $(\mathcal{C}\overset{\boldsymbol{E}}{\underset{\boldsymbol{B}}{\downarrow}},\mathcal{Q})$, a small $\wedge\vee\exists$-fibration. Proposition (5) below is the *completeness theorem* for $\wedge\vee\exists$-fibrations, the fact that there are enough models of $\mathcal{C}$ to distinguish between any pair of different predicates in a fiber. The ones preceding (5) are used for the proof of (5).

Let us write $\mathbf{1}$ for $1_{\mathbf{B}}$, the terminal object of $\mathbf{B}$; and $\mathbf{t}$ for $\mathbf{t_1}$, $\mathbf{f}$ for $\mathbf{f_1}$. $\mathcal{C}$ has the *disjunction property* if for any $X, Y\in\mathcal{C}^{\mathbf{1}}$, if $X\vee Y=\mathbf{t}$, then either $X=\mathbf{t}$, or $Y=\mathbf{t}$. $\mathcal{C}$ has the *existence property* if whenever $(!_A:A\to\mathbf{1})\in\mathcal{Q}$ and $X\in\mathcal{C}^A$, we have that $\exists_{!_A}(X)=\mathbf{t}$ implies the existence of some $c:\mathbf{1}\to A$ such that $c^*(X)=\mathbf{t}$.

(1) Suppose $\mathcal{C}$ has the disjunction and the existence properties, and that $\mathbf{t}\neq\mathbf{f}$ (consistency). Then $\mathrm{Mod}(\mathcal{C})$ has an initial object ; in fact, $M=(M_1, M_2)$ given by $M_1=\hom_{\mathbf{B}}(\mathbf{1},-)$ and for $X\in\mathcal{C}^A$, $M_2(X)=\{c:\mathbf{1}\to A : c^*(X)=\mathbf{t}\}$ is an initial object.

(M may be called the *global-sections model* $\mathcal{C}\to\mathcal{P}(\mathrm{Set})$; we say $c:\mathbf{1}\to A$ *belongs to* X over A if $c^*(X)=\mathbf{t}$.)

The proof is identical to that of 2.2, p. 351 in [M3], although the statement of the latter does not include that of the present proposition.

For a fibration $\mathcal{C}$, $X\in\mathbf{B}$ and $X\in\mathcal{C}^A$, the "slice" fibration $\mathcal{C}/(A, X)$ was described in [M3]. The base-category of $\mathcal{C}/(A, X)$ is $\mathbf{B}/A$; the fiber over $(B\xrightarrow{f}A)\in\mathbf{B}/A$ is $\{Y\in\mathcal{C}^B: Y\leq_f X\}$, ordered as $\mathcal{C}^B$ is. We have a canonical morphism $\delta=\delta_{A,X}:\mathcal{C}\to\mathcal{C}/(A, X)$ that takes $B\in\mathbf{B}$ to $(B\times A\xrightarrow{\pi'}A)$, and $Y\in\mathcal{C}^B$ to $Y\dot{\wedge}X \underset{\mathrm{def}}{=} \pi^*Y\wedge\pi'^*X$ ($\leq_{\pi'} X$; $\pi:B\times A\longrightarrow B$ is the other projection).

For a $\wedge\vee\exists$-fibration $\mathcal{C}$, we define the $\wedge\vee\exists$-fibration $\mathcal{D}=\mathcal{C}/(A, X)$ by also putting

$$\left(\begin{array}{ccc} B & \xrightarrow{f} & C \\ & \searrow\ \circ\ \swarrow & \\ & A & \end{array}\right)\in\mathcal{Q}_{\mathcal{D}} \underset{\mathrm{def}}{\Longleftrightarrow} f\in\mathcal{Q}\,.$$

(2) $\mathcal{C}/(A, X)$ is a $\wedge\vee\exists$-fibration, and $\delta_{A, X}: \mathcal{C} \to \mathcal{C}/(A, X)$ is a map of $\wedge\vee\exists$-fibrations.

The proof is essentially contained in Section 2 of [M3]. It is helpful to add to 2.4(i) and (ii) of [M3] that the forgetful functor $\mathbf{B}/A \to \mathbf{B}$ creates pullbacks; with this, the required instances of the Beck-Chevalley and Frobenius reciprocity conditions become clear.

(3) If $(!_A : A \to \mathbf{1}) \in \mathcal{Q}$ and $X \in \mathcal{C}^A$ such that $\exists_{!_A}(X) = \mathbf{t}$, then $\delta_{A, X}$ is conservative. If $X_1 \vee X_2 =_{\mathbf{t}} \mathbf{t}$, and $Y, Z \in \mathcal{C}^B$, then either $\delta_{\mathbf{t}, X_1}$ or $\delta_{\mathbf{t}, X_2}$ is conservative with respect to (Y, Z).

See 2.7 in [M3].

By a straightforward transfinite iteration of the construction of $\mathcal{C}/(A, X)$ (compare 2.8 in [M3]), we conclude from (2) and (3) that

(4) For any given $A \in \mathbf{B}$, $X, Y \in \mathcal{C}^A$, there are a $\wedge\vee\exists$-fibration $\mathcal{C}^*$ having the disjunction and existence properties, and a map $\mathcal{C} \to \mathcal{C}^*$ of $\wedge\vee\exists$-fibrations which is conservative with respect to (X, Y).

(5) For any given $A \in \mathbf{B}$, $X, Y \in \mathcal{C}^A$, there is $M: \mathcal{C} \to \mathcal{P}(\mathrm{Set})$, a map of $\wedge\vee\exists$-fibrations, which is conservative with respect to (X, Y).

Proof. In $\mathcal{C}/(A, X)$, with $\mathbf{1} = \mathbf{1}_{\mathcal{C}/(A, X)}$ and $\delta = \delta_{A, X}$, we have the global element

$$d_A : \mathbf{1} \longrightarrow \delta(A) \; : \; \begin{array}{ccc} A & \longrightarrow & A \times A \\ & {\scriptstyle 1_A} \searrow \;\; \swarrow {\scriptstyle \pi'} & \\ & A & \end{array}$$

that belongs to $\delta(X)$; moreover, d_A belongs to $\delta(Y)$ over A iff $X\leq Y$. Now, start with X, Y over A in $\mathcal{C}$ such that $X\nleq Y$; pass to $\mathcal{C}'=\mathcal{C}/(A, X)$; in $\mathcal{C}'$, $\mathbf{t}=d_A^*\delta(X) \nleq d_A^*\delta(Y)=Y'$. By (4), there is $\Phi:\mathcal{C}'\to\mathcal{C}^*$ which is conservative with respect to $(\mathbf{t}, Y')$ such that $\mathcal{C}^*$ has the disjunction and existence properties. By (1), we have the global-sections model $N:\mathcal{C}^*\to\mathcal{P}(\mathrm{Set})$. The global-sections model is automatically conservative with respect to any pair $(\mathbf{t}, Z)$ over $\mathbf{1}$ in its domain. We conclude that, for $P=N\circ\Phi : \mathcal{C}'\to\mathcal{P}(\mathrm{Set})$, P is conservative with respect to $(\mathbf{t}, Y')$, that is,

$$P(d_A^*\delta(X)) \nleq P(d_A^*\delta(Y)) \ .$$

It follows that

$$P(\delta(X)) \nleq P(\delta(Y)) \ .$$

For $M = P\circ\delta : \mathcal{C}\to\mathcal{P}(\mathrm{Set})$, this means that $M(X) \nleq M(Y)$.

A $\wedge\vee\to\exists\forall$-*fibration* is a $\wedge\vee\exists$-fibration $\mathcal{C}$ such that

every fiber $\mathcal{C}^A$ is a Heyting algebra, and for all $f:A\to B$, $f^*:\mathcal{C}^B\to\mathcal{C}^A$ is a homomorphism of Heyting algebra; and

for each $q\in\mathcal{Q}_{\mathcal{C}}$, q^* (also) has a right adjoint which satisfies the Beck-Chevalley condition with respect to all (relevant) pullback squares.

For a category $\mathbf{C}$ with pullbacks, to say that $\mathcal{P}(\mathbf{C})$ is a $\wedge\vee\to\exists\forall$-fibration, with $\mathcal{Q}$ the class of all arrows in $\mathbf{C}$, is the same as what we usually express by saying that $\mathbf{C}$ is a Heyting category (see [MR2]). Of course, Set is a Heyting category; more, for any (not necessarily small) category $\mathbf{A}$, $\mathrm{Set}^{\mathbf{A}}$ is a Heyting category. See e.g. [MR2]. The coherent structure in $\mathrm{Set}^{\mathbf{A}}$ (the $\wedge\vee\exists$-fibration structure in $\mathcal{P}(\mathrm{Set}^{\mathbf{A}})$), although not the full Heyting-structure, is "computed pointwise"; that is, the projections $\pi_A:\mathcal{P}(\mathrm{Set}^{\mathbf{A}})\to\mathcal{P}(\mathrm{Set})$ ($A\in\mathbf{A}$) are morphisms of $\wedge\vee\exists$-fibrations.

Given any small $\wedge\vee\exists$-fibration $\mathcal{C}\overset{\boldsymbol{E}}{\underset{\boldsymbol{B}}{\downarrow}}$, we may form $\mathcal{P}(\mathrm{Set}^{\mathrm{Mod}(\mathcal{C})})$, and we have the evaluation morphism

$$\begin{array}{llcl}
e_{\mathcal{C}} : & \mathcal{C} & \longrightarrow & \mathcal{P}(\mathrm{Set}^{\mathrm{Mod}(\mathcal{C})}) \\
& \underline{X} & \longmapsto & \underline{[M \longmapsto M(X)]} \\
& \downarrow & & \downarrow \\
& A & \longmapsto & [M \longmapsto M(A)]
\end{array}$$

of $\wedge\vee\exists$-fibrations.

(6) For a small $\wedge\vee\rightarrow\exists\forall$-fibration $\mathcal{C}$, $e_{\mathcal{C}}:\mathcal{C}\longrightarrow\mathcal{P}(\mathrm{Set}^{\mathrm{Mod}(\mathcal{C})})$ is a morphism of $\wedge\vee\rightarrow\exists\forall$-fibrations.

Proof. The proof is a variant of that of 5.1 in [M3]. The fact that e is conservative follows from (5). We need to show that $e_{\mathcal{C}}$ preserves Heyting implications in the fibers, and $\forall_f$'s; we limit ourselves to the second task. By using the way the $\forall_f$'s are computed in any $\mathcal{P}(\mathrm{Set}^{\boldsymbol{A}})$, our task is as follows.

Assume $M:\mathcal{C}\rightarrow\mathrm{Set}$, a morphism of $\wedge\vee\exists$-fibrations; $(f:A\rightarrow B)\in\mathcal{Q}$, $X\in\mathcal{C}^A$, $\forall_f X\in\mathcal{C}^B$ and $b\in M(B)-M(\forall_f X)$. We want the existence of $N\in\mathrm{Mod}(\mathcal{C})$, a homomorphism $h:M\rightarrow N$ and $a\in N(A)-N(X)$ such that $h_A(b) = (Nf)(a)$.

Let us use ordinary multisorted first-order logic to talk about models of $\mathcal{C}$ and homomorphisms between them. Consider the language $L=\mathrm{L}(\mathcal{C})$ whose sorts are the objects of $\boldsymbol{B}$, operation-symbols are the arrows of $\boldsymbol{B}$, and relation-symbols are all unary, and they correspond to the predicates $P\in\mathcal{C}^A$; P is sorted $P\subset A$. It is clear that every $M\in\mathrm{Mod}(\mathcal{C})$ may be regarded an L-structure; morphisms in $\mathrm{Mod}(\mathcal{C})$ are exactly the morphisms of L-structures. Moreover, there is a (coherent) theory $T=\mathrm{T}(\mathcal{C})$ over L such that $\mathrm{Mod}(\mathcal{C}) = \mathrm{Mod}(T)$.

For a given L-structure M, homomorphisms $h:M\rightarrow N$ with varying N are in a 1-1 correspondence with models of $\mathrm{Diag}^+(M)$, the positive diagram of M, which is a set of atomic sentences in the diagram language $L(|M|)$ in which an individual constant $\underline{a}$ of sort A has been added to L for each sort and $a\in M(A)$; the elements $\mathrm{Diag}^+(M)$ are those atomic sentences that are true in $(M,\underline{a})_{a\in|M|}$. We may also define $\mathrm{Diag}^+(M)$ as $\mathrm{D_b}(M)\cup\mathrm{D_p}(M)$, where $\mathrm{D_b}(M)$ contains all $f(\underline{a})=_B\underline{b}$ for which $f:A\to B$ in $\boldsymbol{B}$, $a\in M(A)$, $b\in M(B)$ and $(Mf)(a)=(b)$; and $\mathrm{D_p}(M)$ contains all $P(\underline{a})$ where $A\in\boldsymbol{B}$, $P\in\mathcal{C}^A$ and $a\in M(P)\subset M(A)$.

Returning to our task, let $\tilde{a}$ be a new individual constant of sort A; under the assumptions, we need the satisfiability of the set

$$T\cup\mathrm{D_b}(M)\cup\mathrm{D_p}(M)\cup\{\neg X(\tilde{a})\}\cup\{\underline{b}=_B f(\tilde{a})\}\ .$$

Assume this fails. By compactness, there are finite subsets $D\subset\mathrm{D_b}(M)$, $D'\subset\mathrm{D_p}(M)$ such that

$$T\cup D\cup D'\ \models\ \underline{b}=_B f(\tilde{a})\longrightarrow X(\tilde{a})\ .$$

Let $\langle c_i\rangle_{i<n}$ be distinct elements of M, $c_i\in M(C_i)$, each distinct from b, such that every $\underline{c}$ that occurs in $D\cup D'$ is one of the $\underline{c}_i$, or is $\underline{b}$. Let z_i be distinct variables, z_i of sort C_i; y a variable of sort B, x one of sort A, all distinct. Let us replace c_i by z_i, b by y; we obtain $\bar{D}$ from D, $\bar{D}'$ from D', and we get that

$$T\models\forall\langle z_i\rangle_{i<n}\forall y\forall x(\bigwedge\bar{D}\wedge\bigwedge\bar{D}'\wedge y=_B f(x)\longrightarrow X(x))\ . \tag{7}$$

Working inside the category $\boldsymbol{B}$ with finite limits, we can construct as an appropriate finite limit an object C together with morphisms $\pi_i:C\to C_i$, $\pi:C\to B$ such that for any L-structure N, $N\models(\bigwedge\bar{D})[\langle\dot{c}_i\rangle_{i<n}\dot{b}/\langle z_i\rangle_{i<n}y]$ iff there is $\dot{c}\in N(C)$ with $N(\pi_i)(\dot{c})=\dot{c}_i$, $N(\pi)(\dot{c})=\dot{b}$ (actually, $\dot{c}$ is then uniquely given). In particular, there is an element $c\in M(C)$ such that $M(\pi_i)(c)=c_i$, $M(\pi)(c)=b$. For any $\alpha\in\bar{D}'$, let α^* be the element of the fiber over C given as follows: if $\alpha:=:P(z_i)$, $\alpha^*\overset{=}{\mathrm{def}}\pi_i^*(P)$; if $\alpha:=:P(y)$, $\alpha^*\overset{=}{\mathrm{def}}\pi^*(P)$. Let $Q=\bigwedge\{\alpha^*:\alpha\in\bar{D}'\}\in\mathcal{C}^C$. Notice that $c\in M(Q)$. Consider the pullback-square

$$\begin{array}{ccc} A\times_B C & \xrightarrow{\rho} & A \\ g\downarrow & & \downarrow f \\ C & \xrightarrow[\pi]{} & B \end{array}$$

We claim that

$$g^*(Q) \le a^*(X) . \tag{8}$$

By (5), it suffices to check that this holds in each model $N\in\mathrm{Mod}(\mathcal{C})$. Assume $N\in\mathrm{Mod}(\mathcal{C})=\mathrm{Mod}(T)$, $\dot{d}\in N(g^*(Q))$, $\dot{c}=(Ng)\dot{d}$, $\dot{a}=(N\rho)\dot{d}$, $\dot{c}_i=(N\pi_i)\dot{c}$, $\dot{b}=(N\pi)\dot{c}$; we have $\dot{b}=(Nf)\dot{a}$, $N\models(\bigwedge\bar{D})[\langle\dot{c}_i\rangle_{i<n}\dot{b}/\langle z_i\rangle_{i<n}y]$ by the defining property of $(C, \langle\pi_i\rangle_i, \pi)$ and $N\models(\bigwedge\bar{D}')[\langle\dot{c}_i\rangle_{i<n}\dot{b}/\langle z_i\rangle_{i<n}y]$ by the definition of Q. Since N satisfies the sentence in (7), it follows that $\dot{a}\in NX$, and thus $\dot{d}\in N(a^*(X))$, which shows the claim.

Since $f\in\mathcal{Q}$, also $g\in\mathcal{Q}$. By (8), $Q\le\forall_g\rho^*(X) = \pi^*\forall_f(X)$. However, in M, $c\in M(Q)$, but $c\notin\pi^*\forall_f(X)$, since $b\notin\forall_f(X)$; this is a contradiction.

A $\wedge\vee\neg\exists$-*fibration* is a $\wedge\vee\exists$-fibration in which every fiber is a Boolean algebra. Every $\wedge\vee\neg\exists$-fibration is a $\wedge\vee\rightarrow\exists\forall$-fibration.

Without essentially changing the concepts, in each of the various kinds of fibrations introduced above, the class $\mathcal{Q}$ of "quantifiable" arrows may be required, in addition, to be closed under composition. If $(\mathcal{C}, \mathcal{Q})$ is a "quantificational" fibration (of one of the four kinds introduced above), then, with $\mathcal{Q}^\circ$ the closure of $\mathcal{Q}$ under composition, $(\mathcal{C}, \mathcal{Q}^\circ)$ is again one of the same kind as the reader will readily see. Also, any morphism $f:(\mathcal{C}, \mathcal{Q})\to(\mathcal{C}', \mathcal{Q}')$ of one of the four kinds is a morphism $f:(\mathcal{C}, \mathcal{Q}^\circ)\to(\mathcal{C}', \mathcal{Q}'^\circ)$ of the same kind.

§4. The syntax of first-order logic with dependent sorts as a fibration

Let $\mathbf{L}$ be a DSV; let $\mathbf{K}$ be the full subcategory of the kinds. Consider the category $\mathbf{B}=\mathbf{B}_{\mathbf{K}}$ which is the free finite-limit completion of $\mathbf{K}$: $i:\mathbf{K}\to\mathbf{B}$, and for any category $\mathbf{S}$ with finite limits, $i^{*}:\mathrm{Lex}(\mathbf{B},\mathbf{S})\longrightarrow\mathrm{Fun}(\mathbf{K},\mathbf{S})$ is an equivalence of categories ($\mathrm{Lex}(\mathbf{B},\mathbf{S})$ is the category of left exact functors $\mathbf{B}\to\mathbf{S}$, $\mathrm{Fun}(\mathbf{K},\mathbf{S}) = \mathbf{S}^{\mathbf{K}}$ the category of all functors $\mathbf{K}\to\mathbf{S}$, i^{*} is defined as composition with i).

It is well-known that for any (small) category $\mathbf{K}$, $\mathbf{B}_{\mathbf{K}}$ can be given as $(\mathrm{Fp}(\mathrm{Set}^{\mathbf{K}}))^{\mathrm{op}}$ ($\mathrm{Fp}(\mathbf{M})$ is the full subcategory of finitely presentable objects of $\mathbf{M}$), with $i:\mathbf{K}\subset\mathbf{B}$ the functor $i:\mathbf{K}\to(\mathrm{Fp}(\mathrm{Set}^{\mathbf{K}}))^{\mathrm{op}}$ induced by Yoneda. (The small-colimit completion of $\mathbf{A}$ is $Y:\mathbf{A}\to\mathrm{Set}^{(\mathbf{A}^{\mathrm{op}})}$; the finite-colimit completion of $\mathbf{A}$ is $Y:\mathbf{A}\to\mathrm{Fp}(\mathrm{Set}^{(\mathbf{A}^{\mathrm{op}})})$; therefore, the finite limit completion of $\mathbf{A}^{\mathrm{op}}$ is $Y:\mathbf{A}^{\mathrm{op}}\to(\mathrm{Fp}(\mathrm{Set}^{(\mathbf{A}^{\mathrm{op}})}))^{\mathrm{op}}$).

Now, for any simple category $\mathbf{K}$, $\mathrm{Fp}(\mathrm{Set}^{\mathbf{K}})$ is the category of *finite* functors $\mathbf{K}\to\mathrm{Set}$; a functor $F:\mathbf{K}\to\mathrm{Set}$ is finite if $\mathrm{El}(F)=\{(K,a): K\in\mathrm{Ob}(\mathbf{K}), a\in FK\}$ is a finite set. Namely, each finite functor is finitely presentable, the finite functors are closed under finite colimits in $\mathrm{Set}^{\mathbf{K}}$, and every functor is the filtered colimit of the collection of its finite subfunctors (the latter uses that $\mathbf{K}$ has finite fan-out); this suffices.

Thus, $\mathbf{B}$ can be taken to be the opposite of the category $\mathrm{Fin}(\mathrm{Set}^{\mathbf{K}})$ of finite functors $\mathbf{K}\to\mathrm{Set}$; the canonical functor $i:\mathbf{K}\to\mathbf{B}$ is (induced by) Yoneda.

Let $\mathrm{Con}[\mathbf{K}]$ be the category whose objects are the contexts (of variables over $\mathbf{K}$), and whose arrows are the specializations. I **claim** that

$$\mathrm{Con}[\mathbf{K}]\simeq\mathrm{Fin}(\mathrm{Set}^{\mathbf{K}}) .$$

Let $F:\mathbf{K}\longrightarrow\mathrm{Set}$ be a finite functor. I define a mapping

$$(K, a) \longmapsto \mathrm{y}^F_{K,a} : \mathrm{El}(F) \longrightarrow \mathrm{VAR}$$

into the class VAR of variables as follows:

$$\mathrm{y}^F_{K,a} \ \overset{=}{\mathrm{def}}\ \langle 2, \mathrm{Y}^F_{K,a}, a\rangle$$

where

$$\mathrm{Y}^F_{K,a} = K(\langle \mathrm{y}^F_{\mathrm{K}_p}, (Fp)(a)\rangle_{p\in K|\boldsymbol{K}}) \ .$$

This is a legitimate definition by recursion on the level of K. $\mathrm{Y}^F_{K,a}$ is a type; this requires that

$$\mathrm{K}_p(\langle \mathrm{y}^F_{\mathrm{K}_{qp}}, (F(qp))(a)\rangle_{q\in \mathrm{K}_p|\boldsymbol{K}}) = \mathrm{Y}^F_{\mathrm{K}_p}, (Fp)(a) \ ,$$

which is true since $(F(qp))(a) = (Fq)((Fp)(a))$. Hence, $\mathrm{y}^F_{K,a}$ is a variable.

We let $\mathcal{Y}_F \overset{=}{\mathrm{def}} \{\mathrm{y}^F_{K,a} : (K,a)\in \mathrm{El}(F)\}$. It is immediate that $\mathcal{Y}_F$ is a context. We have a bijection

$$(K, a) \longmapsto \mathrm{y}^F_{K,a} : \mathrm{El}(F) \xrightarrow{\cong} \mathcal{Y}_F \ .$$

If $h: F \longrightarrow G$ is a natural transformation, we let $s = \mathrm{s}_h : \mathcal{Y}_F \rightarrow \mathcal{Y}_G$ be defined by $s(\mathrm{y}^F_{K,a}) = \mathrm{y}^G_{K, h_K(a)}$. s is a specialization: this requires that $\mathrm{Y}^F_{K,a}|s = \mathrm{Y}^G_{K, h_K(a)}$, which is the same as $h_{\mathrm{K}_p}((Fp)(a)) = (Gp)(h_K a)$ $(p: K \rightarrow \mathrm{K}_p)$, which holds by the naturality of h. It is immediate that we have a bijection

$$h \longmapsto \mathrm{s}_h : \mathrm{Nat}(F, G) \xrightarrow{\cong} \mathrm{Spec}(\mathcal{Y}_F, \mathcal{Y}_G) \ .$$

Also, if $F \xrightarrow{h} G \xrightarrow{k} H$, then $\mathrm{s}_{kh} = \mathrm{s}_k \circ \mathrm{s}_h$, and $\mathrm{s}_{1_F} = 1_{\mathcal{Y}_F}$.

Thus far, we have seen that we have the full and faithful functor

$$\begin{array}{c}F\\ \downarrow h\\ G\end{array} \longmapsto s_{h}\begin{array}{c}\mathcal{Y}_F\\ \downarrow\\ \mathcal{Y}_G\end{array} : \mathtt{Fin}(\mathtt{Set}^{\boldsymbol{K}}) \longrightarrow \mathtt{Con}[\boldsymbol{K}] \qquad (1)$$

Now, given a context $\mathcal{Z}$, define $F=F_{\mathcal{Z}}:\boldsymbol{K}\to\mathtt{Set}$ by $FK=\{z\in\mathcal{Z}: \mathrm{K}_z=K\}$, and $Fp:FK\to F\mathrm{K}_p$ by $(Fp)(z)=\mathrm{x}_{z,p}$; F is a finite functor. Moreover, we have the map

$$s : z \longmapsto \mathrm{y}^F_{\mathrm{K}_z, z} : \mathcal{Z}\longrightarrow\mathcal{Y}_F \ ;$$

s is a specialization since

$$z : \mathrm{K}_z(\langle \mathrm{x}_{z,p}\rangle_{p\in K|\boldsymbol{K}}) , \qquad \mathrm{y}^F_{\mathrm{K}_z, z} : \mathrm{K}_z\{\langle \mathrm{y}^F_{\mathrm{K}_p, (Fp)(z)}\rangle_{p\in \mathrm{K}_z|\boldsymbol{K}} ,$$

and $s(\mathrm{x}_{z,p}) = \mathrm{y}^F_{\mathrm{K}_z, \mathrm{x}_{z,p}} = \mathrm{y}^F_{\mathrm{K}_p, (Fp)(z)}$, by the definition of F.

It is clear that s is a bijection, i.e., an isomorphism in $\mathtt{Con}[\boldsymbol{K}]$.

We have verified that (1) is an equivalence of categories, thus our **claim**.

It is easy to see that the image of (1) consists of those contexts $\mathcal{Z}$ for which $z\in\mathcal{Z} \mapsto (\mathrm{K}_z, \mathrm{a}(z))$ is a 1-1 function.

It is clear that although the categories $\mathtt{Fin}(\mathtt{Set}^{\boldsymbol{K}})$, $\mathtt{Con}[\boldsymbol{K}]$ are large, they are essentially small.

Thus, $\boldsymbol{B}$, the free finite-limit completion of $\boldsymbol{K}$, can be taken to be the opposite of the category $\mathtt{Con}[\boldsymbol{K}]$ of contexts with specializations as arrows. To describe the canonical embedding $i:\boldsymbol{K}\to\boldsymbol{B}$ under the latest construal of the completion $\boldsymbol{B}$, let us define, for any $K\in\boldsymbol{K}$, the context

$$\mathcal{X}_K \overset{=}{\mathrm{def}} \{x^K_p : p\in K|\boldsymbol{K}\} \qquad (2)$$

for which $X = \mathrm{X}_K \overset{=}{\mathrm{def}}\ K(\langle \mathrm{x}^K_p\rangle_{p\in K|\boldsymbol{K}})$ is a sort, and $\mathrm{a}(X)=\langle p\rangle_{p\in K|\boldsymbol{K}}$. In the definition of $\mathcal{X}_K$, the only essential points are that $K(\langle \mathrm{x}^K_p\rangle_{p\in K|\boldsymbol{K}})$ is a sort, and that the mapping $p\mapsto \mathrm{x}^K_p$ is 1-1. X_K is "the most general sort" of the kind K; every other such sort is of the form $\mathrm{X}_K|s$ for some specialization s with domain $\mathcal{X}_K$. Further, let

$$\mathcal{X}^*_K \overset{=}{\mathrm{def}}\ \mathcal{X}_K \dot{\cup} \{\mathrm{x}_K\} \ ,$$

where $\mathrm{x}_K{:}\mathrm{X}_K$, and, for the sake of definiteness, x_K is taken to be the specific variable for which $\mathrm{a}(\mathrm{x}_K)=1_K$. Note that under the equivalence $F\mapsto \mathcal{Y}_F$ between finite functors and contexts, $\mathcal{X}^*_K$ is the context that corresponds to the representable functor $\boldsymbol{K}(K,-):\boldsymbol{K}\to\mathrm{Set}$.

When a context $\mathcal{X}$ is considered an object of $\boldsymbol{B} = (\mathrm{Con}[\boldsymbol{K}])^{\mathrm{op}}$, it is written as $[\mathcal{X}]$. Arrows $s{:}\mathcal{X}\to\mathcal{Y}$ of $\mathrm{Con}[\boldsymbol{K}]$ correspond to arrows $[s]:[\mathcal{Y}]\to[\mathcal{X}]$.

The canonical embedding $i:\boldsymbol{K}\to\boldsymbol{B}$ (having the universal property of the finite limit completion) has $i(K) = [\mathcal{X}^*_K]$.

The morphism $p{:}K\to K_p$ is taken by i to the arrow

$$[s_p]:[\mathcal{X}^*_K]\longrightarrow[\mathcal{X}^*_{\mathrm{K}_p}]$$

for the specialization

$$s_p{:}\mathcal{X}^*_{\mathrm{K}_p}\to\mathcal{X}^*_K:\ \ \mathrm{x}^{\mathrm{K}_p}_q \overset{s}{\longmapsto} \mathrm{x}^K_{qp}\ \ (q{:}\mathrm{K}_p\to\mathrm{K}_q)\ ,\ \ \mathrm{x}_{\mathrm{K}_p} \overset{s}{\longmapsto} \mathrm{x}^K_p\ . \tag{3}$$

Note that in the category $\boldsymbol{B}$, the object $[\mathcal{X}_K]$ is the same as $i[K]$ for the "$\boldsymbol{B}$-valued $\boldsymbol{K}$-structure $i:\boldsymbol{K}\to\boldsymbol{B}$", that is, the limit of the composite $K{\downarrow}(\boldsymbol{K}-\{K\})\xrightarrow{\Phi}\boldsymbol{K}\xrightarrow{i}\boldsymbol{B}$.

We single out four classes of arrows in $\mathrm{Con}[\boldsymbol{K}]$, $\mathcal{Q}_0\subset\mathcal{Q}_1\subset\mathcal{Q}_2\subset\mathcal{Q}_3$. $\mathcal{Q}_0$ consists of the inclusion-arrows $\mathrm{incl}{:}\mathcal{X}_K\to\mathcal{X}^*_K$, where K ranges over the kinds. $\mathcal{Q}_1$ consists of the inclusion-arrows of the form $\mathrm{incl}{:}\mathcal{X}\to\mathcal{X}\dot{\cup}\{x\}$, where $\mathcal{X}$ is any (finite) context, and

$\mathcal{X}\dot{\cup}\{x\}$ is also a context (for this, it is necessary and sufficient that $x\notin\mathcal{X}$ and $\mathtt{Dep}(x)\subset\mathcal{X}$). $\mathcal{Q}_2$ is the class of all 1-1 arrows $i:\mathcal{X}\to\mathcal{Y}$ where $\mathtt{card}(\mathcal{Y})=\mathtt{card}(\mathcal{X})+1$. Finally, $\mathcal{Q}_3$ is the class of all 1-1 arrows $\mathcal{X}\to\mathcal{Y}$.

Every time $s:\mathcal{Y}\to\mathcal{X}$ is a specialization, and $t:\mathcal{Y}\dot{\cup}\{y\}\to\mathcal{X}\dot{\cup}\{x\}$ extends s, with $t(y)=x$, we have the pushout diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\mathtt{incl}} & \mathcal{Y}\dot{\cup}\{y\} \\ {\scriptstyle s}\downarrow & \square & \downarrow{\scriptstyle t} \\ \mathcal{X} & \xrightarrow[\mathtt{incl}]{} & \mathcal{X}\dot{\cup}\{x\} \end{array} \tag{4}$$

in $\mathtt{Con}[\boldsymbol{K}]$. All arrows in $\mathcal{Q}_1$ are pushouts of ones in $\mathcal{Q}_0$. To see this, for a given $\mathcal{X}\xrightarrow{\mathtt{incl}}\mathcal{X}\dot{\cup}\{x\}$, apply (4) to $\mathcal{X}_{K_x}\xrightarrow{\mathtt{incl}}\mathcal{X}^*_{K_x}$ as $\mathcal{Y}\xrightarrow{\mathtt{incl}}\mathcal{Y}\dot{\cup}\{y\}$, and $s:\mathcal{X}_{K_x}\to\mathcal{X}$ given by $s(x^{K_x}_p)=x_{x,p}$.

It is clear that $\mathcal{Q}_2$ is the closure of $\mathcal{Q}_1$ under isomorphisms (meaning that $q:A\to B\in\mathcal{Q}_2$ iff there is $q':A'\to B'\in\mathcal{Q}_1$ with some commutative

$$\begin{array}{ccc} A & \xrightarrow{q} & B \\ \cong\downarrow & \circ & \downarrow\cong \\ A' & \xrightarrow[q']{} & B' \end{array}\ ).$$

(4) shows that any arrow $q:A\to B$ in $\mathcal{Q}_1$ has a pushout along any $a:A\to A'$ that is again in $\mathcal{Q}_1$. Thus, $\mathcal{Q}_2$ is closed under pushout, and in fact it is the closure of $\mathcal{Q}_0$ under pushout.

$\mathcal{Q}_3$ is the closure of $\mathcal{Q}_2$ under composition. Indeed, given any inclusion $i:\mathcal{X}\to\mathcal{Y}$, there is a finite sequence $\mathcal{X}=\mathcal{X}_0\subset\mathcal{X}_1\subset\ldots\subset\mathcal{X}_{n-1}\subset\mathcal{X}_n=\mathcal{Y}$ of contexts such that $\mathtt{card}(\mathcal{X}_{i+1})=\mathtt{card}(\mathcal{X}_i)+1$; enumerate $\mathcal{Y}-\mathcal{X}$ as $\langle y_i\rangle_1^n$ such that the level of K_{y_i} is non-increasing, and put $\mathcal{X}_i=\mathcal{X}\cup\{x_1,\ldots x_i\}$. This shows that every inclusion $i:\mathcal{X}\to\mathcal{Y}$ is the composite of arrows in $\mathcal{Q}_1$; since every 1-1 arrow is isomorphic to an inclusion, the assertion follows.

Without talking about syntax, $[\mathcal{Q}_0] = \{[q] : q \in \mathcal{Q}_0\}$ may be described as the class of arrows of the form $q: iK \to [K]$, where $K \in \boldsymbol{K}$, $i: \boldsymbol{K} \to \boldsymbol{B} = (\mathrm{Fin}(\mathrm{Set}^{\boldsymbol{K}}))^{\mathrm{op}}$ is induced by Yoneda, and $[K]$ is the limit of the composite $K\downarrow(\boldsymbol{K}-\{1_K\}) \xrightarrow{\Phi} \boldsymbol{K} \xrightarrow{i} \boldsymbol{B}$. $[\mathcal{Q}_2]$ is the closure of $[\mathcal{Q}_0]$ under pullback. $[\mathcal{Q}_3]$ is the class of all epimorphisms; also, it is the closure of $[\mathcal{Q}_2]$ under composition.

For the purposes for logic without equality, we let the class $\mathcal{Q}^{\neq}$ of arrows in $\boldsymbol{B}$ be either $[\mathcal{Q}_2]$ ($= \{[q] : q \in \mathcal{Q}_2\}$ or $[\mathcal{Q}_3]$; both $[\mathcal{Q}_2]$ and $[\mathcal{Q}_3]$ are closed under pullback, and the second class is the closure of the first under composition. (According the remarks at the end of the last section, the two possible choices are essentially equivalent).

Corresponding to logic with equality, we have $\mathcal{Q}^{=}$, which is obtained by adding to $\mathcal{Q}^{\neq}$ all isomorphic copies of arrows of the form $[p]$ for p of the form $p: \mathcal{X}\dot\cup\{x, y\} \to \mathcal{X}\dot\cup\{x\}$ such that x and y are distinct variables of the same type, their kind is a maximal one, and p is defined so that $p\restriction\mathcal{X}$ is the identity and $p(x)=p(y)=x$. Categorically, if we put $A=[\mathcal{X}]$, $B=[\mathcal{X}\dot\cup\{x\}]$, and $q: B\to A$, $q=[\mathrm{incl}]$, we have $[p]=\delta=B\to B\times_A B$, the diagonal.

If $s: \mathcal{X}\dot\cup\{x, y\} \longrightarrow \mathcal{Y}$, then for $x'=s(x)$, $y'=s(y)$ and $\mathcal{X}'=\mathcal{Y}-\{x', y'\}$, $\mathcal{X}'$ is a context, since no variable z can have $x' \in \mathrm{Dep}(z)$ or $y' \in \mathrm{Dep}(z)$, by the maximality assumption on the kind of x and y; $\mathcal{Y} = \mathcal{X}'\dot\cup\{x', y'\}$. We have a pushout

$$\begin{array}{ccc}
\mathcal{X}\dot\cup\{x, y\} & \xrightarrow{\;p\;} & \mathcal{X}\dot\cup\{x\} \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\
\mathcal{X}'\dot\cup\{x', y'\} & \xrightarrow[p']{} & \mathcal{X}'\dot\cup\{x'\}
\end{array}$$

with the evident p' and t. It follows that all pullbacks of the additional arrows in $\mathcal{Q}^{=}$ are again of the same form, thus $\mathcal{Q}^{=}$ is closed under pullback. Also, all the additional arrows in $\mathcal{Q}^{=}$ are pullbacks of the specific ones $[p_K]$ where K is a maximal kind, $p_K: \mathcal{X}_K\dot\cup\{x_K, y\} \longrightarrow \mathcal{X}_K\dot\cup\{x_K\}$; here, $\mathcal{X}_K\dot\cup\{x_K\} = \mathcal{X}_K^{*}$ defined above, *etc.*

Suppose $T=(\mathbf{L}, \Sigma)$ is a theory; there are six possibilities for the logic: coherent, intuitionistic, or classical, each with or without equality. We define a fibration $\mathcal{C} = [T] = \mathcal{C}\overset{\mathbf{E}}{\underset{\mathbf{B}}{\downarrow}}$, with a set $\mathcal{Q}=\mathcal{Q}_{\mathcal{C}}$ of distinguished (quantifiable) arrows in $\mathbf{B}$. $\mathbf{B}$ has been given in the foregoing; we use $\mathcal{Q}^{\neq}$ when we exclude equality, $\mathcal{Q}^{=}$ otherwise, as $\mathcal{Q}$.

A *formula-in-a-context* is an ordered pair $(\mathcal{X}, \varphi)$, written as $[\mathcal{X}:\varphi]$, such that $\mathcal{X}$ is a context, and φ is a formula with $\mathrm{Var}(\varphi)\subset\mathcal{X}$. With a given $\mathcal{X}$, $[\mathcal{X}:\varphi]$ is called a *formula-over* $\mathcal{X}$.

To define $\mathcal{C}\overset{\mathbf{E}}{\underset{\mathbf{B}}{\downarrow}}$, for $[\mathcal{X}]\in\mathbf{B}$, the fiber $\mathcal{C}^{[\mathcal{X}]}$ is given as the set of equivalence classes $[\mathcal{X}:\varphi]/\sim_{\mathcal{X}}$ of formulas-over $\mathcal{X}$ under the equivalence relation

$$[\mathcal{X}:\varphi] \sim_{\mathcal{X}} [\mathcal{X}:\psi] \iff T\vdash \varphi\underset{\mathcal{X}}{\Longrightarrow}\psi \text{ and } T\vdash \psi\underset{\mathcal{X}}{\Longrightarrow}\varphi$$

(the range of the formulas φ, ψ, and the deducibility relation $\vdash$ is understood according to the logic in question). In what follows, we will write $[\mathcal{X}:\varphi]$ for $[\mathcal{X}:\varphi]/\sim_{\mathcal{X}}$. $\mathcal{C}^{[\mathcal{X}]}$ is partially ordered by

$$[\mathcal{X}:\varphi] \leq_{\mathcal{X}} [\mathcal{X}:\psi] \iff T\vdash \varphi\underset{\mathcal{X}}{\Longrightarrow}\psi\,;$$

by the rules (Taut) and (Cut) this is well-defined and it is a partial order. Finally, for $s:\mathcal{X}\to\mathcal{Y}$ in $\mathrm{Con}[\mathbf{K}]$, that is, $[s]:[\mathcal{Y}]\longrightarrow[\mathcal{X}]$, $[s]^{*}([\mathcal{X}:\varphi]) \underset{\mathrm{def}}{=} [\mathcal{Y}:\varphi|s]$. By the rule (Subst), $[s]^{*}:\mathcal{C}^{\mathcal{X}}\longrightarrow\mathcal{C}^{\mathcal{Y}}$ is a map of posets.

Since $(\varphi|s)|t = \varphi|ts$, and $\varphi|\mathrm{id} = \varphi$, we have a (pseudo)functor $\mathcal{X}\mapsto\mathcal{C}^{X}$, $([\mathcal{X}]\xrightarrow{[s]}[\mathcal{Y}]) \mapsto [s]^{*}$; thus, we have a fibration. The rules for connectives (not counting the last two) make sure that each fiber has the necessary (propositional) structure, where each operation is given by the corresponding syntactic operation on formulas; e.g., $[\mathcal{X}:\varphi] \wedge_{[\mathcal{X}]} [\mathcal{X}:\psi] = [\mathcal{X}:\varphi\wedge\psi]$.

For $[i]:[\mathcal{X}\dot{\cup}\{x\}]\longrightarrow[\mathcal{X}]$ ($i:\mathcal{X}\longrightarrow\mathcal{X}\dot{\cup}\{x\}$ the inclusion) in $[\mathcal{Q}_1]$ and

$[\mathcal{X}\dot{\cup}\{x\}:\varphi]\in\mathcal{C}^{[\mathcal{X}\dot{\cup}\{x\}]}$, we have $\exists_{[i]}([\mathcal{X}\dot{\cup}\{x\}:\varphi]) = [\mathcal{X}:\exists x\varphi]\in\mathcal{C}^{[\mathcal{X}]}$, and similarly for $\forall$ in place of $\exists$. This follows from the rules $(\exists)$ and $(\forall)$. As we pointed out in Section 2 , if $\mathrm{Var}(\varphi)\subset\mathcal{X}\dot{\cup}\{x\}$, then $\forall x\varphi$, $\exists x\varphi$ are well-formed. Since every arrow q in $[\mathcal{Q}_3]$ is an isomorphic copy of a composite of arrows in $[\mathcal{Q}_1]$, the operation $\exists_q$, or $\forall_q$, will be well-defined, and can be expressed in terms of $\exists_r$, or $\forall_r$, for $r\in[\mathcal{Q}_1]$.

In the case of logic with equality, we have, for

$$\delta:[\mathcal{X}\dot{\cup}\{x\}]\xrightarrow{[p]}[\mathcal{X}\dot{\cup}\{x,y\}] ,$$

an additional arrow in $\mathcal{Q}^{=}$, $\exists_\delta(\mathbf{t}_{[\mathcal{X}\dot{\cup}\{x\}]}) = [\mathcal{X}\dot{\cup}\{x,y\}:x=_{\mathcal{X}}y]$, and more generally, $\exists_{[\delta]}([\mathcal{X}\dot{\cup}\{x\}:\varphi]) = [\mathcal{X}\dot{\cup}\{x,y\}:\ x=_{\mathcal{X}}y\wedge\varphi]$. This is F. W. Lawvere's observation on the definition of equality in hyperdoctrines [L2]. The claimed equality can be deduced by using the rules of equality. We also have that

$$\forall_{[\delta]}([\mathcal{X}\dot{\cup}\{x\}:\varphi]) = [\mathcal{X}\dot{\cup}\{x,y\}:\ x=_{\mathcal{X}}y\longrightarrow\varphi] .$$

The fact that substitution is compatible with the logical operations gives that for any specialization $s:\mathcal{Y}\to\mathcal{X}$, $[s]^*:\mathcal{C}^{\mathcal{X}}\longrightarrow\mathcal{C}^{\mathcal{Y}}$ preserves the (propositional) structure, and that the Beck-Chevalley conditions are fulfilled. We obtain a $\wedge\vee\exists$-fibration, a $\wedge\vee\to\exists\forall$-fibration and a $\wedge\vee\neg\exists$-fibration in the respective cases of coherent logic, intuitionistic logic and classical logic; the presence of the rules $(\wedge\vee)$, $(\wedge\exists)$ ensures this in the coherent case, and that of $(\neg)$ in the classical case.

The construction $[T]$ has the universal property of the fibration of the appropriate kind that is freely generated by T . In what follows, we describe this universal property in a somewhat incomplete way, namely, for "target" fibrations of the form $\mathcal{P}(\boldsymbol{C})$, rather than arbitrary (suitably structured) fibrations.

For a relation $R\in\mathrm{Rel}(\boldsymbol{L})$, we make a definition of the context $\mathcal{X}_R$ analogously to $\mathcal{X}_K$ in (2) : $\mathcal{X}_R \overset{=}{\mathrm{def}} \{x^R_p:p\in R|\boldsymbol{L}\}$ such that $\bar{R} \overset{=}{\mathrm{def}} R(\langle x^R_p\rangle_{p\in R|\boldsymbol{L}})$ is a well-formed atomic formula, and $\alpha(X)=\langle p\rangle_{p\in R|\boldsymbol{L}}$. $\bar{R}$ is the "most general" atomic formula using the relation

R . Moreover, for $p \in R | \boldsymbol{L}$, we let

$$s_p : \mathcal{X}^*_{\mathrm{K}_p} \to \mathcal{X}^*_R : \quad \mathrm{x}_q^{\mathrm{K}_p} \longmapsto \mathrm{x}^R_{qp} \quad (q : \mathrm{K}_p \to \mathrm{K}_q) \ , \quad \mathrm{x}_{\mathrm{K}_p} \longmapsto \mathrm{x}^R_p \ .$$

Changing the meaning of the symbol $\mathrm{Mod}_{\mathcal{D}}(\mathcal{C})$, let us use the notation now in the variable sense of either of $\wedge\vee\exists(\mathcal{C},\mathcal{D})$, $\wedge\vee\to\exists\forall(\mathcal{C},\mathcal{D})$, $\wedge\vee\neg\exists(\mathcal{C},\mathcal{D})$ as the context requires it, according to which logic we are dealing with. In what follows, $\boldsymbol{C}$ is a category having enough structure for the logic at hand: it is a coherent, a Heyting or a Boolean category in the three respective cases.

We have a "forgetful" functor

$$(\ )^- : \mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}([T]) \longrightarrow \mathrm{Mod}_{\boldsymbol{C}}(T) \tag{5}$$

defined as follows. Given $P=(P_1, P_2) \in \mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}([T])$, we define $P^- : \boldsymbol{L} \to \boldsymbol{C}$, $P^- \in \mathrm{Mod}_{\boldsymbol{C}}(T)$, by $P^-(K) = P_1([\mathcal{X}^*_K])$; for $p : K \to \mathrm{K}_p$, $P^-(p) = P_1([s_p])$ (see (3)) (more briefly, $P^- \restriction \boldsymbol{K} = P_1 \circ i$, for the canonical embedding $i : \boldsymbol{K} \to \boldsymbol{B}$); for $R \in \mathrm{Rel}(\boldsymbol{L})$, $P^-(R)$ the domain of a monomorphism m representing the subobject $P_2([\mathcal{X}_R : \bar{R}])$ of $P_1([\mathcal{X}_R])$; and for $p : R \to K_p$, $P^-(p) = P_1([s_p]) \circ m$.

For $h : P \to Q$ in $\mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}([T])$ (that is, $h : P_1 \to Q_1$ with properties), $h^- = h \circ i$; it is easy to see that h^- is an arrow $P^- \to Q^-$.

In the case of coherent logic, the functor (5) is full and faithful, and in the case of intuitionistic and classical logics,

$$(\ )^- : \mathrm{Mod}^{\mathrm{iso}}_{\mathcal{P}(\boldsymbol{C})}([T]) \longrightarrow \mathrm{Mod}^{\mathrm{iso}}_{\boldsymbol{C}}(T) \ , \tag{6}$$

with both categories restricted to have only isomorphisms as arrows (thus, they are groupoids), is full and faithful. The faithfulness is obvious; the fullness requires an easy proof by induction on the complexity of formulas.

In fact, in the case of coherent logic, (5), and in the other two cases, (6), is an equivalence of categories. Indeed, if $M:\mathbf{L}\to\mathbf{C}$ is a model of T, we define

$$[M]\ :\ [T]\longrightarrow\mathcal{P}(\mathbf{C})$$

by $[M]_1([\mathcal{X}])=M[\mathcal{X}]$, $[M]_2([\mathcal{X}:\varphi])=M[\mathcal{X}:\varphi]$. The fact that M is a model ensures that $[M]$ is well-defined (on equivalence classes); the rules of the logic, built into the definition of $[T]$, ensure that $[M]$ is a morphism of fibrations with the appropriate preservation properties. Finally, we have $j_M:[M]^{-}\restriction\mathbf{K}\cong M\restriction\mathbf{K}$ whose components are canonical isomorphisms $M([\mathcal{X}^{*}_{K}])\cong M(K)$, and j_M is in fact an isomorphism $j_M:[M]^{-}\cong M$.

The completeness theorem

$$T\vdash\varepsilon\iff T\models_{\mathrm{Set}}\varepsilon$$

for coherent logic with dependent sorts, with or without equality, is now an immediate consequence of 3.(5). Indeed,

$T\vdash\ \varphi\underset{\mathcal{X}}{\Longrightarrow}\psi$ $\iff$ $[\mathcal{X}:\varphi]\leq_{\mathcal{X}}[\mathcal{X}:\psi]$ in $[T]$

by the construction of $[T]$;

$\iff$ for all $P:[T]\to\mathcal{P}(\mathrm{Set})$, $P[\mathcal{X}:\varphi]\leq P[\mathcal{X}:\psi]$

by 3.(5) ;

$\iff$ for all $M\models T$, $M\models\ \varphi\underset{\mathcal{X}}{\Longrightarrow}\psi$

by the above description of the equivalence $\mathrm{Mod}_{\mathbf{C}}(T)\simeq\mathrm{Mod}_{\mathcal{P}(\mathcal{C})}([T])$,

$\iff$ $T\models_{\mathrm{Set}}\varphi\underset{\mathcal{X}}{\Longrightarrow}\psi$

by definition.

3.(6) gives a proof of the completeness theorem for intuitionistic logic. 3.(6) says that there is a category $\mathbf{K}$, namely $\mathrm{Mod}(T)$, such that T has a conservative Heyting morphism into $\mathrm{Set}^{\mathbf{K}}$; changing here $\mathbf{K}$ into a small category, and then into a poset is an easy matter; see [MR2], [M3].

As it is well-known, completeness for classical logic follows from that for coherent logic directly.

In summary, it is worth emphasizing that the study of first-order logic with dependent sorts without equality is the same as the study of "quantificational" fibrations $(\mathcal{C}\overset{\boldsymbol{E}}{\underset{\boldsymbol{B}}{\downarrow}}, \mathcal{Q})$ where the base category is $\boldsymbol{B}=((\mathrm{Set}^{\boldsymbol{K}})_{\mathrm{fin}})^{\mathrm{op}}$ for a simple category $\boldsymbol{K}$, with $\mathcal{Q}$ being the class of all epimorphisms in $\boldsymbol{B}$. This is a remarkably simple algebraic description of the objects of our interest, even though it is not one that is conjured up immediately by the idea of "first-order logic with dependent sorts".

§5. Equivalence

Let $\boldsymbol{L}$ be a fixed DSV, $\boldsymbol{K}$ the full subcategory of its kinds.

We have defined what an $\boldsymbol{L}$-structure is; even, what a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure is, for any $\boldsymbol{C}$ with finite limits. In what follows, we will make the minimal assumption that $\boldsymbol{C}$ is a regular category (which is equivalent to saying that $\mathcal{P}(\boldsymbol{C})$, with "total" $\mathcal{Q}$, is a $\wedge\exists$-fibration: just ignore $\mathbf{f}$ and $\vee$ in the definition of $\wedge\vee\exists$-fibration).

The category of $\boldsymbol{C}$-valued $\boldsymbol{L}$-structures, $\mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$, has objects the $\boldsymbol{C}$-valued $\boldsymbol{L}$-structures, and morphisms natural transformations; $\mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$ is a full subcategory of $\boldsymbol{C}^{\boldsymbol{L}}$ (with $\boldsymbol{L}$ in its last occurrence understood as a mere category). We write $\mathrm{Str}(\boldsymbol{L})$ for $\mathrm{Str}_{\mathrm{Set}}(\boldsymbol{L})$.

Given $M\in\mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$, we have $M\restriction\boldsymbol{K}:\boldsymbol{K}\longrightarrow\boldsymbol{C}$, its $\boldsymbol{K}$-reduct, the structure of kinds associated to M. For any $R\in\mathrm{Rel}(\boldsymbol{L})$, we have the canonical monomorphism $\mathrm{m}_R:M(R)\rightarrowtail M[R] = (M\restriction\boldsymbol{K})[R]$ (see §1). For a natural transformation $(f:U\longrightarrow V)\in\boldsymbol{C}^{\boldsymbol{K}}$, we have the canonical arrow $f_{[R]}:U[R]\longrightarrow V[R]$ for which

$$
\begin{array}{ccc}
U[R] & \xrightarrow{\quad f_{[R]}\quad} & V[R] \\
{\scriptstyle \pi^U_p}\downarrow & \circ & \downarrow{\scriptstyle \pi^V_p} \\
U(\mathrm{K}_p) & \xrightarrow[\quad h_{\mathrm{K}_p}\quad]{} & V(\mathrm{K}_p)
\end{array}
$$

for all $p\in R|\boldsymbol{L}$. If $(h:M\longrightarrow N)\in\mathrm{Str}(\boldsymbol{L})$, then

$$
\begin{array}{ccc}
M(R) & \xrightarrow{\quad h_R\quad} & N(R) \\
{\scriptstyle \mathrm{m}^M_R}\downarrow & \circ & \downarrow{\scriptstyle \mathrm{m}^N_R} \\
(M\restriction\boldsymbol{K})[R] & \xrightarrow[\quad h_{[R]}\quad]{} & (N\restriction\boldsymbol{K})[R]
\end{array}
$$

which shows that $h\restriction\boldsymbol{K}:M\restriction\boldsymbol{K}\longrightarrow N\restriction\boldsymbol{K}$ determines h (if any).

We have the forgetful functor $\mathcal{E}_{\mathbf{C},\mathbf{L}}=\mathcal{E}:\mathrm{Str}_{\mathbf{C}}(\mathbf{L})\longrightarrow\mathbf{C}^{\mathbf{K}}$; $\mathcal{E}$ is faithful, by the last remark. $\mathcal{E}$ is a fibration. Indeed, given $f:U\to V$ in $\mathbf{C}^{\mathbf{K}}$, and N over V (that is, $N\restriction\mathbf{K}=V$), then the Cartesian arrow $h:M\to N$ over f is obtained by defining M and h such that $M\restriction\mathbf{K}=U$, $h\restriction\mathbf{K}=f$ and, for all $R\in\mathrm{Rel}(\mathbf{L})$,

$$\begin{array}{ccc} M(R) & \xrightarrow{\;h_R\;} & N(R) \\ {\scriptstyle \mathrm{m}^M_R}\downarrow & \square & \downarrow{\scriptstyle \mathrm{m}^N_R} \\ U[R] & \xrightarrow[\;f_{[R]}\;]{} & V[R] \end{array}$$

is a pullback (it is immediate to see that h so defined is Cartesian). As usual with fibrations, let us denote M so defined by $f^*(N)$, and the Cartesian arrow h by $\theta_f:f^*(N)\to N$.

$\mathcal{E}$ is a fibration with fibers that are preorders.

When in particular $\mathbf{C}=\mathrm{Set}$ (which is the most important case), a functor $U:\mathbf{K}\to\mathrm{Set}$ is called *separated* if $U(K)\cap U(K')=\emptyset$ whenever K, K' are distinct objects of $\mathbf{K}$. For a separated U, we define $|U|=\bigcup_{K\in\mathbf{K}}U(K)$; for a general U, we would put $|U|=\coprod_{K\in\mathbf{K}}U(K)=\{(K,a):k\in\mathbf{K},\ a\in U(K)\}$. Of course, every functor is isomorphic to a separated one. When $f:U\to V$, and U is separated, for $a\in|U|$ we may write $h(a)$ without ambiguity for $h_K(a)$ for which $a\in U(K)$. For notational simplicity, we will restrict attention to separated functors $\mathbf{K}\to\mathrm{Set}$.

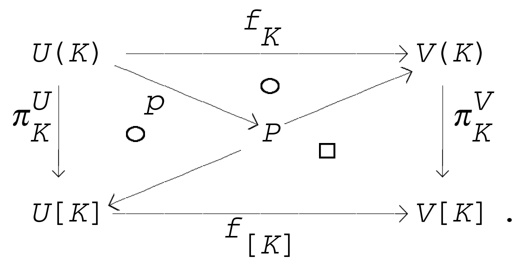
I will now isolate a property of a natural transformation $f:U\to V$ in $\mathbf{C}^{\mathbf{K}}$. Let first $\mathbf{C}=\mathrm{Set}$. We say that f is *very surjective* if whenever $K\in\mathbf{K}$ ▯ $\langle a_p\rangle_{p\in K|\mathbf{K}}\in U[K]$, the mapping

$$f_{\langle a_p\rangle_{p\in K|\mathbf{K}}}:UK(\langle a_p\rangle_{p\in K|\mathbf{K}})\longrightarrow VK(\langle fa_p\rangle_{p\in K|\mathbf{K}})\ :\ a\longmapsto f(a)$$

(see (3) in §1) is surjective.

For a general $\mathbf{C}$ (assumed to be regular), $f:U\to V$ in $\mathbf{C}^{\mathbf{K}}$ is *very surjective* if for every

$K\in\boldsymbol{K}$, the canonical map $p: U(K)\to P=U[K]\times_{V[K]}V(K)$ from the diagram below is surjective (a regular epimorphism):



It is clear that if f is an isomorphism (in $\boldsymbol{C}^{\boldsymbol{K}}$), then it is very surjective. It is easy to see (by induction on the level of $K\in\boldsymbol{K}$) that very surjective implies surjective (being a regular epimorphism in $\boldsymbol{C}^{\boldsymbol{K}}$), but not necessarily conversely.

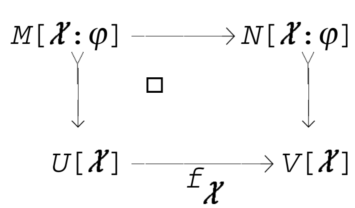
In this section, we consider logic with dependent sorts only without equality; all $\boldsymbol{L}$-formulas are without equality.

(1) Let $f:U\to V$ in $\boldsymbol{C}^{\boldsymbol{K}}$ be very surjective, and any $N\in\mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$ over V. Let $h=\theta_f: M=f^*(N)\to N$.

(a) Let first $\boldsymbol{C}=\mathrm{Set}$. h is *elementary with respect to logic without equality* in the sense that for any context $\mathcal{X}$ and $\boldsymbol{L}$-formula φ (in logic with dependent sorts and without equality) with $\mathrm{Var}(\varphi)\subset\mathcal{X}$, and any $\langle a_x\rangle_{x\in\mathcal{X}}\in M[\mathcal{X}]$,

$$M\models\varphi[\langle a_x\rangle_{x\in\mathcal{X}}]\quad\Longleftrightarrow\quad N\models\varphi[\langle ha_x\rangle_{x\in\mathcal{X}}]\quad .$$

(b) For a general $\boldsymbol{C}$ which is a Heyting category (to interpret all $\boldsymbol{L}$-formulas), for any φ and $\mathcal{X}$ as above, there is a pullback



(1b)

(the vertical monomorphisms are representatives for the subobjects $M[\mathcal{X}:\varphi]\in S(U[\mathcal{X}])$, $N[\mathcal{X}:\varphi]\in S(V[\mathcal{X}])$; in other words, (1b) says $M[\mathcal{X}:\varphi] = (f_{\mathcal{X}})^{*}N[\mathcal{X}:\varphi]$).

here, $f_{\mathcal{X}}$ is the canonical map determined through by the definition of $U[\mathcal{X}]$, $V[\mathcal{X}]$ as limits in $\boldsymbol{C}$.

Obviously, (b) generalizes (a).

The proof for (a) can be given as a straightforward induction on the complexity of φ. The clause for atomic formulas is essentially the definition of M. For the propositional connectives, the induction step is automatic. By the duality in Set between $\exists$ and $\forall$, it is enough to handle the inductive step involving $\exists$, which is done using the "very surjective" assumption. In Appendix B, I will take a "fibrational" view of the notion of equivalence, and give a detailed proof of the more general form (b) .

Let M, N be $\boldsymbol{C}$-valued $\boldsymbol{L}$-structures. We say that they are $\boldsymbol{L}$-*equivalent*, and we write $M\sim_{\boldsymbol{L}}N$, if there is a diagram

$$M \xleftarrow{\;\bar{m}\;} P \xrightarrow{\;\bar{n}\;} N$$

in $\mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$ such that $\bar{m}\restriction\boldsymbol{K}$, $\bar{n}\restriction\boldsymbol{K}$ are very surjective, and $\bar{m}$ and $\bar{n}$ are Cartesian arrows in the fibration $\mathcal{E}_{\boldsymbol{C},\,\boldsymbol{L}}$. Paraphrased, this means that there exists a functor $W\in\boldsymbol{C}^{\boldsymbol{K}}$ and very surjective maps $m:W\to M\restriction\boldsymbol{K}$, $n:W\to N\restriction\boldsymbol{K}$ such that $m^{*}(M) = n^{*}(N)$, that is, for all $R\in\mathrm{Rel}(\boldsymbol{L})$,

$$\begin{array}{ccccccc}
M(R) & \longleftarrow & M(R)\times_{M[R]}W[R] & = & N(R)\times_{N[R]}W[R] & \longrightarrow & N(R)\\
\downarrow & & \square & \downarrow & \square & & \downarrow\\
M[R] & \xleftarrow[\;m_{[R]}\;]{} & & W[R] & & \xrightarrow[\;n_{[R]}\;]{} & N[R]
\end{array} \qquad (1')$$

(where the equality means equality of subobjects of $W[R]$). In case $\boldsymbol{C}=\mathrm{Set}$, (1') means that if $R\in\mathrm{Rel}(\boldsymbol{L})$, $\langle c_p\rangle_{p\in R|\boldsymbol{K}}\in W[R]$, then

$$\langle mc_p\rangle_{p\in R|\boldsymbol{K}}\in M(R) \iff \langle nc_p\rangle_{p\in R|\boldsymbol{K}}\in N(R) \quad . \qquad (1")$$

The data (W, m, n) are said to form an $\boldsymbol{L}$-*equivalence* of M and N; in notation, $(W, m, n):M \xleftrightarrow[\boldsymbol{L}]{} N$.

It is easy to see that the relation $\sim_{\boldsymbol{L}}$ is an equivalence relation (for a proof, see Appendix B). It is also clear that isomorphism of $\boldsymbol{L}$-structures implies $\boldsymbol{L}$-equivalence.

Let us write $M\equiv_{\boldsymbol{L}} N$ for: $M\vDash\sigma \iff N\vDash\sigma$ for all $\boldsymbol{L}$-sentences in logic with dependent sorts and without equality. We have

(2)(a) $M\sim_{\boldsymbol{L}} N \Longrightarrow M\equiv_{\boldsymbol{L}} N$.

This immediately follows from (1).

The word "equivalence" is used in " $\boldsymbol{L}$-equivalence" because of the relationship to the various notions of "equivalence" used in category theory; see later.

At this point, the reader may want to look at Appendix C, which may help understand the concept of $\boldsymbol{L}$-equivalence.

We now will exploit the fact that we have specified variables "with arbitrary parameters". In what follows, a *context* is a, not necessarily finite, set $\mathcal{Y}$ of variables such that $y\in\mathcal{Y}$, $x\in\mathrm{Dep}(y)$ imply that $x\in\mathcal{Y}$. When we want to refer to the previous sense of "context", we will say "finite context". A *specialization* is a map of contexts whose restriction to all finite subcontexts of the domain is a specialization in the original sense. Just as in case of finite contexts, there is a correspondence between contexts and functors $F:\boldsymbol{K}\to\mathrm{Set}$ which is an equivalence of the categories $\mathrm{Set}^{\boldsymbol{K}}$ and $\mathrm{Con}_{\infty}[\boldsymbol{K}]$, the category of all (small) contexts and specializations.

Given a context $\mathcal{Y}$ and an $\boldsymbol{K}$-structure M, the set $M[\mathcal{Y}]$ is defined by the formula (1), §1 (which was the definition of $M[\mathcal{Y}]$ for finite $\mathcal{Y}$). Given a formula φ with $\mathrm{Var}(\varphi)\subset\mathcal{Y}$, $M[\mathcal{Y}:\varphi]$ is the subset of $M[\mathcal{Y}]$ for which, for any $\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}]$,

$$\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\varphi] \iff \langle a_y\rangle_{y\in\mathcal{Y}'}\in M[\mathcal{Y}':\varphi]$$

for any (equivalently, some) finite context $\mathcal{Y}'$ with $\mathrm{Var}(\varphi)\subset\mathcal{Y}'\subset\mathcal{Y}$. As before, we write also $M\models\varphi[\langle a_y\rangle_{y\in\mathcal{Y}}]$ for $\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}:\varphi]$.

Suppose $\mathcal{X}$ is a context, M, N $\boldsymbol{L}$-structures, $\vec{a}=\langle a_x\rangle_{x\in\mathcal{X}}\in M[\mathcal{X}]$, $\vec{b}=\langle b_x\rangle_{x\in\mathcal{X}}\in N[\mathcal{X}]$. We write

$$(W,m,n):(M,\vec{a})\xleftrightarrow[\boldsymbol{L}]{}(N,\vec{b}) \qquad (3)$$

if $(W,m,n):M\xleftrightarrow[\boldsymbol{L}]{}N$ and there is $\langle s_x\rangle_{x\in\mathcal{X}}\in W[\mathcal{X}]$ such that $ms_x=a_x$ and $ns_x=b_x$ for all $x\in\mathcal{X}$. We write $(M,\vec{a})\sim_{\boldsymbol{L}}(N,\vec{b})$ if there is (W,m,n) such that (3) holds.

With $M, N, \mathcal{X}, \vec{a}, \vec{b}$ as above, we write $(M,\vec{a})\equiv_{\boldsymbol{L}}(N,\vec{b})$ for: for all $\boldsymbol{L}$-formulas φ with $\mathrm{Var}(\varphi)\subset\mathcal{X}$, we have $M\models\varphi[\langle a_x\rangle_{x\in\mathcal{X}}] \iff N\models\varphi[\langle b_x\rangle_{x\in\mathcal{X}}]$.

We have the following generalization of (2)(a) :

(2)(b) $(M,\vec{a})\sim_{\boldsymbol{L}}(N,\vec{b}) \implies (M,\vec{a})\equiv_{\boldsymbol{L}}(N,\vec{b})$;

this also follows immediately from (1) .

Let $\mathcal{Y}$ be a context, x a variable such that $x\notin\mathcal{Y}$ but $\mathcal{Y}\dot{\cup}\{x\}$ is a context (thus, $\mathrm{x}_{x,p}\in\mathcal{Y}$ for all $p\in\mathrm{K}_x|\boldsymbol{K}$), and let Φ be a set of formulas in logic with dependent sorts over $\boldsymbol{L}$ such that $\mathrm{Var}(\Phi)=\bigcup_{\varphi\in\Phi}\mathrm{Var}(\varphi)\subset\mathcal{Y}\dot{\cup}\{x\}$; such Φ is called a *$\mathcal{Y}$-set* (of formulas; with x any variable as described with respect to $\mathcal{Y}$). Let M be an $\boldsymbol{L}$-structure, and $\vec{a}=\langle a_y\rangle_{y\in\mathcal{Y}}\in M[\mathcal{Y}]$. We say that Φ is *satisfiable in* $(M,\vec{a})$ if there is $a\in|M|$ (more precisely, $a\in M K_x[\langle a_{x_{x,p}}\rangle_{p\in K_x|\boldsymbol{K}}]$) such that $M\models\varphi[\vec{a}, a/x]$ (of course, $\vec{a}, a/x$ stands for $\langle a'_y\rangle_{y\in\mathcal{Y}\dot\cup\{x\}}$ for which $a'_y=a_y$ for $y\in\mathcal{Y}$, and $a'_x=a$). Φ is *finitely satisfiable in* $(M, \vec{a})$ if every finite subset of Φ is satisfiable in $(M, \vec{a})$. M is said to be *$\mathcal{Y}$-$\boldsymbol{L}$-saturated* if for every $\vec{a}\in M[\mathcal{Y}]$ and every $\mathcal{Y}$-set Φ, if Φ is finitely satisfiable in $(M, \vec{a})$, then Φ is satisfiable in $(M, \vec{a})$.

Let κ be an infinite cardinal. We say that M is *κ, $\boldsymbol{L}$-saturated* if it is $\mathcal{Y}$-$\boldsymbol{L}$-saturated for every context $\mathcal{Y}$ with cardinality smaller than κ.

For saturated models for ordinary first order logic, see [CK]. In [MR2], one can find a detailed introduction to saturated and special models for multisorted logic; the basic facts and their proofs in the multisorted context do not essentially differ from the original one-sorted versions.

κ, $\boldsymbol{L}$-saturation is κ-saturation with respect to $\boldsymbol{L}$-formulas. Since $\boldsymbol{L}$-formulas form a part of the multisorted formulas over $|\boldsymbol{L}|$, it is clear that if M, an $\boldsymbol{L}$-structure, is κ-saturated as a structure for the similarity type $|\boldsymbol{L}|$, then M is κ, $\boldsymbol{L}$-saturated. More generally, suppose that we have "interpreted" $\boldsymbol{L}$ in a theory S in ordinary multisorted first-order logic; that is, we have a $\boldsymbol{C}$-valued $\boldsymbol{L}$-structure $I:\boldsymbol{L}\longrightarrow\boldsymbol{C}$, for $\boldsymbol{C}$ the Lindenbaum-Tarski category $[S]$ of S (see [MR]; $[S]$ is a Boolean category). Then if M is a model of S, or equivalently, $M:\boldsymbol{C}\to\mathrm{Set}$ is a coherent functor, and M is κ-saturated in the ordinary sense, then the $\boldsymbol{L}$-structure $M\restriction\boldsymbol{L}=MI:\boldsymbol{L}\to\mathrm{Set}$ is κ, $\boldsymbol{L}$-saturated.

By the *cardinality of* the structure M, $\#M$, we mean the cardinality of its underlying set $|M|$.

(4) Suppose the $\boldsymbol{L}$-structures M, N are κ, $\boldsymbol{L}$-saturated, and both are of cardinality $\leq\kappa$. Then the converses of (2)(a) and (2)(b) hold:

$$M\equiv_{\boldsymbol{L}}N \implies M\sim_{\boldsymbol{L}}N\ ;$$

and more generally, if $\mathcal{X}$ is a context of size $<\kappa$, $\vec{a}\in M[\mathcal{X}]$, $\vec{b}\in N[\mathcal{X}]$, then

$$(M, \vec{a}) \equiv_{\boldsymbol{L}} (N, \vec{b}) \implies (M, \vec{a}) \sim_{\boldsymbol{L}} (N, \vec{b}) \quad .$$

Proof.

For a given infinite cardinal κ, and a given context $\mathcal{X}$ of cardinality less than κ, let $\mathcal{U}=\mathcal{U}[\kappa, \mathcal{X}]$ be a context such that $\#\mathcal{U} = \kappa$, $\mathcal{X}\subset\mathcal{U}$, and for every sort X with $\mathrm{Var}(X)\subset\mathcal{U}$, the cardinality of the set of variables $x\in\mathcal{U}$ with $x:X$ is equal to κ. It is easy to see that such an $\mathcal{U}$ exists; we define contexts $\mathcal{U}_i$ by recursion on $i\le k$ for k the height of $\boldsymbol{K}$; let $\mathcal{U}_0=\emptyset$; if $\mathcal{U}_i$ has been defined, pick, for every sort X whose kind is of level i and for which $\mathrm{Var}(X) \subset \mathcal{U}_i$, a set V_X of variables $v:X$ such that $\#V_X=\kappa$, and let $\mathcal{U}_{i+1}$ be the union of $\mathcal{U}_i$ and all the V_X for all such X; if $k=\omega$, let $\mathcal{U}_\omega=\bigcup_{i<\omega}\mathcal{U}_i$; let $\mathcal{U}=\mathcal{U}_k$.

Next, enumerate $\mathcal{U}$ as a sequence $\langle u_\alpha\rangle_{\alpha<\kappa}$ in such a way that for each $\beta<\kappa$, $\langle u_\alpha\rangle_{\alpha<\beta}$ is a context; equivalently, such that for each $\beta<\kappa$, $\mathrm{Dep}(u_\beta)\subset\{u_\alpha:\alpha<\beta\}$. Note first of all that for any finite context $\mathcal{Y}$, there is an enumeration $\mathcal{Y}=\{y_i:i<n\}$ such that $\langle y_i\rangle_{i<j}$ is a context for all $j<n$; enumerate first the level-0 variables, next the level-1 ones, etc. Call such an enumeration of $\mathcal{Y}$ "good". Now, take first an arbitrary enumeration $\langle v_\alpha\rangle_{\alpha<\kappa}$ of $\mathcal{U}$; define the increasing sequence $\langle\beta_\alpha\rangle_{\alpha<\kappa}$ of ordinals and the partial enumeration $\langle u_\gamma\rangle_{\gamma<\beta_\alpha}$ by induction on α as follows. For a limit ordinal α, $\beta_\alpha=\lim_{\delta<\alpha}\beta_\delta$. For $\alpha=\delta+1$, let $\langle u_{\beta_\delta+i}\rangle_{i<n}$ be a good enumeration of $\mathrm{Dep}(v_\delta)\cup\{v_\delta\}$, and let $\beta_\alpha=\beta_\delta+n$.

For every sort X such that $\mathrm{Var}(X)\subset\mathcal{U}$, let $\langle u_{\alpha_{X,\,\nu}}\rangle_{\nu<\kappa}$ be an enumeration in increasing order of all u_α of sort X for which $u_\alpha\notin\mathcal{X}$. Finally, for any $\alpha<\kappa$, let $\nu[\alpha]$ be the ordinal ν for which $\alpha_{X,\,\nu} = \alpha$ where X is the sort of u_α.

Assume $\mathcal{X}$ is a context of size $<\kappa$, $\#M,\#N\le\kappa$, $\vec{a}=\langle a_x\rangle_{x\in\mathcal{X}}\in M[\mathcal{X}]$, $\vec{b}=\langle b_x\rangle_{x\in\mathcal{X}}\in N[\mathcal{X}]$, and $(M, \vec{a}) \equiv_{\boldsymbol{L}} (N, \vec{b})$. For any M-sort $MK(\langle c_p\rangle_{p\in K|\boldsymbol{K}})=MK(\vec{c})$, let us fix an enumeration $\langle e_\xi\rangle_{\xi<\lambda} = \langle e_{K,\,\vec{c},\,\xi}\rangle_{\xi<\lambda_{K,\,\vec{c}}}$ of the set $MK(\vec{c})$; here, $\lambda_{K,\,\vec{c}}\le\kappa$.

Consider $\mathcal{U}=\mathcal{U}[\kappa, \mathcal{X}]$ constructed above.

We define a context $\mathcal{Z}$, a subset of $\mathcal{U}$, by deciding, recursively on $\alpha<\kappa$, whether u_α belongs to $\mathcal{Z}$ or not; furthermore, we also define, for each $u_\alpha\in\mathcal{Z}$, elements $c_\alpha\in|M|$ and $d_\alpha\in|N|$. Let $\mathcal{Z}_\alpha$ denote the set of all u_β with $\beta<\alpha$ for which $u_\beta\in\mathcal{Z}$, and $\vec{c}[\alpha]$ be the sequence $\langle c_z\rangle_{z\in\mathcal{X}\cup\mathcal{Z}_\alpha}\in M[\mathcal{X}\cup\mathcal{Z}_\alpha]$ for which $c_x=a_x$ ($x\in\mathcal{X}$) and $c_{u_\beta}=c_\beta$ ($u_\beta\in\mathcal{Z}_\alpha$). Similarly, we have $\vec{d}[\alpha]\in N[\mathcal{X}\cup\mathcal{Z}_\alpha]$. The induction hypothesis of the construction is that

$$(M,\vec{c}[\alpha+1])\equiv_{\boldsymbol{L}}(M,\vec{d}[\alpha+1])\,. \tag{5}$$

Suppose $\alpha<\kappa$, and $\mathcal{Z}_\alpha$, $\vec{c}[\alpha]$, $\vec{d}[\alpha]$ have been defined so that, for all $\beta<\alpha$, $(M,\vec{c}[\beta+1])\equiv_{\boldsymbol{L}}(M,\vec{d}[\beta+1])$. Since in the definition of "$\equiv_{\boldsymbol{L}}$", formulas with finitely many free variables are involved, we can conclude that

$$(M,\vec{c}[\alpha])\equiv_{\boldsymbol{L}}(M,\vec{d}[\alpha])\,. \tag{6}$$

Look at the variable u_α and its sort X. If $u_\alpha\in\mathcal{X}$, we let $u_\alpha\in\mathcal{Z}$, $c_\alpha=a_{u_\alpha}$, $d_\alpha=b_{u_\alpha}$. (5) is now an automatic consequence of (6).

If not all the variables in X (which are u_β's for $\beta<\alpha$) are in $\mathcal{Z}$, then $u_\alpha\notin\mathcal{Z}$, and we are finished with the stage α.

Assume that $u_\alpha\notin\mathcal{X}$ and all the variables in X are in $\mathcal{Z}$. Look at the ordinal $\nu=\nu[\alpha]$; write ν in the form $\nu=2\cdot\mu$ or $\nu=2\cdot\mu+1$ as the case may be. Let first $\nu=2\cdot\mu$. With $X=K(\langle u_{\beta_p}\rangle_{p\in K|\boldsymbol{K}})$, consider the M-sort $MK(\langle c_{\beta_p}\rangle_{p\in K|\boldsymbol{K}})=MK(\vec{c})$ and its previously fixed enumeration $\langle e_\xi\rangle_{\xi<\lambda}$ ($=\langle e_{K,\vec{c},\xi}\rangle_{\xi<\lambda_{K,\vec{c}}}$). If $\mu\geq\lambda$, then again $u_\alpha\notin\mathcal{Z}$. If, however, $\mu<\lambda$, then $u_\alpha\in\mathcal{Z}$. Moreover, $c_\alpha \overset{\text{def}}{=} e_\mu$.

Let Φ be the $\mathcal{X}\cup\mathcal{Z}_\alpha$-set of all formulas φ with $\mathrm{Var}(\varphi)\subset\mathcal{X}\cup\mathcal{Z}_\alpha\dot{\cup}\{u_\alpha\}$ for which $M\models\varphi[\vec{c}[\alpha],e_\mu/u_\alpha]$. I claim that Φ is finitely satisfiable in $(N,\vec{d}[\alpha])$. Let Ψ be a finite subset of Φ. For $\varphi=\bigwedge\Psi$, we have $M\models\varphi[\vec{c}[\alpha],e_\mu/u]$, hence, $M\models(\exists u_\alpha\varphi)[\vec{c}[\alpha]]$ (note that $\exists u_\alpha\varphi$ is well-formed, since for every $z\in\mathrm{Var}(\varphi)$, $z\neq u_\alpha$, we have $z\in\mathcal{X}\cup\mathcal{Z}_\alpha$, hence $\mathrm{Dep}(z)\subset\mathcal{X}\cup\mathcal{Z}_\alpha$, and $u_\alpha\notin\mathrm{Dep}(z)$). As a consequence, by (6),

$N \models (\exists u\varphi)[\vec{d}[\alpha]]$. This means that Ψ is satisfiable in $(N, \vec{d}[\alpha])$ as desired.

Since $\#(\mathcal{X} \cup \mathcal{Z}_\alpha) < \kappa$, and N is κ, $\boldsymbol{L}$-saturated, Φ is satisfiable in $(N, \vec{d}[\alpha])$, by $d_\alpha \in NK(\langle d_{\beta_p} \rangle_{p \in K|\boldsymbol{K}})$, say. The choice of Φ ensures that (5) holds.

In case $\nu = 2 \cdot \mu + 1$, we proceed similarly, with the roles of M and N interchanged.

With the construction completed, we put $\mathcal{Z} = \bigcup_{\alpha<\kappa} \mathcal{Z}_\alpha$. We let W be the functor $F_{\mathcal{Z}}: \boldsymbol{K} \to \mathrm{Set}$ associated with the context $\mathcal{Z}$ (see §4). $m: W \to M \restriction \boldsymbol{K}$, $n: W \to N \restriction \boldsymbol{K}$ are defined by $m(u_\alpha) = c_\alpha$, $n(u_\alpha) = d_\alpha$ ($u_\alpha \in \mathcal{Z}$) . The definition ensures that $\mathcal{X} \subset \mathcal{Z}$ and $m(x) = a_x$, $n(x) = b_x$ ($x \in \mathcal{X}$) .

Let us see that m is very surjective. Let $K \in \boldsymbol{K}$. $W[K]$ is the set of all tuples $\langle z_p \rangle_{p \in K|\boldsymbol{K}}$ for which each $z_p \in \mathcal{Z}$, and $X = K(\langle z_p \rangle_{p \in K|\boldsymbol{K}})$ is a (well-formed) sort; $WK(\langle z_p \rangle_{p \in K|\boldsymbol{K}})$ is the set of all $z \in \mathcal{Z}$ such that $z : X$. So, assume that $X = K(\langle z_p \rangle_{p \in K|\boldsymbol{K}}) = K(\langle u_{\beta_p} \rangle_{p \in K|\boldsymbol{K}})$ is a sort, and

$$a \in MK(\langle mz_p \rangle_{p \in K|\boldsymbol{K}}) = MK(\langle c_{\beta_p} \rangle_{p \in K|\boldsymbol{K}}) = MK(\vec{c}) \quad .$$

Then $a = e_{K, \vec{c}, \mu}$ for some $\mu < \lambda_{K, \vec{c}}$, and for $\alpha = \alpha_{X, 2 \cdot \mu}$, the construction at stage α puts $u_\alpha : X$ into $\mathcal{Z}$; that is, $u_\alpha \in WK(\langle z_p \rangle_{p \in K|\boldsymbol{K}})$, with $a = c_\alpha = mu_\alpha$ as desired.

The fact that n is very surjective is seen analogously.

We have that $(W, m, n) : M \xleftrightarrow[\boldsymbol{L}]{} N$, since (1") is a consequence of (5) being true for all $\alpha < \kappa$; one has to apply (5) to atomic formulas.

This completes the proof of (4).

Let $\boldsymbol{C}$ be a small Boolean category. By a *model of* $\boldsymbol{C}$ we mean a functor $M: \boldsymbol{C} \to \mathrm{Set}$ preserving the Boolean structure (that is, M is a coherent functor). We write $M \models \boldsymbol{C}$ to say that M is a model of $\boldsymbol{C}$.

There is a theory $T_{\mathbf{C}}=(L_{\mathbf{C}},\Sigma_{\mathbf{C}})$ in multisorted first-order logic, with $L_{\mathbf{C}}$ the underlying graph of $\mathbf{C}$, such that the models of $\mathbf{C}$ are *the same* as the models of $T_{\mathbf{C}}$ (note that both the models of $\mathbf{C}$ and the models of $T_{\mathbf{C}}$ are particular diagrams $L_{\mathbf{C}}\to \mathrm{Set}$). Moreover, for any subobject $\varphi\in S_{\mathbf{C}}(A)$, $A\in\mathbf{C}$, there is a (simply defined) $L_{\mathbf{C}}$-formula $\underline{\varphi}(x)$ with a single free variable $x{:}A$ such that for every $M\models\mathbf{C}$ and $a\in M(A)$, $M\models\underline{\varphi}[a]$ ($\Longleftrightarrow M\models\underline{\varphi}[a/x]$) iff $a\in M(\varphi)$ ($\subset M(A)$). See [MR].

For $\sigma\in S(1_{\mathbf{C}})$, a subobject of the terminal object in $\mathbf{C}$, we write $M\models\sigma$ for $M(\sigma)=1$ in Set . We will call a subobject of $1_{\mathbf{C}}$ a *sentence* in $\mathbf{C}$.

Let $I{:}\mathbf{L}\to\mathbf{C}$ a $\mathbf{C}$-valued $\mathbf{L}$-structure (in particular, $I{:}\mathbf{L}\to\mathbf{C}$ is a functor from $\mathbf{L}$ as a category). When $\mathbf{C}$ is the Lindenbaum-Tarski category $[S]$ of a theory $S=(L_S,\Sigma_S)$ in ordinary multisorted logic (see [MR] or [M?]), then such an I is what we should consider an *interpretation* of the DS vocabulary $\mathbf{L}$ in the theory S. An example is obtained by taking $S=(|\mathbf{L}|,\Sigma[\mathbf{L}])$ (for $\Sigma[\mathbf{L}]$, see §1), and for $I{:}\mathbf{L}\to[S]$ the $[S]$-structure defined by $I(A)=[a{:}\mathbf{t}]$ for $A\in\mathbf{L}$ where $a{:}A$, and for $f{:}A\to B$, $I(f)=\langle a\mapsto b{:}\,fa=b\rangle{:}[a{:}\mathbf{t}]\to[b{:}\mathbf{t}]$. $I{:}\mathbf{L}\to[S]$ is the *canonical* interpretation of logic with dependent types in multisorted logic. In this case, for any formula φ of FOLDS over $\mathbf{L}$, with $\mathrm{Var}(\varphi)\subset\mathcal{X}$, we have $I[\mathcal{X}{:}\varphi]=m^*[\mathcal{X}{:}\varphi^*]$; here, $m{:}I[\mathcal{X}{:}\varphi]\rightarrowtail\{\mathcal{X}\}\overset{=}{\mathrm{def}}\prod_{x\in\mathcal{X}}K_x$ is the canonical monomorphism, m^* denotes pulling back along m; φ^* was defined in §1.

For a general $I{:}\mathbf{L}\to\mathbf{C}$, and for an $\mathbf{L}$-sentence θ, let us write $I(\theta)$ for the sentence $I[\emptyset{:}\theta]$ of $\mathbf{C}$. In case $\mathbf{C}=[S]$, $I(\theta)$ also stands for any one of the S-equivalent L_S-sentences which are the representatives of the $\mathbf{C}$-subobject $I(\theta)$.

When $M\models\mathbf{C}$, the composite $MI{:}\mathbf{L}\to\mathrm{Set}$ is an $\mathbf{L}$-structure. We also write $M\upharpoonright\mathbf{L}$ for MI ; $M\upharpoonright\mathbf{L}$ is the *$\mathbf{L}$-reduct of* M (via I).

Let $\mathbf{C}$ and $\mathbf{D}$ be small Boolean categories, $I{:}\mathbf{L}\to\mathbf{C}$ and $J{:}\mathbf{L}\to\mathbf{D}$. Notational conventions introduced above for $I{:}\mathbf{L}\to\mathbf{C}$ are valid for $J{:}\mathbf{L}\to\mathbf{D}$, *mutatis mutandis.*

(7)(a) Assume that σ is a sentence of $\mathbf{C}$, τ a sentence of $\mathbf{D}$, and for all $M\models\mathbf{C}$, $N\models\mathbf{D}$,

$$M \models \sigma \ \ \& \ \ M{\restriction}\boldsymbol{L} \sim_{\boldsymbol{L}} N{\restriction}\boldsymbol{L} \ \ \Longrightarrow \ \ N \models \tau \ .$$

Then there is an $\boldsymbol{L}$-sentence θ in logic with dependent sorts without equality such that for all $M \models \boldsymbol{C}$, $N \models \boldsymbol{D}$, we have

$$M \models \sigma \Longrightarrow M{\restriction}\boldsymbol{L} \models \theta \qquad \text{and} \qquad N{\restriction}\boldsymbol{L} \models \theta \Longrightarrow N \models \tau \ .$$

For a more general formulation, consider a finite $\boldsymbol{L}$-context $\mathcal{X}$, and the object $I[\mathcal{X}] \in \boldsymbol{C}$. $I[\mathcal{X}]$ is defined as a finite limit in $\boldsymbol{C}$; see the end of §1; let $\pi_{[x]} : I[\mathcal{X}] \to I(\mathrm{K}_x)$ be the limit projections ($x \in \mathcal{X}$). Given any $M \models \boldsymbol{C}$, we have similar projections $\rho_{[x]} : (M{\restriction}\boldsymbol{L})[\mathcal{X}] \to MI(\mathrm{K}_x)$ in $\mathtt{Set}$, and a canonical isomorphism $\mu : (M{\restriction}\boldsymbol{L})[\mathcal{X}] \xrightarrow{\cong} M(I[\mathcal{X}])$ making each diagram

$$\begin{array}{ccccc} (M{\restriction}\boldsymbol{L})[\mathcal{X}] & & \xrightarrow[\cong]{\ \ \mu\ \ } & & M(I[\mathcal{X}]) \\ & {\scriptstyle \rho_{[x]}} \searrow & \circ & \swarrow {\scriptstyle M(\pi_{[x]})} & \\ & & MI(\mathrm{K}_x) & & \end{array} \tag{7'}$$

commute. If $\vec{a} = \langle a_x \rangle_{x \in \mathcal{X}} \in (M{\restriction}\boldsymbol{L})[\mathcal{X}]$, we write $\langle \vec{a} \rangle$ for $\mu(\vec{a}) \in M(I[\mathcal{X}])$. Once again, similar conventions apply in the context of $J : \boldsymbol{L} \to \boldsymbol{D}$.

(7)(b) Assume that $\mathcal{X}$ is a finite $\boldsymbol{L}$-context, $\sigma \in \mathrm{S}_{\boldsymbol{C}}(I[\mathcal{X}])$, $\tau \in \mathrm{S}_{\boldsymbol{D}}(J[\mathcal{X}])$, and for all $M \models \boldsymbol{C}$, $N \models \boldsymbol{D}$, $\vec{a} \in (M{\restriction}\boldsymbol{L})[\mathcal{X}]$, $\vec{b} \in (N{\restriction}\boldsymbol{L})[\mathcal{X}]$,

$$\langle \vec{a} \rangle \in M(\sigma) \ \ \& \ \ (M{\restriction}\boldsymbol{L}, \vec{a}) \sim_{\boldsymbol{L}} (N{\restriction}\boldsymbol{L}, \vec{b}) \ \ \Longrightarrow \ \ \langle \vec{b} \rangle \in N(\tau) \ . \tag{8}$$

Then there is an $\boldsymbol{L}$-formula θ in logic with dependent sorts without equality with $\mathrm{Var}(\varphi) \subset \mathcal{X}$ such that

$$\sigma \leq_{I[\mathcal{X}]} I[\mathcal{X}:\theta] \ , \qquad J[\mathcal{X}:\theta] \leq_{J[\mathcal{X}]} \tau \ . \tag{8'}$$

Note that (8') may be written equivalently as for all $M \models \boldsymbol{C}$, $N \models \boldsymbol{D}$, $\vec{a} \in (M \upharpoonright \boldsymbol{L})[\mathcal{X}]$ and $\vec{b} \in (N \upharpoonright \boldsymbol{L})[\mathcal{X}]$,

$$\langle \vec{a} \rangle \in M(\sigma) \Longrightarrow M \upharpoonright I \models \theta[\vec{a}] \text{ and } N \upharpoonright J \models \theta[\vec{b}] \Longrightarrow \langle \vec{b} \rangle \in N(\tau) .$$

Proof. Let us extend the vocabulary $L_{\boldsymbol{C}}$ to $L_{\boldsymbol{C}}(c)$ by adding a single new individual constant c of sort $A \overset{=}{\text{def}} I[\mathcal{X}]$. For any $\varphi \in S_{\boldsymbol{C}}(A)$, let $\varphi(c)$ denote $\underline{\varphi}(c/x)$, the result of substituting c for x in $\underline{\varphi}(x)$. For an $\boldsymbol{L}$-formula θ with $\mathrm{Var}(\theta) \subset \mathcal{X}$, let $\theta(c)$ stand for $(I[\mathcal{X}:\theta])(c)$. Similarly, we introduce $d:B \overset{=}{\text{def}} J[\mathcal{X}]$; for $\psi \in S_{\boldsymbol{D}}(B)$, $\psi(d)$ and for θ as before, $\theta(d)$.

Let Θ be the set of all $\boldsymbol{L}$-formulas θ with $\mathrm{Var}(\theta) \subset \mathcal{X}$ such that $\sigma \le_A I[\mathcal{X}:\theta]$. Consider the set $\Sigma \overset{=}{\text{def}} \Sigma_{\boldsymbol{D}} \cup \{\theta(d) : \theta \in \Theta\}$ of $L_{\boldsymbol{D}}(d)$-sentences. I claim that

$$(L_{\boldsymbol{D}}(d), \Sigma) \models \tau(d) . \tag{9}$$

Once the claim is proved, by compactness there are finitely many $\theta_i \in \Theta$ ($i<n$) such that $(L_{\boldsymbol{D}}(d),\ \Sigma_{\boldsymbol{D}} \cup \{\theta_i(d) : i<n\}) \models \tau(d)$, which means, for $\theta = \bigwedge_{i<n} \theta_i \in \Theta$ that $(L_{\boldsymbol{D}}(d), \Sigma_{\boldsymbol{D}}) \models \theta(d) \to \tau(d)$, that is, $(L_{\boldsymbol{D}}(d), \Sigma_{\boldsymbol{D}}) \models \forall x{:}B.(\underline{\theta}(x) \to \underline{\tau}(x))$, which means $J[\mathcal{X}:\theta] \le_B \tau$; thus, it is enough to see the claim.

Assume that there is an infinite cardinal $\lambda \ge \#L_{\boldsymbol{C}}$ such that $\lambda^+ = 2^\lambda$ (see below for the legitimacy of this assumption). Let $\kappa = \lambda^+$. According to the existence theorem for saturated models (see [CK], [MR2]), any $L_{\boldsymbol{D}}(d)$-structure is elementarily equivalent to a κ-saturated structure of cardinality $\le \kappa$. Therefore, to show (9), take $(N, b/d)$, a κ-saturated model of cardinality $\le \kappa$ of $(L_{\boldsymbol{D}}(d), \Sigma)$, to show $(N, b/d) \models \tau(d)$.

Let Φ be the set of $\boldsymbol{L}$-formulas φ with $\mathrm{Var}(\varphi) \subset \mathcal{X}$ such that $b \in N(I[\mathcal{X}:\varphi]) \subset NB$; for every $\boldsymbol{L}$-formula φ with $\mathrm{Var}(\varphi) \subset \mathcal{X}$, exactly one of φ, $\neg\varphi$ belongs to Φ. Since $(N, b/d)$ is a model of $(L_{\boldsymbol{D}}(d), \Sigma)$, with Σ defined as it is, we have $\Theta \subset \Phi$. I make the subclaim that the theory

$$(L_{\boldsymbol{C}}(c),\ \Sigma_{\boldsymbol{C}} \cup \{\sigma(c)\} \cup \{\varphi(c) : \varphi \in \Phi\}) \tag{10}$$

is consistent. Consider a finite subset $\{\varphi_i : i<n\}$ of Φ. If

$(L_{\boldsymbol{C}}(c),\ \Sigma_{\boldsymbol{C}}\cup\{\sigma(c)\}\cup\{\varphi_i(c):i<n\})$ were not consistent, then we would have, for $\varphi=\bigwedge_{i<n}\varphi_i\in\Phi$, that $\sigma\leq_A I[\mathcal{X}:\neg\varphi]$, which would mean that $\neg\varphi\in\Theta\subset\Phi$, contradicting $\varphi\in\Phi$. This shows the subclaim.

Now, let $(M,\ a/c)$ be a κ-saturated model of (10) of cardinality $\leq\kappa$. Let $\vec{a}\in(M\restriction\boldsymbol{L})[\mathcal{X}]$ such that $a=\langle\vec{a}\rangle$ (see (7')) and $\vec{b}\in(N\restriction\boldsymbol{L})[\mathcal{X}]$ such that $b=\langle\vec{b}\rangle$. Then, for any $\boldsymbol{L}$-formula θ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that $M\restriction\boldsymbol{L}\models\theta[\vec{a}]$, we have $\neg\theta\notin\Phi$, hence $\theta\in\Phi$, hence $N\restriction\boldsymbol{L}\models\theta[\vec{b}]$. This says that $(M\restriction\boldsymbol{L},\ \vec{a})\equiv_{\boldsymbol{L}}(N\restriction\boldsymbol{L},\ \vec{b})$. By (4), $(M\restriction\boldsymbol{L},\ \vec{a})\sim_{\boldsymbol{L}}(N\restriction\boldsymbol{L},\ \vec{b})$, and by the (8), the assumption of the proposition, $\langle\vec{b}\rangle\in N(\tau)$, that is, $N\models\underline{\tau}[\langle\vec{b}\rangle/x]$, that is, $(N,\ b/d)\models\tau(d)$ as promised.

The set-theoretic assumption used in the proof is redundant, by a general absoluteness theorem (arithmetic statements are absolute with respect to the constructible universe, in which the Generalized Continuum Hypothesis (GCH) holds; see [J]). On the other hand, one may use "special" models in place of saturated ones, and avoid the use of GCH; see [CK], [MR2].

(11)(a) Assume that S is a theory in multisorted logic, and $I:\boldsymbol{L}\to[S]$ is an interpretation of the DSV $\boldsymbol{L}$ in S. Suppose that the class $\mathrm{Mod}(S)$ of models of S is invariant under $\boldsymbol{L}$-equivalence in the sense that for any L_S-structures M and N, $M\in\mathrm{Mod}(S)$ and $M\restriction\boldsymbol{L}\sim_{\boldsymbol{L}}N\restriction\boldsymbol{L}$ imply that $N\in\mathrm{Mod}(S)$. Then S is $\boldsymbol{L}$-axiomatizable; that is, for a set Θ of $\boldsymbol{L}$-sentences, $\mathrm{Con}_{L_S}(\{I(\theta):\theta\in\Theta\})=\mathrm{Con}_{L_S}(\Sigma_S)$; here, $\mathrm{Con}_L(\Phi)$ is the set of L-sentences that are consequences of the theory $(L,\ \Phi)$.

Note that the conclusion can also be expressed by saying that for any L_S-structure M, $M\models\Sigma_S$ iff $M\restriction\boldsymbol{L}\models\Theta$.

(11)(b) More generally, assume, in addition to S and $I:\boldsymbol{L}\to[S]$, a theory T in a language extending that of S ($L_S\subset L_T$) such that

> for any $M,\ N\in\mathrm{Mod}(T)$, $M\restriction L_S\in\mathrm{Mod}(S)$ and $M\restriction\boldsymbol{L}\sim_{\boldsymbol{L}}N\restriction\boldsymbol{L}$ imply that

$N\restriction \mathrm{L}_S \in \mathrm{Mod}(S)$.

Then, there is a set Θ of $\boldsymbol{L}$-sentences such that, for any $M\models T$, $M\models\Sigma_S$ iff $M\restriction\boldsymbol{L}\models\Theta$.

(11)(a) is the special case when $T=(\mathrm{L}_S, \emptyset)$.

Proof of (11)(b). For any $\tau\in\Sigma_S$, $M\models T$ and $N\models T$, we have

$$M\models\Sigma_S \ \&\ M\restriction\boldsymbol{L}\sim_{\boldsymbol{L}}N\restriction\boldsymbol{L} \implies N\models\tau .$$

By appropriately coding the condition $M\restriction\boldsymbol{L}\sim_{\boldsymbol{L}}N\restriction\boldsymbol{L}$ in first order logic with suitable additional primitives, and by applying compactness, we can find $\sigma[\tau]$, a finite conjunction of elements of Σ_S, such that for any $M\models T$ and $N\models T$,

$$M\models\sigma[\tau] \ \&\ M\restriction\boldsymbol{L}\sim_{\boldsymbol{L}}N\restriction\boldsymbol{L} \implies N\models\tau .$$

Then by (7)(a), applied to $\boldsymbol{C}=\boldsymbol{D}=[T]$, and $I=J:\boldsymbol{L}\xrightarrow{I}[S]\xrightarrow{\mathrm{incl}}[T]$, we can find $\theta[\tau]$, an $\boldsymbol{L}$-sentence, such that $T\models\sigma[\tau]\longrightarrow I(\theta[\tau])$, $T\models I(\theta[\tau])\longrightarrow\tau$. Clearly, $\Theta=\{\theta[\tau]:\ \tau\in\Sigma_S\}$ is then appropriate for the assertion.

We leave it to the reader to formulate a version of (11) with formulas in a given context $\mathcal{X}$ instead of sentences.

The following, which is a special case of (7)(b), says that a first-order property invariant under $\boldsymbol{L}$-equivalence is expressible in logic with dependent types over $\boldsymbol{L}$.

(12) Let $I:\boldsymbol{L}\to\boldsymbol{C}$ be as before. Assume that $\mathcal{X}$ is a finite $\boldsymbol{L}$-context, $\sigma\in\mathrm{S}(I[\mathcal{X}])$, and for all $M, N\models\boldsymbol{C}$ and $\vec{a}\in(M\restriction\boldsymbol{L})[\mathcal{X}]$, $\vec{b}\in(N\restriction\boldsymbol{L})[\mathcal{X}]$,

$$\langle\vec{a}\rangle\in M(\sigma) \ \ \&\ \ (M\restriction\boldsymbol{L},\vec{a})\sim_{\boldsymbol{L}}(N\restriction\boldsymbol{L},\vec{b}) \implies \langle\vec{b}\rangle\in N(\sigma) .$$

Then there is an $\boldsymbol{L}$-formula θ in logic with dependent sorts without equality with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that $\sigma=_{I[\mathcal{X}]}I[\mathcal{X}:\theta]$.

The notion of $\mathbf{L}$-equivalence as defined is relevant to FOLDS without equality. However, frequently we deal with FOLDS with restricted equality. As explained in §1, when M is an $\mathbf{L}$-structure, it can be considered as an $\mathbf{L}^{\mathrm{eq}}$-structure, with the additional relations E_K interpreted as true equality; let us write M for the resulting "standard" $\mathbf{L}^{\mathrm{eq}}$-structure as well. What does it mean to have an equivalence $(W, m, n) : M \xleftrightarrow[\mathbf{L}^{\mathrm{eq}}]{} N$ for $\mathbf{L}$-structures M, N? Clearly, this is to say that $(W, m, n) : M \xleftrightarrow[\mathbf{L}]{} N$ and, for any maximal kind K, and $\vec{c} \in W[K]$, $c_1, c_2 \in WK(m\vec{c})$, we have that $mc_1 = mc_2$ iff $nc_1 = nc_2$. Let us write $(W, m, n) : M \xleftrightarrow[\mathbf{L}]{\approx} N$ for $(W, m, n) : M \xleftrightarrow[\mathbf{L}^{\mathrm{eq}}]{} N$, and let us call such (W, m, n) an $\mathbf{L}$, $\approx$-equivalence; also, write $M \approx_{\mathbf{L}} N$ for $M \sim_{\mathbf{L}^{\mathrm{eq}}} N$; note that throughout, M and N are $\mathbf{L}$-structures.

Let us define $M \equiv_{\mathbf{L}^=} N$ as we did $M \equiv_{\mathbf{L}} N$ above, except that we refer to logic with equality. Then, using the translation $\varphi \mapsto \hat{\varphi}$ mentioned in §1, we obviously have $M \equiv_{\mathbf{L}^=} N \iff M \equiv_{\mathbf{L}^{\mathrm{eq}}} N$. Thus, by (2)(a) we have

(13) For $\mathbf{L}$-structures M and N, $M \approx_{\mathbf{L}} N \implies M \equiv_{\mathbf{L}^=} N$.

$\mathbf{L}$,$\approx$-equivalences can be "normalized" in a certain way, which will be useful for us later.

Let $U, V \in \mathrm{Set}^{\mathbf{K}}$. A very surjective morphism $f : U \to V$ is *normal* if for any maximal kind K, and any $\vec{a} \in U[K]$, " f is 1-1 in the fiber over $\vec{a}$ ", that is, if $b, c \in UK(\vec{a})$, then $f(b) = f(c)$ implies $b = c$. Together with the very surjective condition, this says that f induces a bijection $UK(\vec{a}) \xrightarrow{\cong} VK(f\vec{a})$.

Let M, N be $\mathbf{L}$-structures. A *normal* $\mathbf{L}$,$\approx$-equivalence $(W, m, n) : M \xleftrightarrow[\mathbf{L}]{\approx} N$ is an $\mathbf{L}$,$\approx$-equivalence in which both m and n are normal. We have the fact

(14) For any $\mathbf{L}$-structures M, N, if $M \approx_{\mathbf{L}} N$, then there is a normal $\mathbf{L}$,$\approx$-equivalence

$(W, m, n):M\overset{\approx}{\underset{\boldsymbol{L}}{\leftrightarrow}}N$.

The argument is as follows. Start with any $\boldsymbol{L}$,≈-equivalence $(W, m, n):M\overset{\approx}{\underset{\boldsymbol{L}}{\leftrightarrow}}N$. Define $W'\in\mathrm{Set}^{\boldsymbol{K}}$ by setting $W'K=WK$ for all $K\in\boldsymbol{K}$ except the maximal ones; for a maximal K, $W'K\underset{\mathrm{def}}{=}WK/\sim$, where $\sim$ is the equivalence relation on WK for which $b\sim c$ iff b and c are over the same $\vec{a}\in W[K]$, and $m(b)=m(c)$. When in this definition, we replace m by n, the result is the same; this is because (W, m, n) being an $\boldsymbol{L}$,≈-equivalence, $m(b)=m(c)$ iff $n(b)=n(c)$ for b, c over the same element in $W[K]$. For an arrow $p:K\to K_p$, $W'(p)=W(p)$ when K is not maximal (in which case K_p is not maximal either); and for K maximal, $(W'p)(b/\sim)=(Wp)(b)$; the latter is well-defined, since by the definition of $\sim$, if $b\sim c$, then $(Wp)(b)=(Wp)(c)$. Clearly, $W':\boldsymbol{K}\to\mathrm{Set}$ is well-defined, and we have obvious maps $p:W\to W'$, $m':W'\to M\restriction\boldsymbol{K}$, $n':W'\to N\restriction\boldsymbol{K}$ such that

$$\begin{array}{ccccc}
 & & W & & \\
 & \overset{m}{\swarrow} & \downarrow & \overset{n}{\searrow} & \\
M\restriction\boldsymbol{K} & \circ & p & \circ & N\restriction\boldsymbol{K} \\
 & \underset{m'}{\nwarrow} & \downarrow & \underset{n'}{\nearrow} & \\
 & & W' & &
\end{array}$$

I claim that $(W', m', n'):M\overset{\approx}{\underset{\boldsymbol{L}}{\leftrightarrow}}N$; the normality condition is clearly satisfied. Consider a relation R in $\boldsymbol{L}$. In the commutative diagram

$$\begin{array}{ccccc}
(m^{*}M)R & \xrightarrow{q} & (m'^{*}M)R & \longrightarrow & MR \\
\downarrow & & \downarrow & & \downarrow \\
W[R] & \xrightarrow[p_{[R]}]{} & W'[R] & \xrightarrow[m'_{[R]}]{} & M[R]
\end{array}$$

the outside rectangle and the right-hand square are pullbacks. It follows that the left-hand square is a pullback too. Obviously, $p_{[R]}$ is surjective. It follows that q is surjective. This determines the subobject $(m'^{*}M)R\rightarrowtail W'[R]$ as the image of $(m^{*}M)R\rightarrowtail W[R]$ under $p_{[R]}$. Switching to N from M, $(n'^{*}N)R\rightarrowtail W'[R]$ is the image of $(n^{*}N)R\rightarrowtail W[R]$

under $p_{[R]}$. Since $(m^* M)R =_{W[R]} (n^* N)R$, it follows that
$(m'^* M)R =_{W'[R]} (n'^* N)R$ as desired. The additional condition concerning equality is clearly satisfied.

Notice that the above proof works for an essentially arbitrary $\boldsymbol{C}$ in place of $\mathtt{Set}$.

Note that if $m: W \to M \restriction \boldsymbol{K}$ is normal, then $m^* M$ formed from M as a standard $\boldsymbol{L}^{\mathrm{eq}}$-structure is a standard $\boldsymbol{L}^{\mathrm{eq}}$-structure too. Put in another way, the standard fiberwise equality relations on the maximal kinds in $m^* M$ are formed by the same pullback operation from the corresponding relation on M as any primitive $\boldsymbol{L}$-relation.

We have the following variant of (12).

(15) Let $\boldsymbol{C}$ be a small Boolean category, $I: \boldsymbol{L} \to \boldsymbol{C}$. Assume that $\mathcal{X}$ is a finite $\boldsymbol{L}$-context, $\sigma \in S(I[\mathcal{X}])$, and for all $M, N \models \boldsymbol{C}$ and $\vec{a} \in (M \restriction \boldsymbol{L})[\mathcal{X}]$, $\vec{b} \in (N \restriction \boldsymbol{L})[\mathcal{X}]$,

$$\langle \vec{a} \rangle \in M(\sigma) \;\;\&\;\; (M \restriction \boldsymbol{L}, \vec{a}) \approx_{\boldsymbol{L}} (N \restriction \boldsymbol{L}, \vec{b}) \quad \Longrightarrow \quad \langle \vec{b} \rangle \in N(\sigma) \quad .$$

Then there is an $\boldsymbol{L}$-formula θ in logic with dependent sorts *with* equality with $\mathrm{Var}(\theta) \subset \mathcal{X}$ such that $\sigma =_{I[\mathcal{X}]} I[\mathcal{X}:\theta]$.

Proof. By definition, for each maximal K , $I[\mathrm{E}_K] = I(K) \times_{I[K]} I(K)$. Let us form $I^{\mathrm{eq}}: \boldsymbol{L}^{\mathrm{eq}} \longrightarrow \boldsymbol{C}$ extending $I: \boldsymbol{L} \to \boldsymbol{C}$ by specifying that, $I^{\mathrm{eq}}(\mathrm{E}_K) = I[E_K]$, with $I^{\mathrm{eq}}(\mathrm{e}_{K0}) = I^{\mathrm{eq}}(\mathrm{e}_{K1}) = 1_{I[\mathrm{E}_K]}$. We apply (12) to $I^{\mathrm{eq}}: \boldsymbol{L}^{\mathrm{eq}} \to \boldsymbol{C}$. For $M \models \boldsymbol{C}$, $M \restriction \boldsymbol{L}^{\mathrm{eq}} = M \circ I^{\mathrm{eq}}$ is, clearly, the same as $M \restriction \boldsymbol{L}$ as a standard $\boldsymbol{L}^{\mathrm{eq}}$-structure. Thus,

$$(M \restriction \boldsymbol{L}^{\mathrm{eq}}, \vec{a}) \sim_{\boldsymbol{L}^{\mathrm{eq}}} (N \restriction \boldsymbol{L}^{\mathrm{eq}}, \vec{b}) \quad \Longleftrightarrow \quad (M \restriction \boldsymbol{L}, \vec{a}) \approx_{\boldsymbol{L}} (N \restriction \boldsymbol{L}, \vec{b}) \quad .$$

Thus, from the hypothesis of (15), that of (12) follows. By (12), we have some θ in FOLDS without equality over $\boldsymbol{L}^{\mathrm{eq}}$ such that $\sigma =_{I[\mathcal{X}]} I^{\mathrm{eq}}[\mathcal{X}:\theta]$; but clearly, for θ' in FOLDS with equality over $\boldsymbol{L}$ such that $\hat{\theta}' = \theta$, we have $I[\mathcal{X}:\theta'] = I^{\mathrm{eq}}[\mathcal{X}:\theta]$; thus $\sigma =_{I[\mathcal{X}]} I[\mathcal{X}:\theta']$ as required.

§6. Equivalence of categories, and of diagrams of categories

The simplest application of the results of the last section is to invariance under equivalence of categories of first order properties of diagrams of objects and arrows in a category. In what follows, until further notice, $\boldsymbol{L}$ stands for $\mathbf{L}_{\mathtt{cat}}$, the DSV for categories introduced in §1; a category $\boldsymbol{A}$ may be regarded an $\boldsymbol{L}$-structure. A context of variables for $\boldsymbol{L}$ is essentially a functor $\boldsymbol{K}=\mathbf{L}_{\mathtt{graph}}\to\mathtt{Set}$, that is, a graph; we are mainly interested in finite contexts, although for the notions to be introduced next, there is no need to confine attention to finite contexts.

For a context $\mathcal{X}$, an *augmented* category of type $\mathcal{X}$ is a pair $(\boldsymbol{A}, \vec{a})$, with $\boldsymbol{A}$ a category, and $\vec{a}\in\boldsymbol{A}[\mathcal{X}]$ (that is, $\vec{a}$ a diagram of type the graph $\mathcal{X}$). Until further notice, notations such as $(\boldsymbol{A}, \vec{a})$, $(\boldsymbol{B}, \vec{b})$ denote augmented categories. Mere categories are considered special cases of augmented categories of type $\emptyset$; $\boldsymbol{A}$, $\boldsymbol{B}$ *etc.* denote categories.

For augmented categories $(\boldsymbol{A}, \vec{a})$, $(\boldsymbol{B}, \vec{b})$ of the same type, we write

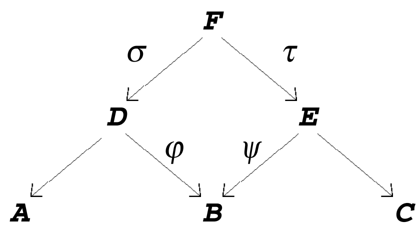
$$(\boldsymbol{A}, \vec{a}) \stackrel{\sim}{\longrightarrow} (\boldsymbol{B}, \vec{b})$$

if there is an equivalence functor $F:\boldsymbol{A}\xrightarrow{\simeq}\boldsymbol{B}$ (F is full and faithful, and essentially surjective on objects) that maps $\vec{a}$ to $\vec{b}$; we may also write $(\boldsymbol{B}, \vec{b}) \stackrel{\sim}{\longleftarrow} (\boldsymbol{A}, \vec{a})$ for the same. Note that the relation $\stackrel{\sim}{\longrightarrow}$ is reflexive and transitive but not symmetric (an equivalence functor $\boldsymbol{A}\xrightarrow{\simeq}\boldsymbol{B}$ may take two different objects $A\neq A'$ in $\boldsymbol{A}$ to the same B in $\boldsymbol{B}$; then $(\boldsymbol{A}, \langle A, A'\rangle) \stackrel{\sim}{\longrightarrow} (\boldsymbol{B}, \langle B, B\rangle)$ but not *vice versa*). The special case when the type $\mathcal{X}$ is $\emptyset$, is, however, symmetric; $\boldsymbol{A}\stackrel{\sim}{\longrightarrow}\boldsymbol{B}$ implies $\boldsymbol{A}\stackrel{\sim}{\longleftarrow}\boldsymbol{B}$ since every equivalence functor has a quasi-inverse (by the Axiom of Choice); $\boldsymbol{A}\stackrel{\sim}{\longrightarrow}\boldsymbol{B}$ is the same as equivalence of categories, $\boldsymbol{A}\simeq\boldsymbol{B}$.

The equivalence relation generated by the relation $\stackrel{\sim}{\longrightarrow}$ is only "one step away" from $\stackrel{\sim}{\longrightarrow}$; it is $\stackrel{\sim}{\longleftrightarrow}$ defined as

$(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftrightarrow} (\boldsymbol{B}, \vec{b}) \underset{\text{def}}{\Longleftrightarrow}$ there is $(\boldsymbol{C}, \vec{c})$ such that $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftarrow} (\boldsymbol{C}, \vec{c}) \overset{\sim}{\longrightarrow} (\boldsymbol{B}, \vec{b})$.(1^{-})

To see the transitivity of the relation $\overset{\sim}{\longleftrightarrow}$, assume $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftarrow} (\boldsymbol{D}, \vec{d}) \overset{\sim}{\longrightarrow} (\boldsymbol{B}, \vec{b})$ and $(\boldsymbol{B}, \vec{b}) \overset{\sim}{\longleftarrow} (\boldsymbol{E}, \vec{e}) \overset{\sim}{\longrightarrow} (\boldsymbol{C}, \vec{c})$, and consider the diagram



where the quadrangle has $\boldsymbol{F}$ the "isomorphism-comma" category, with objects $(D, E, \varphi D \xrightarrow{\cong} \psi E)$, and arrows the usual commutative squares, with $\sigma : \boldsymbol{F} \to \boldsymbol{D}$, $\tau : \boldsymbol{F} \to \boldsymbol{E}$ the forgetful functors. Since φ , ψ are equivalence functors, so are σ , τ . Let $\vec{f} = \langle f_x \rangle_{x \in \mathcal{X}} \in \boldsymbol{F}[\mathcal{X}]$ be defined as follows. For $x \in \mathcal{X}$, $x : \mathrm{O}$, let $f_x = (d_x, e_x, \mathrm{id} : \varphi d_x \xrightarrow{\cong} \psi e_x)$; note that $\varphi d_x = \psi e_x$ by assumption. For $x \in \mathcal{X}$, $x : \mathrm{A}(y, z)$, let $f_x = (d_x : d_y \to d_z, e_x : e_y \to e_z) : f_y \to f_z$; note that $\varphi d_x = \psi e_x$ by assumption. We have that $(\boldsymbol{F}, \vec{f}) \overset{\sim}{\longrightarrow} (\boldsymbol{D}, \vec{d})$, $(\boldsymbol{F}, \vec{f}) \overset{\sim}{\longrightarrow} (\boldsymbol{E}, \vec{e})$. Using the composites $\boldsymbol{F} \to \boldsymbol{A}$, $\boldsymbol{F} \to \boldsymbol{C}$, we obtain $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftrightarrow} (\boldsymbol{B}, \vec{b})$ as desired.

Recall the relation $\approx_{\boldsymbol{L}}$ of the last section; $\approx_{\boldsymbol{L}}$ is, in particular, a relation between augmented categories. We have that $\approx_{\boldsymbol{L}}$ is the same as $\overset{\sim}{\longleftrightarrow}$.

(1) **(a)** $(\boldsymbol{A}, \vec{a}) \overset{\sim}{\longleftrightarrow} (\boldsymbol{B}, \vec{b}) \iff (\boldsymbol{A}, \vec{a}) \approx_{\boldsymbol{L}} (\boldsymbol{B}, \vec{b})$;

(b) $\boldsymbol{A} \simeq \boldsymbol{B} \iff \boldsymbol{A} \approx_{\boldsymbol{L}} \boldsymbol{B}$.

Proof. Assume $(\boldsymbol{A}, \vec{a}) \approx_{\boldsymbol{L}} (\boldsymbol{B}, \vec{b})$. By §5, there is a normal $\boldsymbol{L}$,≈-equivalence $(W, u, v) : (\boldsymbol{A}, \vec{a}) \underset{\boldsymbol{L}}{\overset{\approx}{\longleftrightarrow}} (\boldsymbol{B}, \vec{b})$. Then, $\boldsymbol{C} = u^{*}(\boldsymbol{A}) = v^{*}(\boldsymbol{B})$ is a category, since, by 5.(1), as a standard $\boldsymbol{L}$-structure, $\boldsymbol{C}$ satisfies all the axioms of category which are formulated in FOLDS (see 5.(2)(a)). Furthermore, clearly, $\theta_u:\boldsymbol{C}\longrightarrow\boldsymbol{A}$, $\theta_v:\boldsymbol{C}\longrightarrow\boldsymbol{B}$ are surjective equivalence functors. This shows the right-to-left direction in (a). For the proof of the other direction, we prove the implication

$$(\boldsymbol{A},\vec{a})\overset{\sim}{\longrightarrow}(\boldsymbol{B},\vec{b})\quad\Longrightarrow\quad(\boldsymbol{A},\vec{a})\sim_{\boldsymbol{L}}(\boldsymbol{B},\vec{b})\ ;$$

to this end, we "saturate" the given equivalence appropriately; we will do this proof in a more general situation below.

Knowing the transitivity of the relation $\sim_{\boldsymbol{L}}$, the transitivity of $\overset{\sim}{\longleftrightarrow}$ also follows from (1)(a).

(b) is a special case of (a).

Recall the translation $\varphi\mapsto\varphi^*$ in §1; this is just to say that any formula φ of FOLDS over $\boldsymbol{L}$ may be regarded a formula (φ^*) over $|\boldsymbol{L}|$ in ordinary multisorted logic.

Let $\mathtt{T}_{\mathtt{cat}}=(|\boldsymbol{L}|,\Sigma_{\mathtt{cat}})$ the theory of categories in ordinary multisorted logic ($\Sigma_{\mathtt{cat}}$ can be taken to be $\Sigma[\mathbf{L}_{\mathtt{cat}}]\cup\{\theta^*:\theta\in\Theta\}$; $\Sigma[\boldsymbol{L}]$ for any DSV $\boldsymbol{L}$ was defined in §1 ; Θ is the set of axioms in FOLDS for categories as given in §1.). When T is a theory extending $\mathtt{T}_{\mathtt{cat}}$ ($|\boldsymbol{L}|\subset\mathrm{L}_T$, $\Sigma_{\mathtt{cat}}\subset\Sigma_T$), and $M\models T$, we write $|M|$ for $M\restriction\boldsymbol{L}$, the underlying category of M.

(2)(a) Let T be a theory extending $\mathtt{T}_{\mathtt{cat}}$. Let $\mathcal{X}$ be a finite context over $\mathbf{L}_{\mathtt{cat}}$, σ an L_T-formula such that $\mathtt{Var}(\sigma)\subset\mathcal{X}$. If

for any $M, N\models T$ and diagrams $\vec{a}\in|M|[\mathcal{X}]$, $\vec{b}\in|N|[\mathcal{X}]$, $M\models\sigma[\vec{a}]$ and $(|M|,\vec{a})\overset{\sim}{\longleftrightarrow}(|N|,\vec{b})$ imply $N\models\sigma[\vec{b}]$,

then

there is θ in FOLDS with restricted equality over $\mathbf{L}_{\mathtt{cat}}$ with $\mathtt{Var}(\theta)\subset\mathcal{X}$ such that for all $M\models T$ and diagrams $\vec{a}\in|M|[\mathcal{X}]$, we have $M\models\sigma[\vec{a}]$ iff $M\models\theta^*[\vec{a}]$.

(**b**) In particular, if σ is a sentence over L_T, and for any $M, N\models T$, $M\models\sigma$ and $|M|\simeq|N|$ imply $N\models\sigma$, then there is a sentence θ of FOLDS over $\mathbf{L}_{\mathtt{cat}}$ such that for any $M\models T$, $M\models\sigma$ iff $M\models\theta^{*}$.

Proof. We apply 5.(15) to $\mathbf{C}=[T]$, with $I:\mathbf{L}\to\mathbf{C}$ the composite of $I:\mathbf{L}\to[\mathrm{T}_{\mathrm{cat}}]$ defined in §5 before (7)(a) and the inclusion $[\mathrm{T}_{\mathrm{cat}}]\to[T]$; moreover, we take σ in 5.(15) to be $m^{*}([\mathcal{X}:\sigma])\rightarrowtail I[\mathcal{X}]$ ($m:I[\mathcal{X}]\rightarrowtail\{\mathcal{X}\}$; see §5 before (7)(a)). By (1)(a), the assumption implies that

$$M\models\sigma[\vec{a}]\ \ \&\ \ (|M|,\vec{a})\approx_{\mathbf{L}}(|N|,\vec{b})\quad\Longrightarrow\quad N\models\sigma[\vec{b}]\ .$$

The conclusion of 5.(15) is what we want. (b) is a special case of (a).

We say that a theory T extending $\mathrm{T}_{\mathrm{cat}}$ is *normal* if for any $M\models T$ and any category $\mathbf{A}$, if $\mathbf{A}\simeq|M|$, then there is a model $N\models T$ such that $\mathbf{A}=|N|$. In other words, normality of T says that the property of being the $\mathbf{L}_{\mathrm{cat}}$-reduct of a model of T is invariant under equivalence of categories. Most theories of categories (possibly) with additional structure are normal. E.g., so is the theory of monoidal categories, or the theory of categories with specified finite limits. Of course, $\mathrm{T}_{\mathrm{cat}}$ itself is normal.

Let $\mathcal{X}$ be a finite context, and σ be a formula over L_{T} with $\mathrm{Var}(\sigma)\subset\mathcal{X}$. Let us say that σ is *preserved along equivalence functors between models of* T if the following holds: whenever $M, N\models T$, $\vec{a}\in M[\mathcal{X}]$, $\vec{b}\in N[\mathcal{X}]$, then $M\models\sigma[\vec{a}]$ and $(|M|,\vec{a})\xrightarrow{\sim}(|N|,\vec{b})$ imply $N\models\sigma[\vec{b}]$. When in this definition, $(|M|,\vec{a})\xrightarrow{\sim}(|N|,\vec{b})$ is replaced by $(|M|,\vec{a})\xleftarrow{\sim}(|N|,\vec{b})$, we obtain the notion of being *reflected* along equivalence functors. Now, notice that for T a normal theory, the hypothesis of (2)(a) holds iff σ is preserved and reflected along equivalence functors of models of T (the point is that, in case T is normal, in (1^{-}), when $\mathbf{A}$ (and $\mathbf{B}$) are reducts of models of T, $\mathbf{C}$ can also be expanded to a model of T). Thus, we obtain the following variant of (2)(a):

(3) Let T be a normal theory of categories (possibly) with additional structure. Let $\mathcal{X}$ be a finite context over $\mathbf{L}_{\mathrm{cat}}$. Suppose that the first-order formula σ over L_{T} with free variables all in $\mathcal{X}$ is preserved and reflected along equivalence functors of models of T. Then there is a formula φ in FOLDS with restricted equality over $\mathbf{L}_{\mathrm{cat}}$ with $\mathrm{Var}(\varphi)\subset\mathcal{X}$ such that σ is equivalent to φ^{*} in models of T.

Freyd's and Blanc's characterization (see [F], [FS], [B]) of first order properties of finite diagrams invariant under equivalence is (3) for $T=\mathrm{T}_{\mathrm{cat}}$. In fact, the general result (3) can also be obtained by their methods, which is very different from the methods of this paper (we will comment on this in Appendix C). It seems however that the more general result (2), in particular, (2)(b), cannot be obtained by the Freyd's and Blanc's methods (although I should concede that the added generality in (2)(b) consisting in a reference to not-necessarily normal theories does not seem very important).

The results of §5 that are more general than 5.(15) (*e.g.*, the "interpolation-style" result (7)(b)) will also have consequences for equivalences of categories; we leave their formulation to the reader.

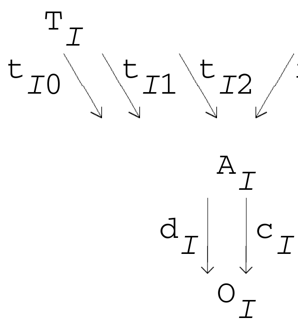
Extending the Freyd-Blanc result to more complex categorical structures will involve a new element. For instance, in the case of structures consisting of two categories and a functor between them (an example of which is a fibration), the first-order properties invariant under equivalence (in the appropriate standard sense; see also below) are not those expressible in FOLDS directly, but rather, those that are expressible in FOLDS in the language of the so-called saturated anafunctor associated with the given functor. Anafunctors are treated in [M2]; explanations will be given presently.

We now proceed to giving the framework for dealing with structures consisting of several (possibly infinitely many) categories, functors between them, and natural transformations between the latter. We will return to the simplest special case of two categories and a functor between them afterwards.

Let $\boldsymbol{I}$ be a small 2-graph; $\boldsymbol{I}:\mathbf{L}_{\text{2-graph}}\to\mathrm{Set}$. We associate the graph $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ with $\boldsymbol{I}$; $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ serves as a similarity type for diagrams $\boldsymbol{I}\to\mathrm{Cat}$ of (small) categories, functors and natural transformations. The objects of $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ are as follows:

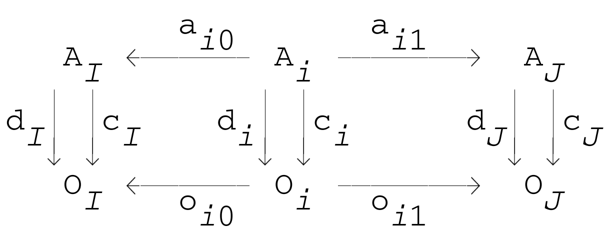
O_I , A_I , I_I , T_I ($I \in Ob(\boldsymbol{I})$)
O_i , A_i ($i \in Arr(\boldsymbol{I})$)
O_α ($\alpha \in 2\text{-}Cell(\boldsymbol{I})$)

The arrows of $L_{diag}[\boldsymbol{I}]$ are shown in the following three diagrams:

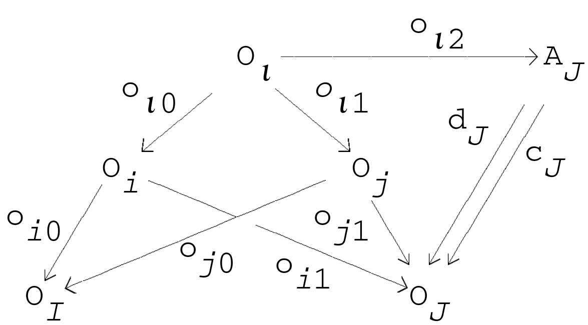


(3)

displaying the arrows associated with an object I ;



which displays the arrows associated with an arrow $i:I \to J$ in $\boldsymbol{I}$; and

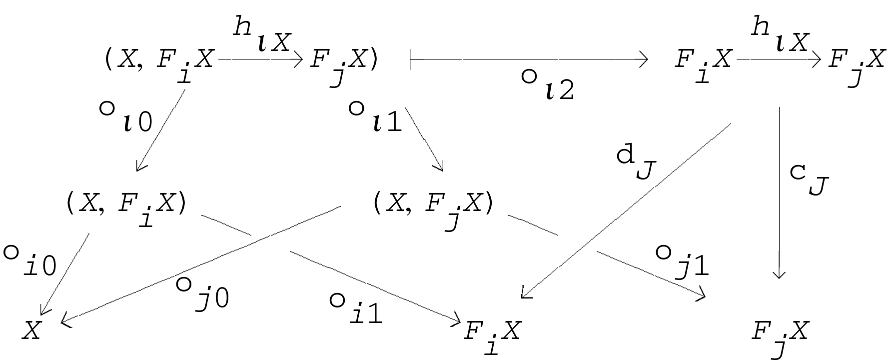


which displays the ones associated with the 2-cell $\iota : i \to j$ ($I \overset{i}{\underset{j}{\rightrightarrows}} J$, $\downarrow \iota$) .

Given a $\boldsymbol{I}$-diagram

$$D{:}\,\mathbf{I}\to\mathrm{Cat} : (\langle \mathbf{C}_I\rangle_{I\in\mathbf{I}},\ \langle F_i:\mathbf{C}_I\to\mathbf{C}_J\rangle_{i:I\to J},\ \langle h_\iota:F_i\to F_j\rangle_{I\,\overset{i}{\underset{j}{\rightrightarrows}}\,J\ (\iota:i\Rightarrow j)}) \tag{4}$$

of categories, functors and natural transformations, we construe D as an $\mathrm{L}_{\mathrm{diag}}[\mathbf{I}]$-structure as follows. (3) is interpreted as the category $\mathbf{C}_I$. When $i:I\to J$, O_i is the set of pairs (X, F_iX) with $X\in\mathrm{Ob}(\mathbf{C}_I)$, with $(X, F_iX)\overset{\mathrm{o}_{i0}}{\longmapsto}X$, $(X, F_iX)\overset{\mathrm{o}_{i1}}{\longmapsto}F_iX$. A_i is the set of pairs $(f, F_if)=(X\xrightarrow{f}Y,\ F_iX\xrightarrow{F_if}F_iY)$, with $(f, F_if)\overset{\mathrm{d}_i}{\longmapsto}(X, F_iX)$, $(f, F_if)\overset{\mathrm{c}_i}{\longmapsto}(Y, F_iY)$, $(f, F_if)\overset{\mathrm{a}_{i0}}{\longmapsto}f$, $(f, F_if)\overset{\mathrm{a}_{i1}}{\longmapsto}F_if$. For $I\,\overset{i}{\underset{j}{\rightrightarrows}}\,J$ $(\downarrow\iota)$, O_ι is the set of pairs $(X,\ F_iX\xrightarrow{h_{\iota X}}F_jX)$. The effect of the remaining arrows, as well as the corresponding commutativities, are shown by the following picture:



Let $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ be the DSV defined as follows. The underlying simple category of $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ is generated by the graph $\mathrm{L}_{\mathrm{diag}}[\mathbf{I}]$, subject to the following equalities between arrows:

the ones ensuring that (3) generates a copy of $\mathbf{L}_{\mathrm{cat}}$ (see §1);

$\mathrm{o}_{i0}\mathrm{d}_i=\mathrm{d}_I\mathrm{a}_{i0}$, $\mathrm{o}_{i1}\mathrm{d}_i=\mathrm{d}_J\mathrm{a}_{i1}$, $\mathrm{o}_{i0}\mathrm{c}_i=\mathrm{c}_I\mathrm{a}_{i0}$, $\mathrm{o}_{i1}\mathrm{c}_i=\mathrm{c}_J\mathrm{a}_{i1}$,

$$\mathrm{o}_{i0}\mathrm{o}_{\iota 0}=\mathrm{o}_{j0}\mathrm{o}_{\iota 1},\quad \mathrm{d}_J\mathrm{o}_{\iota 2}=\mathrm{o}_{i1}0_{\iota 0},\quad \mathrm{c}_J\mathrm{o}_{\iota 2}=\mathrm{o}_{j1}0_{\iota 1}\ . \tag{5}$$

The relations of $\mathbf{L}$ are exactly its top-level objects; that is, T_I, I_I, A_i, O_ι, for I, i and ι ranging over the 0-cells, 1-cells and 2-cells of $\mathbf{I}$, respectively.

The equalities on arrows are suggested by what is true for $\mathbf{I}$-diagrams as structures. In fact, every $\mathbf{I}$-diagram is a functor $D:\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]\longrightarrow\mathrm{Set}$, that is, the equalities listed are true in it (as identities). Also, the relations of $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ are interpreted in D relationally (the corresponding family of functions is monomorphic). In summary, every $\mathbf{I}$-diagram is an $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$-structure.

$\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ is the similarity type of what we call the "anadiagrams" of type $\mathbf{I}$. An *anadiagram* $M:\mathbf{I}\xrightarrow{\mathrm{a}}\mathrm{Cat}$ is an $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$-structure satisfying the following axioms (A0) to (A6) in FOLDS with equality ($I\overset{i}{\underset{j}{\rightrightarrows}}J$ with 2-cell ι range over objects, arrows and 2-cells in $\mathbf{I}$ as shown; the unique existential quantifiers in (A2) and (A5) are abbreviations in the usual way, and they refer to equality on the sorts $\mathrm{A}_J(\cdot,\cdot)$).

(A0): axioms expressing that for each $I\in\mathrm{Ob}(\mathbf{I})$, the part of M referring to I is a category .

(A1) $\forall X:\mathrm{O}_I.\exists A:\mathrm{O}_J.\exists s:\mathrm{O}_i(X,A).\,\mathbf{t}$.

(A2)
$$\forall X,Y:\mathrm{O}_I.\forall A,B:\mathrm{O}_J\forall s:\mathrm{O}_i(X,A).\forall t:\mathrm{O}_i(Y,B).\forall f:\mathrm{A}_I(X,Y).$$
$$\exists!g:\mathrm{A}_J(A,B).\mathrm{A}_i(s_{\mathrm{d}_i},t_{\mathrm{c}_i},f_{\mathrm{a}_{i0}},g_{\mathrm{a}_{i1}})\ .$$

(A3)
$$\forall X:\mathrm{O}_I.\forall A:\mathrm{O}_J.\forall s:\mathrm{O}_i(X,A).\forall\alpha:\mathrm{A}_I(X,X).\forall\bar{\alpha}:\mathrm{A}_J(A,A)$$
$$[\mathrm{A}_i(s,s,\alpha,\bar{\alpha})\longrightarrow(\mathrm{I}_I(X_{\mathrm{d}_I\mathrm{i}_I},\alpha_{\mathrm{i}_I})\longrightarrow\mathrm{I}_J(\mathrm{A},\bar{\alpha}))]\ .$$

(A4)
$$\forall X,Y,Z:\mathrm{O}_I.\forall A,B,C:\mathrm{O}_J.\forall s:\mathrm{O}_i(X,A).\forall t:\mathrm{O}_i(Y,B).\forall u:\mathrm{O}_i(Z,C)$$
$$\forall f:\mathrm{A}_I(X,Y).\forall g:\mathrm{A}_I(Y,Z).\forall h:\mathrm{A}_I(X,Z)$$
$$\forall\bar{f}:\mathrm{A}_I(A,B).\forall\bar{g}:\mathrm{A}_I(B,C).\forall\bar{h}:\mathrm{A}_I(A,C)$$
$$[((\mathrm{A}_i(s,t,f,\bar{f})\wedge\mathrm{A}_i(t,u,g,\bar{g})\wedge\mathrm{A}_i(s,u,h,\bar{h}))\longrightarrow$$
$$(\mathrm{T}_I(f,g,h)\longrightarrow\mathrm{T}_J(\bar{f},\bar{g},\bar{h}))]\ .$$

(A5)
$$\forall X:\mathrm{O}_I.\forall A:\mathrm{O}_J.\forall B:\mathrm{O}_J.\forall s:\mathrm{O}_i(X,A).\forall t:\mathrm{O}_j(X,B)$$

$$\exists ! f{:}\mathrm{A}_J(A, B).\mathrm{O}_\iota(s_{\iota 0}, t_{\iota 1}, f_{\iota 2}) .$$

(A6) $\forall X, Y{:}\mathrm{O}_I . \forall A, B, C, G{:}\mathrm{O}_J$
$\forall s{:}\mathrm{O}_i(X, A) \forall t{:}\mathrm{O}_i(Y, B) . \forall u{:}\mathrm{O}_j(X, C) . \forall v{:}\mathrm{O}_j(Y, G)$
$\forall f{:}\mathrm{A}_I(X, Y) . \forall g{:}\mathrm{A}_J(A, B) . \forall k{:}\mathrm{A}_J(C.G)$
$\forall \ell{:}\mathrm{A}_J(A, C) . \forall m{:}\mathrm{A}_J(B, G)$
$[(\mathrm{O}_i(s, t, f, g) \wedge \mathrm{O}_j(u, v, f, k) \wedge \mathrm{O}_\iota(s, u, \ell) \wedge \mathrm{O}_\iota(t, v, m)) \longrightarrow$
$\exists n{:}\mathrm{A}_J(B, C) . (\mathrm{T}_J(g, m, n) \wedge \mathrm{T}_J(\ell, k, n))]$.

For a less formal explanation of the notion of anadiagram, I refer to [M2]. In that paper, I introduce the notion of *anafunctor* between categories, a generalization of the notion of functor. An anafunctor defines its values on objects only up to isomorphism. Formally, the definition of anafunctor is obtained by specializing the definition of "anadiagram" to the case when $\boldsymbol{I}$ is the (2-)graph $0 \xrightarrow{\langle 0, 1\rangle} 1$ (without 2-cells). Anadiagrams have anafunctors instead of functors as 1-cells, and natural transformations of anafunctors as 2-cells.

Note that any $\boldsymbol{I}$-diagram $D{:}\boldsymbol{I} \to \mathrm{Cat}$ is an anadiagram; all the axioms for "anadiagram" are satisfied in D (as an $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$-structure). In fact, the diagrams are essentially the same as those anadiagrams M in which the sorts O_i ($i \in \mathrm{Arr}(\boldsymbol{I})$) are interpreted *relationally*, that is, the family $\langle Mp \rangle_{p:\mathrm{O}_i \to \mathrm{K}_p}$ is jointly monomorphic.

On the other hand, any anadiagram gives rise to a diagram, obtained by making some non-canonical choices. Let M be an anadiagram $M{:}\boldsymbol{I} \xrightarrow{a} \mathrm{Cat}$; we construct $D{:}\boldsymbol{I} \longrightarrow \mathrm{Cat}$; we use the notation (4) for the ingredients of D. For $I \in \mathrm{Ob}(\boldsymbol{I})$, the category $\boldsymbol{C}_I$ is given directly by the data in M corresponding to I (see (A0)). By (A1), for any $i{:}I \to J$ in $\boldsymbol{I}$ and $X \in \mathrm{Ob}(\boldsymbol{C}_I) = M\mathrm{O}_I$, we make a choice of $A^i_X = A_X \in M\mathrm{O}_J$ and $s^i_X = s_X \in M\mathrm{O}_i(X, A)$; we put $F_i X = A_X$. Starting with $f{:}X \to Y$, and using (A2) with $A = A_X$, $B = A_Y$, $s = s_X$, $t = s_Y$, we let $F_i f = g$ whose unique existence (A2) states. (A3) and (A4) assure that F_i so defined is a functor $F_i{:}\boldsymbol{C}_I \to \boldsymbol{C}_J$. Using (A5) with $A = A^i_X$, $B = A^j_X$, $s = s^i_X$, $t = s^j_X$, we put $h_{\iota X} = f$ for the f whose existence (5) asserts. (A6) ensures that h_ι is a natural transformation $h_\iota{:}F_i \to F_j$. Let us refer D as the diagram obtained from M by *cleavage* (in analogy to the terminology used with fibration); of course, it is not uniquely determined.

Next, we describe the *saturation* $D^{\#}$ of a diagram $D:\mathbf{I}\to\mathrm{Cat}$, an anadiagram canonically associated with D. (As a matter of fact, the components corresponding to the 1-cells $i:I\to J$ will be the "saturated anafunctors" $F_i^{\#}$ associated with the given functors F_i, in the sense of [M2].)

In $D^{\#}$, the interpretation of each part of $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ as in (3) is the same as in D.

For $i:I\to J$, a 1-cell in $\mathbf{I}$, $D^{\#}O_i$ is the set of triples $\mu=(X, A, F_iX\xrightarrow[\mu]{\cong}A)$ with $X\in\mathbf{C}_I$, $A\in\mathbf{C}_J$ and μ an isomorphism as shown; $\mu\overset{O_{i0}}{\longmapsto}X$, $\mu\overset{O_{i1}}{\longmapsto}A$. $D^{\#}A_i$ is the set of all entities

$$\left(\begin{array}{c}X\\ \downarrow f\\ Y\end{array},\quad \begin{array}{ccc}F_iX & \xrightarrow[\mu]{\cong} & A\\ \downarrow F_if & \circ & \downarrow g\\ F_iY & \xrightarrow[\nu]{\cong} & B\end{array}\right)$$

with the displayed entity mapped to (X, A, μ) by d_i, to (Y, B, ν) by c_i, to f by a_{i0}, and to g by a_{i1}. For $I\underset{j}{\overset{i}{\rightrightarrows}}J$, $D^{\#}O_\iota$ consists of all

$$\left(X,\quad \begin{array}{ccc}F_iX & \xrightarrow[\cong]{\mu} & A\\ h_{\iota X}\downarrow & \circ & \downarrow g\\ F_jX & \xrightarrow[\rho]{\cong} & B\end{array}\right),$$

and the displayed item is mapped to (X, A, μ) by $\mathrm{o}_{\iota 0}$, to (X, B, ρ) by $\mathrm{o}_{\iota 1}$, and to g by $\mathrm{o}_{\iota 2}$.

We leave it to the reader to verify that $D^{\#}$ so defined is an anadiagram.

$D^{\#}$ satisfies a property that distinguishes it from diagrams; it is *saturated*, by which we mean that it satisfies, for each $i:I\to J$ in $\mathbf{I}$, the FOLDS sentence

(A7) $$\forall X{:}\mathrm{O}_I.\forall A, B{:}\mathrm{O}_J.\forall s{:}\mathrm{O}_i(X, A).\forall f{:}\mathrm{A}_J(A, B)$$
$$(\mathrm{Iso}(f) \longrightarrow \exists! t{:}\mathrm{O}_i(X, B).\exists g{:}\mathrm{A}_I(X, X).(\mathrm{I}_I(g) \wedge \mathrm{A}_i(s, t, g, f))\,;$$

here, $\mathrm{Iso}(f)$ abbreviates

$$\exists h{:}\mathrm{A}_J(B, A)\exists k{:}\mathrm{A}_J(A, A)\exists \ell{:}\mathrm{A}_J(B, B).(\mathrm{I}_J(k) \wedge \mathrm{I}_J(\ell) \wedge \mathrm{T}_J(f, h, k) \wedge \mathrm{T}_J(h, f, \ell))$$
.

In fact, it can be proved (although we will not need this result) that, up to *isomorphism* as $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$-structures, the saturated $\boldsymbol{I}$-anadiagrams are precisely the ones of the form $D^{\#}$, for some diagram D.

Given D as in (5), and another $\boldsymbol{I}$-type diagram

$$\hat{D}{:}\boldsymbol{I} \to \mathrm{Cat} : (\langle \hat{\boldsymbol{C}}_I \rangle_{I \in \boldsymbol{I}}, \langle \hat{F}_i{:}\hat{\boldsymbol{C}}_I \to \hat{\boldsymbol{C}}_J \rangle_{i:I \to J}, \langle \hat{h}_\iota{:}\hat{F}_i \to \hat{F}_j \rangle_{I \underset{j}{\overset{i}{\rightrightarrows}} J \,(\iota: i \to j)})\,, \tag{6}$$

we say that D and $\hat{D}$ are *equivalent*, and write $D \simeq \hat{D}$, if there exist a family $\langle E_I{:}\boldsymbol{C}_I \xrightarrow{\simeq} \hat{\boldsymbol{C}}_I \rangle_{I \in \boldsymbol{I}}$ of equivalence functors, and a family $\langle e_i \rangle_{i:I \to J}$ of natural isomorphisms as in

$$\begin{array}{ccc} \boldsymbol{C}_I & \xrightarrow{E_I} & \hat{\boldsymbol{C}}_I \\ {\scriptstyle F_i}\downarrow & \cong \swarrow e_i & \downarrow {\scriptstyle \hat{F}_i} \\ \boldsymbol{C}_J & \xrightarrow[E_J]{} & \hat{\boldsymbol{C}}_J \end{array} \qquad e_i{:}\hat{F}_i \circ E_I \xrightarrow{\cong} E_J \circ F_i\,,$$

satisfying the additional naturality condition:

$$\begin{array}{ccc} E_J \circ F_i & \xleftarrow{e_i} & \hat{F}_i \circ E_I \\ {\scriptstyle E_J \circ h_\iota}\downarrow & \bigcirc & \downarrow {\scriptstyle \hat{h}_\iota \circ E_I} \\ E_J \circ F_j & \xleftarrow[e_j]{} & \hat{F}_j \circ E_I \end{array}$$

for every $I \underset{j}{\overset{i}{\rightrightarrows}} J$ (with $\Downarrow\iota$) in $\boldsymbol{I}$. The data $E=(\langle E_I\rangle_{I\in\boldsymbol{I}}, \langle e_i\rangle_{i\in\mathrm{Arr}(\boldsymbol{I})})$ form an *equivalence* of D and $\hat{D}$, in notation,

$$E = (\langle E_I\rangle_{I\in\boldsymbol{I}}, \langle e_i\rangle_{i\in Arr(\boldsymbol{I})}) : D \xrightarrow{\simeq} \hat{D}. \tag{7}$$

This notion of equivalence is a "bicategorical" notion; it is the equivalence in the internal sense of the bicategory (actually, 2-category) $\mathrm{Hom}(\langle\boldsymbol{I}\rangle, \mathrm{Cat})$ of homomorphisms of bicategories, pseudo-natural transformations and modifications, with $\langle\boldsymbol{I}\rangle$ the 2-category generated by the 2-graph $\boldsymbol{I}$. (The main part of the fact that the "one-way" formulation of equivalence given above as the definition, and the "internal" concept just mentioned coincide, is the symmetry of the relation $\simeq$; an outline of the proof of the symmetry of $\simeq$ is given below.) It is the "good" notion of equivalence, the one that comes up in practice. For instance, in Chapter 4 of [MP], diagrams of sketches, and diagrams of accessible categories are dealt with, and the present notion of equivalence is the one which is operative. Specifically, the Uniform Sketchability Theorem, one of the main results of [MP] (4.4.1 in [MP]) says that a small diagram of accessible categories is *equivalent* to one obtained from a diagram of sketches by taking the categories of models of the sketches involved.

Although the fact is well-known, I outline the proof that the relation $D\simeq\hat{D}$ is symmetric. Since it is easily seen to be transitive and reflexive, $\simeq$ is an equivalence relation.

Assume data as in (7); see also (4) and (6). We define $\hat{E}:\hat{D}\xrightarrow{\simeq}D$. With $I\in\mathrm{Ob}(\boldsymbol{I})$, $A\in\mathrm{Ob}(\hat{\boldsymbol{C}}_I)$, choose $X_A^I=X_A\in\mathrm{Ob}(\boldsymbol{C}_I)$ and $\varepsilon_A^I=\varepsilon_A : E_I X_A \xrightarrow{\cong} A \in \mathrm{Arr}(\hat{\boldsymbol{C}}_I)$. Put $\hat{E}_I A = X_A$. For $f:A\to B\in\mathrm{Arr}(\hat{\boldsymbol{C}}_I)$, $\hat{E}_I f$ is the arrow that makes the square

$$\begin{array}{ccc} E_I X_A & \xrightarrow[\cong]{\varepsilon_A} & A \\ {\scriptstyle \hat{E}_I f}\downarrow & \circ & \downarrow{\scriptstyle f} \\ E_I X_B & \xrightarrow[\varepsilon_B]{\cong} & B \end{array}$$

commute. $\hat{E}_I$ so defined is a functor $\hat{E}_I:\hat{\boldsymbol{C}}_I\longrightarrow\boldsymbol{C}_I$; it is an equivalence functor; it is a quasi-inverse of E_I : we have $\varepsilon_I : E_I\hat{E}_I \xrightarrow{\cong} 1_{\hat{\mathbf{C}}_I}$ with components the ε^I_A, and $\eta_I : 1_{\mathbf{C}_I} \xrightarrow{\cong} \hat{E}_I E_I$ with components $\eta_{I,X}$ for which $E_I(\eta_{I,X}) = (\varepsilon^I_{E_I X})^{-1}$. For $i:I\to J$ in $\mathbf{I}$, we define $\hat{e}_i : F_i\hat{E}_J \xrightarrow{\cong} \hat{E}_J\hat{F}_i$ as the composite

$$F_i\hat{E}_J \xrightarrow[\cong]{\eta_J F_i \hat{E}_J} \hat{E}_J E_J F_i \hat{E}_J \xrightarrow[\cong]{\hat{E}_J e_i^{-1} \hat{E}_I} \hat{E}_J \hat{F}_i E_I \hat{E}_J \xrightarrow[\cong]{\hat{E}_J F_i \varepsilon_I} \hat{E}_J \hat{F}_i \quad .$$

$\langle \hat{e}_i \rangle_{i\in \mathrm{Arr}(\mathbf{I})}$ will be compatible with the h_ι, and give $\hat{E}:\hat{D} \xrightarrow{\simeq} D$ as desired.

Let $\mathbf{K}_0$ be the full subcategory of $\mathbf{L}_{\mathrm{anadiag}}[\mathbf{I}]$ consisting of the objects O_I and A_I for all $I\in\mathrm{Ob}(\mathbf{I})$. A *restricted* context is a context over $\mathbf{K}_0$. We have $D\restriction\mathbf{K}_0 = D^{\#}\restriction\mathbf{K}_0$, and hence, for a restricted context $\mathcal{X}$, $D[\mathcal{X}] = D^{\#}[\mathcal{X}]$.

With $\mathcal{X}$ a restricted context, an *augmented* $\mathbf{I}$*-diagram of type* $\mathcal{X}$ is a pair $(D, \vec{a})$ where $D:\mathbf{I}\to\mathrm{Cat}$, and $\vec{a}\in D[\mathcal{X}]$; notations such as $(D, \vec{a})$, $(\hat{D}, \vec{b})$ denote augmented $\mathbf{I}$-diagrams. We write $E:(D, \vec{a}) \xrightarrow{\simeq} (\hat{D}, \vec{b})$ for the following: $E:D\xrightarrow{\simeq}\hat{D}$ with E as in (7) such that $E(\vec{a})=\vec{b}$ in the obvious sense that $E_I(a_X)=b_X$. The relation $\xrightarrow{\sim}$ between augmented diagrams is defined thus:

$$(D, \vec{a}) \xrightarrow{\sim} (\hat{D}, \vec{b}) \iff \text{there exists } E:(D, \vec{a}) \xrightarrow{\simeq} (\hat{D}, \vec{b}) \quad .$$

We write $(D, \vec{a}) \xleftrightarrow{\sim} (\hat{D}, \vec{b})$ for: there exists $(\tilde{D}, \vec{c})$ such that $(D, \vec{a}) \xleftarrow{\sim} (\tilde{D}, \vec{c}) \xrightarrow{\sim} (\hat{D}, \vec{b})$. The relation $\xleftrightarrow{\sim}$ is the equivalence relation generated by $\xrightarrow{\sim}$; this can be seen directly, but it also follows from (8) below. In particular, when $\mathcal{X}=\emptyset$, the relation $\xleftrightarrow{\sim}$ coincides with $\simeq$ for $\mathbf{I}$-diagrams (since $\simeq$ is an equivalence relation).

(8) For augmented $\mathbf{I}$-diagrams $(D, \vec{a})$, $(\hat{D}, \vec{b})$ of the same type, we have

$$(D,\vec{a})\overset{\sim}{\longleftrightarrow}(\hat{D},\vec{b}) \quad\Longleftrightarrow\quad (D^{\#},\vec{a})\approx_{\boldsymbol{L}}(\hat{D}^{\#},\vec{b})\ ;$$

here, $\boldsymbol{L}=\mathbf{L}_{\mathtt{anadiag}}[\boldsymbol{I}]$.

As a special case, for (mere) $\boldsymbol{I}$-diagrams D and $\hat{D}$,

$$D\simeq\hat{D} \quad\Longleftrightarrow\quad D^{\#}\approx_{\boldsymbol{L}}\hat{D}^{\#}\ .$$

Proof. (A)($\Longleftarrow$:) Let $(\mathcal{R}, r_0, r_1):(D^{\#},\vec{a})\underset{\boldsymbol{L}}{\leftrightarrow}(\hat{D}^{\#},\vec{b})$ be a normal $\boldsymbol{L}$, $\approx$-equivalence (see 5.(2")). Let $\vec{c}\in\mathcal{R}[\mathcal{X}]$ ($\mathcal{X}$ the type of $(D^{\#},\vec{a})$, $(\hat{D}^{\#},\vec{b})$) for which $r_0(\vec{c})=\vec{a}$, $r_1(\vec{c})=\vec{b}$.

Let $M=r_0^{*}(D^{\#})=r_1^{*}(\hat{D}^{\#})$, a standard $\boldsymbol{L}$-structure. Since $D^{\#}$ is an anadiagram, and the concept of "anadiagram" is elementary in FOLDS over $\boldsymbol{L}$, by 5.(1)(a), M is an anadiagram. Let $\tilde{D}$ be obtained from M by cleavage. We show that there is an equivalence $E:\tilde{D}\xrightarrow{\simeq}D$ which extends

$$m\restriction\boldsymbol{K}_0=\theta_m\restriction\boldsymbol{K}_0:M\restriction\boldsymbol{K}_0=\tilde{D}\restriction\boldsymbol{K}_0\rightarrow D^{\#}\restriction\boldsymbol{K}_0=D\restriction\boldsymbol{K}_0$$

(that is, $E_I=(\theta_m)_I$ for all $I\in\mathrm{Ob}(\boldsymbol{I})$; here, we used the notation (7) for E), and similarly, there is $\hat{E}:\tilde{D}\xrightarrow{\simeq}\hat{D}$ extending $n\restriction\boldsymbol{K}_0$. In particular, it will follow that $E(\vec{c})=\vec{a}$, $\hat{E}(\vec{c})=\vec{b}$ and $(D,\vec{a})\overset{\sim}{\longleftarrow}(\tilde{D},\vec{c})\overset{\sim}{\longrightarrow}(\hat{D},\vec{b})$ as desired.

We use a notation for $\tilde{D}$ that is analogous to (6). The functor $E_I:\tilde{\boldsymbol{C}}_I\longrightarrow\boldsymbol{C}_I$ is defined by the effect m_{O_I} and m_{A_I}; since $\theta_m:M\rightarrow D$ preserves the relations I_I and T_I, E_I is a functor. By the normality of r_0, E_I induces bijections on hom-sets, and by the surjectivity of r_0 on O_I, E_I is a surjective equivalence.

Let $i:I\rightarrow J$. Looking back at how the cleavage $\tilde{D}$ was defined, we see that $\tilde{F}_i X=A_X$, with $s_X \in M\mathrm{O}_i(X, A_X)$. Then $m(s_X) \in D^{\#}\mathrm{O}_i(mX, mA_X) = D^{\#}\mathrm{O}_i(E_I X, E_J\tilde{F}_i X)$. By the definition of $D^{\#}$, this means that $ms_X : F_i E_I X \xrightarrow{\cong} E_J\tilde{F}_i X$. We put $e_{iX} = ms_X$. To see that $e_i = \langle e_{iX}\rangle_{X\in \mathrm{Ob}(\boldsymbol{C}_I)}$ is a natural transformation $e_i : F_i E_I \xrightarrow{\cong} E_J\tilde{F}_i$, let $f: X \to Y \in \tilde{\boldsymbol{C}}_I$. We see that $\tilde{F}_i f$ is defined by the property that $M(\mathrm{A}_i)(s_X, s_Y, f, \tilde{F}_i f)$ should hold. But θ_m preserves A_i ; hence, $D^{\#}(A_i)(e_{iX}, e_{iY}, E_I f, E_J\tilde{F}_i f)$, which, by the definition of $D^{\#}$, means

$$\begin{array}{ccc} F_i E_I X & \xrightarrow{e_{iX}} & E_J\tilde{F}_i X \\ {\scriptstyle F_i E_I f}\downarrow & \circ & \downarrow{\scriptstyle E_J\tilde{F}_i f} \\ F_i E_I Y & \xrightarrow[e_{iY}]{} & E_J\tilde{F}_i Y \end{array} ,$$

which is the naturality of e_i .

Let $I \overset{i}{\underset{j}{\rightrightarrows}} J$ (with $\iota : i \to j$) be given. The naturality condition on (e_i, e_j) with respect to $\iota : i \to j$ is seen as follows. Let $X \in \mathrm{Ob}(\tilde{\boldsymbol{C}}_I)$. The definition of the component $\tilde{h}_{\iota X} : \tilde{F}_i X \to \tilde{F}_j X$ is defined (in the process of cleavage) by the condition $M\mathrm{O}_\iota(s^i_X, s^j_X, \tilde{h}_{\iota X})$. The map $\theta_m : M \to D^{\#}$ preserves the relation O_ι . It follows that $D^{\#}\mathrm{O}_\iota(ms^i_X, ms^j_X, m\tilde{h}_{\iota X})$ holds; that is, $D^{\#}\mathrm{O}_\iota(e_{iX}, e_{jX}, E_J\tilde{h}_{\iota X})$ holds. Considering the definition of $D^{\#}\mathrm{O}_\iota$, this says that

$$\begin{array}{ccc} E_J\tilde{F}_i X & \xleftarrow{e_{iX}} & F_i E_I X \\ {\scriptstyle E_J\tilde{h}_{\iota X}}\downarrow & \circ & \downarrow{\scriptstyle h_{\iota E_I X}} \\ E_J\tilde{F}_j X & \xleftarrow[e_{jX}]{} & F_j E_I X \end{array}$$

which is what we wanted.

(B)("only if") We show that $(D, \vec{a}) \xrightarrow{\sim} (\hat{D}, \vec{b})$ implies $(D, \vec{a}) \approx_{\boldsymbol{L}} (\hat{D}, \vec{b})$. Since $\approx_{\boldsymbol{L}}$ is an equivalence relation, the desired assertion will follow.

Suppose that $E:(D,\vec{a})\xrightarrow{\simeq}(\hat{D},\vec{b})$; E is taken in the notation in (7) ; we construct $(\mathcal{R},r_0,r_1):(D^{\#},\vec{a})\underset{\boldsymbol{L}}{\overset{\approx}{\longleftrightarrow}}(\hat{D}^{\#},\vec{b})$. The kinds of $\boldsymbol{L}$ are as in

$$\begin{array}{ccccc} \mathrm{A}_I & & & & \mathrm{A}_J \\ {\scriptstyle d_I}\downarrow\ \downarrow{\scriptstyle c_I} & & & & {\scriptstyle d_J}\downarrow\ \downarrow{\scriptstyle c_J} \\ \mathrm{O}_I & \xleftarrow[\mathrm{o}_{i0}]{} & \mathrm{O}_i & \xrightarrow[\mathrm{o}_{i1}]{} & \mathrm{O}_J \end{array}$$

with $i:I\to J$ in $\boldsymbol{I}$; we have to define $\mathcal{R}$ on these kinds.

We put $\mathcal{R}\mathrm{O}_I=\{(X,\hat{X},\sigma):X\in D\mathrm{O}_I,\ \hat{X}\in\hat{D}\mathrm{O}_I,\ \sigma:E_IX\xrightarrow{\cong}\hat{X}\}$, with $(X,\hat{X},\sigma)\overset{r_0}{\longmapsto}X$, $(X,\hat{X},\sigma)\overset{r_1}{\longmapsto}\hat{X}$. The "very surjective" condition on r_0 , r_1 at O_I holds by the essential surjectivity of E_I .

$$\mathcal{R}\mathrm{A}_I \overset{=}{\mathrm{def}} \{((X,\hat{X},\sigma),(Y,\hat{Y},\tau),\ \begin{matrix}X\\ f\downarrow\\ Y\end{matrix},\ \begin{matrix}\hat{X}\\ \hat{f}\downarrow\\ \hat{Y}\end{matrix}):$$

$$(X,\hat{X},\sigma),\ (Y,\hat{Y},\tau)\in\mathcal{R}\mathrm{O}_I,\ \begin{array}{ccc} E_IX & \xrightarrow{\sigma} & \hat{X} \\ {\scriptstyle E_If}\downarrow & \circ & \downarrow{\scriptstyle \hat{f}} \\ E_IY & \xrightarrow[\tau]{} & \hat{Y} \end{array}\} ,$$

with the displayed item being mapped to $(X,\hat{X},\sigma)$ by $\mathcal{R}\mathrm{d}_I$, to $(Y,\hat{Y},\tau)$ by $\mathcal{R}\mathrm{c}_I$, to f by r_0 , and to $\hat{f}$ by r_1 . The mapping

$$f\longmapsto\hat{f}\ :\ D^{\#}\mathrm{A}(X,Y)\longrightarrow\hat{D}^{\#}\mathrm{A}(\hat{X},\hat{Y})$$

so defined, with fixed $(X,\hat{X},\sigma),\ (Y,\hat{Y},\tau)\in\mathcal{R}\mathrm{O}_I$, is a bijection; this holds since E_I is an equivalence of categories. This shows the "very surjective" condition for r_0 , r_1 at A_I , as well as the preservation of $\dot{\mathrm{E}}_{\mathrm{A}_I}$.

$$\mathcal{R}\mathrm{O}_i \overset{=}{\mathrm{def}} \{((X,\hat{X},\sigma),(A,\hat{A},\alpha),\mu):(X,\hat{X},\sigma)\in\mathcal{R}\mathrm{O}_I,\ (A,\hat{A},\alpha)\in\mathcal{R}\mathrm{O}_J,\ \mu:F_iX\xrightarrow{\cong}A\} ,$$

with the displayed item being mapped to $(X, \hat{X}, \sigma)$ by $\mathcal{R}\mathrm{o}_{i0}$, to $(A, \hat{A}, \alpha)$ by $\mathcal{R}\mathrm{o}_{i1}$, to $(X, A, \mu)\in D\mathrm{O}_i$ by r_0, and to $(\hat{X}, \hat{A}, \hat{\mu})\in \hat{D}\mathrm{O}_i$ by r_1 where $\hat{\mu}$ is determined by the following commutativity:

$$\begin{array}{ccc}
\hat{F}_i E_I X & \xrightarrow[\cong]{\sigma} & \hat{F}_i \hat{X} \\
{\scriptstyle e_{iX}}\downarrow{\scriptstyle \cong} & & \Big\downarrow \\
E_J F_i X & \circ & \hat{\mu} \\
{\scriptstyle E_J\mu}\downarrow{\scriptstyle \cong} & & \Big\downarrow \\
E_J A & \xrightarrow[\alpha]{\cong} & \hat{A}
\end{array} \quad . \qquad (9)$$

Note that since all given arrows are isomorphisms, $\hat{\mu}$ is uniquely determined, and it is an isomorphism. Moreover, since E_J is an equivalence of categories, the mapping

$$\mu \mapsto \hat{\mu} : D^{\#}\mathrm{O}_i(X, A) \longrightarrow \hat{D}^{\#}\mathrm{O}_i(\hat{X}, \hat{A})$$

so defined (with the rest of the data fixed) is a bijection, which shows the "very surjective" condition at O_i, and the preservation of $\dot{\mathrm{E}}_{\mathrm{O}_i}$.

This completes the data for $(\mathcal{R}, r_0, r_1)$; it remains to verify the necessary properties.

Let us consider the preservation of the relation A_i by $(\mathcal{R}, r_0, r_1)$. What we have to do is this. We take four items

$$x_{\mathrm{d}_i}\in\mathcal{R}\mathrm{O}_i\,,\ x_{\mathrm{c}_i}\in\mathcal{R}\mathrm{O}_i\,,\ x_{\mathrm{a}_{i0}}\in\mathcal{R}\mathrm{A}_I\,,\ x_{\mathrm{a}_{i1}}\in\mathcal{R}\mathrm{A}_J$$

such that $(x_{\mathrm{d}_i}, x_{\mathrm{c}_i}, x_{\mathrm{a}_{i0}}, x_{\mathrm{a}_{i1}})\in\mathcal{R}[\mathrm{A}_i]$, that is,

(10)
$$\begin{array}{l}
\mathcal{R}\mathrm{o}_{i0}(x_{\mathrm{d}_i}) = \mathcal{R}\mathrm{d}_I(x_{\mathrm{a}_{i0}})\ ,\ \ \mathcal{R}\mathrm{o}_{i1}(x_{\mathrm{d}_i}) = \mathcal{R}\mathrm{d}_J(x_{\mathrm{a}_{i1}})\ , \\
\mathcal{R}\mathrm{o}_{i0}(x_{\mathrm{c}_i}) = \mathcal{R}\mathrm{c}_I(x_{\mathrm{a}_{i0}})\ ,\ \ \mathcal{R}\mathrm{o}_{i1}(x_{\mathrm{c}_i}) = \mathcal{R}\mathrm{c}_J(x_{\mathrm{a}_{i1}})\ ;
\end{array}$$

we consider their r_0 and r_1-projections; and we have to show that

$$(r_0 x_{\mathtt{d}_i}, r_0 x_{\mathtt{c}_i}\rangle, r_0 x_{\mathtt{a}_{i0}}, r_0 x_{\mathtt{a}_{i1}}) \in D^{\#}\mathtt{A}_i \tag{11}$$

if and only if

$$(r_1 x_{\mathtt{d}_i}, r_1 x_{\mathtt{c}_i}, r_1 x_{\mathtt{a}_{i0}}, r_1 x_{\mathtt{a}_{i1}}) \in \hat{D}^{\#}\mathtt{A}_i\ . \tag{12}$$

Let $x_{\mathtt{d}_i} = ((X, \hat{X}, \sigma), (A, \hat{A}, \alpha), \mu)$

with $\sigma: E_I X \xrightarrow{\cong} \hat{X}$, $\alpha: E_J A \xrightarrow{\cong} \hat{A}$, $\mu: F_i X \xrightarrow{\cong} A$;

$x_{\mathtt{c}_i} = ((Y, \hat{Y}, \tau), (B, \hat{B}, \beta), \nu)$

with $\tau: E_I Y \xrightarrow{\cong} \hat{Y}$, $\beta: E_J B \xrightarrow{\cong} \hat{B}$, $\nu: F_i Y \xrightarrow{\cong} B$;

note that $x_{\mathtt{d}_i} \xmapsto{\circ_{i0}} \sigma$, $x_{\mathtt{d}_i} \xmapsto{\circ_{i1}} \alpha$, $x_{\mathtt{c}_i} \xmapsto{\circ_{i0}} \tau$, $x_{\mathtt{c}_i} \xmapsto{\circ_{i1}} \beta$.

The first and third of the above conditions (10) force the first two components of $x_{\mathtt{a}_{i0}}$ to be $(X, \hat{X}, \sigma)$ and $(Y, \hat{Y}, \tau)$, respectively. Let

$$x_{\mathtt{a}_{i0}} = ((X, \hat{X}, \sigma), (Y, \hat{Y}, \tau), f: X \to Y, \hat{f}: \hat{X} \to \hat{Y})\ ;$$

we have

$$\begin{array}{ccc} E_I X & \xrightarrow{\ \sigma\ } & \hat{X} \\ {\scriptstyle E_I f}\downarrow & \circ & \downarrow{\scriptstyle \hat{f}} \\ E_I Y & \xrightarrow[\ \tau\ ]{} & \hat{Y} \end{array}\ . \tag{13}$$

Similarly,

$$x_{\mathtt{a}_{i1}} = ((A, \hat{A}, \alpha), (B, \hat{B}, \beta), g: A \to B, \hat{g}: \hat{A} \to \hat{B})$$

with

$$\begin{array}{ccc} E_J A & \xrightarrow{\alpha} & \hat{A} \\ {\scriptstyle E_J g}\downarrow & \circ & \downarrow{\scriptstyle \hat{g}} \\ E_J B & \xrightarrow[\beta]{} & \hat{B} \end{array} \quad . \tag{14}$$

(11) means

$$\begin{array}{ccc} F_i X & \xrightarrow{\mu} & A \\ \downarrow{\scriptstyle F_i f} & \circ & \downarrow{\scriptstyle g} \\ F_i Y & \xrightarrow[\nu]{} & B \end{array} \quad , \tag{15}$$

whereas (12) means

$$\begin{array}{ccc} \hat{F}_i \hat{X} & \xrightarrow{\hat{\mu}} & \hat{A} \\ \downarrow{\scriptstyle \hat{F}_i \hat{f}} & \circ & \downarrow{\scriptstyle \hat{g}} \\ \hat{F}_i \hat{Y} & \xrightarrow[\hat{\nu}]{} & \hat{B} \end{array} \tag{16}$$

where $\hat{\mu}$ and $\hat{\nu}$ are defined as $\hat{\mu}$ is in (9); we want to see that (15) iff (16). Consider the following diagram:

$$\begin{array}{ccccccccc}
\hat{F}_i \hat{X} & & & & \xrightarrow{\qquad\qquad\hat{\mu}\qquad\qquad} & & & & \hat{A} \\
& \nwarrow{\scriptstyle \hat{F}_i \sigma} & & & 1 & & & \nearrow{\scriptstyle \alpha} & \\
& & \hat{F}_i E_I X & \xrightarrow{e_{iX}} & E_J F_i X & \xrightarrow{E_J \mu} & E_J A & & \\
{\scriptstyle \hat{F}_i \hat{f}}\downarrow & 2 & {\scriptstyle \hat{F}_i E_I f}\downarrow & 3 & {\scriptstyle E_J F_i f}\downarrow & 4 & {\scriptstyle E_J g}\downarrow & 5 & \downarrow{\scriptstyle \hat{g}} \\
& & \hat{F}_i E_I Y & \xrightarrow[e_{iY}]{} & E_J F_i Y & \xrightarrow[E_J \nu]{} & E_J B & & \\
& \swarrow{\scriptstyle \hat{F}_i \tau} & & & 6 & & & \searrow{\scriptstyle \beta} & \\
\hat{F}_i \hat{Y} & & & & \xrightarrow[\qquad\qquad\hat{\nu}\qquad\qquad]{} & & & & \hat{B}
\end{array} \quad .$$

The cells 1 and 6 commute, by the definitions of $\hat{\mu}$ and $\hat{\nu}$ (see (9)). 2 commutes by (13), 3 by the naturality of e_i, and 5 by (14). Note that all arrows except the vertical ones are isomorphisms. If (15) commutes, then so does 4; the resulting commutativity of the outside square is (16) as desired. Conversely, if (16) commutes, then so does 4 (using the isomorphisms in the diagram), and since E_J is faithful, so does (15).

Let us look at the similar verification of preservation of $\circ_\iota$; $I \underset{j}{\overset{i}{\rightrightarrows}} J$. We take $(x_{\circ_{\iota 0}}, x_{\circ_{\iota 1}}, x_{\circ_{\iota 2}}) \in \mathcal{R}[\circ_\iota]$, that is,

$$x_{\circ_{\iota 0}} = ((X, \hat{X}, \sigma), (A, \hat{A}, \alpha), \mu) \in \mathcal{R}\circ_i$$

with $\sigma : E_I X \xrightarrow{\cong} \hat{X}$, $\alpha : E_J A \xrightarrow{\cong} \hat{A}$, $\mu : F_i X \xrightarrow{\cong} A$;

$$x_{\circ_{\iota 1}} = ((X, \hat{X}, \sigma), (B, \hat{B}, \beta), \rho) \in \mathcal{R}\circ_j$$

with the same σ as above, and $\beta : E_J B \xrightarrow{\cong} \hat{B}$, $\rho : F_i X \xrightarrow{\cong} B$

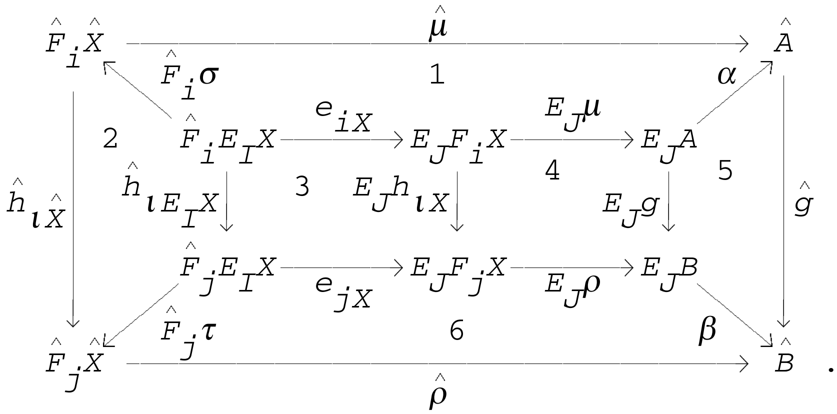
(since we must have $\mathcal{R}\circ_{i0}(x_{\circ_{\iota 0}}) = \mathcal{R}\circ_{j0}(x_{\circ_{\iota 1}})$ (see the first equation in (4)), the first components of $x_{\circ_{\iota 0}}$ and $x_{\circ_{\iota 1}}$ have to agree);

$$x_{\circ_{\iota 2}} = ((A, \hat{A}, \alpha), (B, \hat{B}, \beta), g : A \to B, \hat{g} : \hat{A} \to \hat{B})$$

with (9) holding (see the other two equations in (4)). Looking at the definition of $D^{\#}\circ_\iota$, $\hat{D}^{\#}\circ_\iota$, what we have to see is that

$$\begin{array}{ccc} F_i X & \xrightarrow[\cong]{\mu} & A \\ {\scriptstyle h_{\iota X}}\downarrow & \circ & \downarrow{\scriptstyle g} \\ F_j X & \xrightarrow[\rho]{\cong} & B \end{array} \qquad \text{iff} \qquad \begin{array}{ccc} \hat{F}_i \hat{X} & \xrightarrow[\cong]{\hat{\mu}} & \hat{A} \\ {\scriptstyle \hat{h}_{\iota X}}\downarrow & \circ & \downarrow{\scriptstyle \hat{g}} \\ \hat{F}_j \hat{X} & \xrightarrow[\rho]{\cong} & \hat{B} \end{array} \quad . \tag{17}$$

Consider



The cells 1 and 6 commute for reasons as before. 2 commutes because of the naturality of $\hat{h}_\iota$, 3 because of the naturality of (e_i, e_j) with respect to $\iota: i \to j$, 5 because of (14). 4 is the antecedent of (17) with E_J applied to it, the outer square is the succedent of (17). The assertion in (17) follows.

The remaining properties are the preservation of the T_I, I_I, and of the equalities on the A_I, O_i. These are immediately seen.

We need that $(\mathcal{R}, r_0, r_1)$ "relates $\vec{a}$ to $\vec{b}$". For $\mathcal{X}$ the restricted context involved, $\vec{a}=\langle a_x\rangle_{x\in\mathcal{X}}$, $\vec{b}=\langle b_x\rangle_{x\in\mathcal{X}}$; we want $\vec{c}=\langle c_x\rangle_{x\in\mathcal{X}}\in\tilde{D}[\mathcal{X}]$ such that $r_0(\vec{c})=\vec{a}$, $r_1(\vec{c})=\vec{b}$. For $x\in\mathcal{X}$, $\mathrm{K}_x=\mathrm{O}_I$, define $c_x=1_{E_Ia_x}: E_Ia_x \xrightarrow{\cong} b_x \in \mathcal{R}\mathrm{O}_I$; we have $r_0c_x=a_x$, $r_1c_x=b_x$. For $x\in\mathcal{X}$, $x:\mathrm{A}_I(y, z)$, define $c_x=(c_y, c_z, a_x, b_x)\in\mathcal{R}\mathrm{A}_I$; $c_x\in\mathcal{R}\mathrm{A}_I$ indeed holds since this means

$$\begin{array}{ccc} E_I(a_y) & \xrightarrow{c_y} & b_y \\ {\scriptstyle E_I(a_x)}\downarrow & \circ & \downarrow{\scriptstyle b_x} \\ E_I(a_z) & \xrightarrow[c_z]{} & b_z \end{array}$$

and this holds since $E_I(a_x)=b_x$; also, $r_0(c_x)=a_x$, $r_1(c_x)=b_x$.

This completes the proof of (8).

Let $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}] = (\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}], \Sigma_{\mathrm{diag}}[\boldsymbol{I}])$ be the theory of $\boldsymbol{I}$-diagrams of categories, functors, and natural transformations. $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$ is a theory in ordinary multisorted logic with equality. The models of $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$ are those $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$-structures that are isomorphic to some $D:\boldsymbol{I}\to\mathrm{Cat}$ as an $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$-structure (see above). Indeed, we can easily write down a set of axioms $\Sigma_{\mathrm{diag}}[\boldsymbol{I}]$ over $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ whose models are, up to isomorphism, precisely the D's. Now, the construction $D\mapsto D^{\#}$ is related to an interpretation

$$\Phi:\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}] \longrightarrow [\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]] \qquad (19)$$

of the DSV $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ in the theory $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$; namely, $D^{\#}\cong\bar{D}\circ\Phi$; here, $\bar{D}:[\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]]\longrightarrow\mathrm{Set}$ is the coherent functor induced by $D:\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]\to\mathrm{Set}$.

To describe Φ, I first introduce certain specific formulas over the language $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$. We refer to the (arbitrary) objects, arrows and 2-cell $I \overset{i}{\underset{j}{\rightrightarrows}} J$ in $\boldsymbol{I}$.

$$\bar{\mathrm{I}}_I(\kappa) \overset{=}{\underset{\mathrm{def}}{}} \exists x\in\mathrm{I}_I.\,\mathrm{i}_I(x)=\kappa \qquad (\kappa:\mathrm{A}_I)$$

$$\dot{\mathrm{I}}_I(X,\kappa) \overset{=}{\underset{\mathrm{def}}{}} \bar{\mathrm{I}}_I(\kappa)\wedge\mathrm{d}_I(\kappa)=X \qquad (X:\mathrm{O}_I,\ \kappa:\mathrm{A}_I)$$

$$\bar{\mathrm{T}}_I(f,g,h) \overset{=}{\underset{\mathrm{def}}{}} \exists x\in\mathrm{T}_I.\,\mathrm{t}_{I0}(x)=f\wedge\mathrm{t}_{I1}(x)=g\wedge\mathrm{t}_{I0}(x)=h \qquad (f,g,h:\mathrm{A}_I)$$

$$\dot{\mathrm{T}}_I(X,Y,Z,f,g,h) \overset{=}{\underset{\mathrm{def}}{}}$$
$$\mathrm{d}_I(f)=X\wedge\mathrm{c}_I(f)=Y\wedge\mathrm{d}_I(g)=Y\wedge\mathrm{c}_I(g)=Z\wedge\mathrm{d}_I(h)=X\wedge\mathrm{c}_I(h)=Z\wedge\bar{\mathrm{T}}_I(f,g,h)$$
$$(X,Y,Z:\mathrm{O}_I;\ f,g,h:\mathrm{A}_I)$$

$$\mathrm{Iso}_I(\mu) \overset{=}{\underset{\mathrm{def}}{}} \exists\nu,\kappa,\lambda\in\mathrm{A}_I.\,\bar{\mathrm{I}}_I(\kappa)\wedge\bar{\mathrm{I}}_I(\lambda)\wedge\bar{\mathrm{T}}_I(\mu,\nu,\kappa)\wedge\bar{\mathrm{T}}_I(\nu,\mu,\lambda)$$
$$(\mu:\mathrm{A}_I)$$

$$\dot{\mathrm{O}}_i(X,A,\mu) \overset{=}{\underset{\mathrm{def}}{}} \mathrm{Iso}_J(\mu)\wedge\mathrm{c}_J(x)=A\wedge\exists x\in\mathrm{O}_i.\,\mathrm{o}_{i0}(x)=X\wedge\mathrm{d}_J(\mu)=\mathrm{o}_{i1}(x).$$
$$(X:\mathrm{O}_I,\ A:\mathrm{O}_J,\ \mu:\mathrm{A}_J)$$

$$\mathrm{Comm}_J(\mu,g,h,\nu) \overset{=}{\underset{\mathrm{def}}{}} \exists k\in\mathrm{A}_J.\,\bar{\mathrm{T}}_J(\mu,g,k)\wedge\bar{\mathrm{T}}_J(h,\nu,k)$$
$$(\mu,g,h,\nu:\mathrm{A}_J)$$

$$\dot{\mathrm{A}}_i(X,Y,A,B,\mu,\nu,f,g) \overset{=}{\underset{\mathrm{def}}{}}$$
$$\dot{\mathrm{O}}_i(X,A,\mu)\wedge\dot{\mathrm{O}}_i(Y,B,\nu)\wedge\exists x\in\mathrm{A}_i.\,\mathrm{a}_{i0}(x)=f\wedge\mathrm{Comm}_J(\mu,g,\mathrm{a}_{i1}(x),\nu).$$
$$(X,Y:\mathrm{O}_I,\ A,B:\mathrm{O}_J,\ f:\mathrm{A}_I,\ \mu,\nu,g:\mathrm{A}_J)$$

$$\dot{\mathrm{O}}_{\iota}(X, A, B, \mu, \nu, h) \ \overline{\overline{\mathrm{def}}}$$

$$\dot{\mathrm{O}}_{i}(X, A, \mu)\wedge\dot{\mathrm{O}}_{j}(X, B, \nu)\wedge\exists x\in\mathrm{O}_{\iota}.\mathrm{o}_{i0}\mathrm{o}_{\iota 0}(x)=X\wedge\mathrm{Comm}_{J}(\mu, h, \mathrm{o}_{\iota 2}(x), \nu).$$
$$(X:\mathrm{O}_{I}, \ A, B:\mathrm{O}_{J}, \ \mu, \nu, h:\mathrm{A}_{J})$$

This is the description of the effect of Φ on objects:

$$\Phi(\mathrm{O}_{I}) = [X\in\mathrm{O}_{I}: \mathbf{t}]$$
$$\Phi(\mathrm{A}_{I}) = [X\in\mathrm{A}_{I}: \mathbf{t}]$$
$$\Phi(\mathrm{I}_{I}) = [X\in\mathrm{O}_{I}, \kappa\in\mathrm{A}_{I}: \dot{\mathrm{I}}_{I}(X, \kappa)]$$
$$\Phi(\mathrm{T}_{I}) = [X, Y, Z\in\mathrm{O}_{I}; f, g, h\in\mathrm{A}_{I}: \dot{\mathrm{T}}_{I}(X, Y, Z, f, g, h)]$$
$$\Phi(\mathrm{O}_{i}) = [X\in\mathrm{O}_{I}, A\in\mathrm{O}_{J}, \mu\in\mathrm{A}_{J}: \dot{\mathrm{O}}_{i}(X, A, \mu)]$$
$$\Phi(\mathrm{A}_{i}) = [X, Y\in\mathrm{O}_{I}; A, B\in\mathrm{O}_{J}; f\in\mathrm{A}_{I}; \mu, \nu, g\in\mathrm{A}_{J}: \dot{\mathrm{A}}_{i}(X, Y, A, B, \mu, \nu, f, g)]$$
$$\Phi(\mathrm{O}_{\iota}) = [X\in\mathrm{O}_{I}; A, B\in\mathrm{O}_{J}; \mu, \nu, h\in\mathrm{A}_{J}: \dot{\mathrm{O}}_{\iota}(X, A, B, \mu, \nu, h)]$$

To complete the definition of Φ as in (19), we should also specify the effect of Φ on arrows; this is done in the way straightforwardly suggested by our intentions with Φ.

The fact mentioned above that $D^{\#} \cong \bar{D}\circ\Phi$ holds will be seen directly. In fact, if we define $\bar{D}$ in the standard way (among the possibilities that differ by isomorphisms only), we obtain an equality $D^{\#} = \bar{D}\circ\Phi$.

Next, we explain a translation of formulas to formulas induced by Φ. Temporarily, let us call a FOLDS variable μ *special* if $\mu:\mathrm{O}_{i}(X, A)$ for (unique) suitable $i:I\to J\in\mathrm{Arr}(\boldsymbol{I})$ and $X:\mathrm{O}_{I}$, $A:\mathrm{O}_{J}$. Let us fix a 1-1 mapping $\mu\mapsto\mu^{*}$ of special variables μ to variables μ^{*} in ordinary multisorted logic over $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ such that, when μ is as above, $\mu^{*}:\mathrm{A}_{J}$. The non-special variables over $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ are considered variables over $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$; if $x:\mathrm{O}_{I}$, $x:\mathrm{O}_{I}$ in the sense of multisorted logic, and if $x:\mathrm{A}_{I}(y, z)$, then $x:\mathrm{A}_{I}$ in the sense of multisorted logic.

For a special variable μ as above, we have the formula $\varphi_{[\mu]} \ \overline{\overline{\mathrm{def}}} \ \dot{\mathrm{O}}_{i}(X, A, \mu^{*})$, with the latter formula introduced above. For a finite context $\mathcal{X}$, we let $\mathcal{X}^*=\mathcal{X}-\{\mu\in\mathcal{X}:\ \mu$ special$\}\cup\{\mu^*\in\mathcal{X}:\ \mu$ special$\}$ (exchange every special variable μ for μ^*), and consider the formula $\varphi_{[\mathcal{X}]} \overset{=}{\text{def}} \bigwedge\{\varphi_{[\mu]}:\mu\in\mathcal{X}\text{ special}\}$; $\mathrm{Var}(\varphi_{[\mathcal{X}]})=\mathcal{X}^*$. For a finite set $\mathcal{Y}$ of variables over $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$, we write $\{\mathcal{Y}\}$ for the product-object $[\mathcal{Y}:\mathbf{t}]=\prod_{y\in\mathcal{Y}}[y\in\mathrm{K}_Y:\ \mathbf{t}]$ in $[\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]]$, where $y:\mathrm{K}_Y$.

Recall that, with Φ as in (19), for any finite context $\mathcal{X}$, we have the object $\Phi[\mathcal{X}]$ defined as a certain pullback. Inspection shows that $\Phi[\mathcal{X}]$ can be taken to be $|[\mathcal{X}^*:\varphi_{[\mathcal{X}]}]|$, the domain of the subobject $[\mathcal{X}^*:\varphi_{[\mathcal{X}]}]$ of $\{\mathcal{X}^*\}$; we have a canonical monomorphism $m:\Phi[\mathcal{X}]\rightarrowtail\{\mathcal{X}^*\}$. Thus, for any θ in FOLDS with restricted equality, with $\mathrm{Var}(\theta)\subset\mathcal{X}$, $\Phi[\mathcal{X}:\theta]\rightarrowtail\Phi[\mathcal{X}]$ may be regarded a subobject $\Phi[\mathcal{X}:\theta]\rightarrowtail\Phi[\mathcal{X}]\overset{m}{\rightarrowtail}\{\mathcal{X}^*\}$ of $\{\mathcal{X}^*\}$. We can produce a formula θ^* such that $\mathrm{Var}(\theta^*)=\mathrm{Var}(\theta)^*$ and

$$\Phi[\mathcal{X}:\theta] \underset{\{\mathcal{X}^*\}}{=} [\mathcal{X}^*:\theta^*]$$

(equality of subobjects of $\{\mathcal{X}^*\}$) as follows. We have, for atomic formulas

$$(\mathrm{I}_I(X,\kappa))^* \equiv \dot{\mathrm{I}}_I(X,\kappa)$$
$$(X:\mathrm{O}_I,\ \kappa:\mathrm{A}_I)$$
$$(\mathrm{T}_I(X,Y,Z,f,g,h))^* \equiv \dot{\mathrm{T}}_I(X,Y,Z,f,g,h)$$
$$(X,Y,Z:\mathrm{O}_I;\ f:\mathrm{A}_I(X,Y);\ g:\mathrm{A}_I(Y,Z);\ h:\mathrm{A}_I(X,Z))$$
$$(\mathrm{A}_i(X,Y,A,B,\mu,\nu,f,g))^* \equiv \dot{\mathrm{A}}_i(X,Y,A,B,\mu^*,\nu^*,f,g)$$
$$(X,Y:\mathrm{O}_I;\ A,B:\mathrm{O}_J;\ \mu:\mathrm{O}_i(X,A),\ \nu:\mathrm{O}_i(Y,B),\ f:\mathrm{A}_I(X,Y),\ g:\mathrm{A}_J(A,B))$$
$$(\mathrm{O}_\iota(X,A,B,\mu,\nu,h))^* \equiv \dot{\mathrm{O}}_\iota(X,A,B,\mu^*,\nu^*,h)$$
$$(X:\mathrm{O}_I,\ A,B:\mathrm{O}_J;\ \mu:\mathrm{O}_i(X,A),\ \nu:\mathrm{O}_i(Y,B),\ h:\mathrm{A}_J(A,B))$$
$$(f=_{\mathrm{A}_I(X,Y)}g)^* \equiv \mathrm{d}_I(f)=\mathrm{d}_I(g)=X\wedge\mathrm{c}_I(f)=\mathrm{c}_I(g)=Y\wedge f=_{\mathrm{A}_I}g$$
$$(X,Y:\mathrm{O}_I;\ f:\mathrm{A}_I(X,Y),\ g:\mathrm{A}_I(X,Y))$$
$$(\mu=_{\mathrm{O}_i(X,A)}\nu)^* \equiv \dot{\mathrm{O}}_i(X,A,\mu)\wedge\dot{\mathrm{O}}_i(X,A,\nu)\wedge\mu^*=_{\mathrm{A}_J}\nu^*$$
$$(X:\mathrm{O}_I;\ A:\mathrm{O}_J;\ \mu,\nu:\mathrm{O}_i(X,A));$$

for connectives

$$\mathbf{t}^* \equiv \mathbf{t}$$

$$\mathbf{f}^* \equiv \mathbf{f}$$

$$(\theta\wedge\rho)^* \equiv \theta^*\wedge\rho^*$$

$$(\theta\vee\rho)^* \equiv \theta^*\vee\rho^*$$

$$(\theta\rightarrow\rho)^* \equiv \varphi_{[\mathcal{X}]}\wedge(\theta^*\rightarrow\rho^*) \qquad (\ \mathcal{X} = \mathtt{Var}(\theta\rightarrow\rho)\ )$$

and for quantifiers

$$(\forall x\theta)^* \equiv \forall x\in \mathtt{O}_{I}.\theta^* \qquad (\ x\mathtt{:O}_{I}\ )$$

$$(\forall x\theta)^* \equiv \forall x\in \mathtt{A}_{I}.((\mathtt{d}_{I}(x)=y\wedge \mathtt{c}_{I}(x)=z)\longrightarrow\theta^*) \qquad (\ x\mathtt{:A}_{I}(y,z)\ )$$

$$(\forall x\theta)^* \equiv \forall x^*\in \mathtt{A}_{J}.(\dot{\mathtt{O}}_{i}(y,z,x^*)\longrightarrow\theta^*) \qquad (\ i\mathtt{:}I\rightarrow J,\ x\mathtt{:O}_{i}(y,z)\ );$$

the existential quantifier is dealt with correspondingly.

Notice that if $\mathtt{Var}(\theta)$ is a restricted context, then $\mathtt{Var}(\theta^*)=\mathtt{Var}(\theta)$.

The upshot of all this as follows. For an $\boldsymbol{I}$-diagram $D\mathtt{:}\boldsymbol{I}\rightarrow\mathtt{Cat}$, and its saturation $D^{\#}$, if $\mathcal{X}$ is a finite restricted context over $\mathbf{L}_{\mathtt{anadiag}}[\boldsymbol{I}]$, θ is a FOLDS formula with $\mathtt{Var}(\theta)\subset\mathcal{X}$, and $\vec{a}\in D[\mathcal{X}]$, then

$$D^{\#}\vDash\theta[\vec{a}] \iff D\vDash\theta^*[\vec{a}] \quad .$$

For a structure M over a language extending $\mathtt{L}_{\mathtt{diag}}[\boldsymbol{I}]$, $|M|$ denotes its reduct to $\mathtt{L}_{\mathtt{diag}}[\boldsymbol{I}]$; $|M|$ is the underlying $\boldsymbol{I}$-diagram of M.

(20)(a) Let T be a theory extending $\mathtt{T}_{\mathtt{diag}}[\boldsymbol{I}]$. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathtt{anadiag}}[\boldsymbol{I}]$, σ an $\mathtt{L}_{T}$-formula such that $\mathtt{Var}(\sigma)\subset\mathcal{X}$. The following two conditions (i), (ii) are equivalent.

(i) For any $M, N\models T$ and tuples $\vec{a}\in |M|[\mathcal{X}]$, $\vec{b}\in |N|[\mathcal{X}]$, $M\models\sigma[\vec{a}]$ and $(|M|, \vec{a}) \overset{\sim}{\longleftrightarrow} (|N|, \vec{b})$ imply $N\models\sigma[\vec{b}]$.

(ii) There is θ in FOLDS over $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that for all $M\models T$ and tuples $\vec{a}\in |M|[\mathcal{X}]$, we have $M\models\sigma[\vec{a}]$ iff $M\models\theta^{*}[\vec{a}]$.

(**b**) In particular, if σ is a sentence over L_T, and for any $M, N\models T$, $M\models\sigma$ and $|M| \simeq |N|$ imply $N\models\sigma$, then there is a sentence θ of FOLDS over $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ such that for any $M\models T$, $M\models\sigma$ iff $M\models\theta^{*}$.

Proof. ((ii)$\longrightarrow$(i)) Given θ as (ii), we have

$$M\models\sigma[\vec{a}] \iff M\models\theta^{*}[\vec{a}] \iff |M|\models\theta^{*}[\vec{a}] \iff |M|^{\#}\models\theta[\vec{a}]$$

and similarly,

$$N\models\sigma[\vec{b}] \iff |N|^{\#}\models\theta[\vec{b}] \quad .$$

Assume the hypotheses of (i), in particular, $(|M|, \vec{a}) \overset{\sim}{\longleftrightarrow} (|N|, \vec{b})$. By (8), for $\boldsymbol{L}=\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$, $(|M|^{\#}, \vec{a})\approx_{\boldsymbol{L}}(|N|^{\#}, \vec{b})$, hence, by 5.(2)(b), $|M|^{\#}\models\theta[\vec{a}] \iff |N|^{\#}\models\theta[\vec{b}]$. By what we saw above, this means $M\models\sigma[\vec{a}] \iff N\models\sigma[\vec{b}]$ as desired.

((i)$\longrightarrow$(ii)) Assume (i). We apply 5.(15) with $\bar{\sigma}=m^{*}([\mathcal{X}:\sigma]\in S(\Phi[\mathcal{X}])$ in place of σ; $m:\Phi[\mathcal{X}] \rightarrowtail \{\mathcal{X}\}$ as above. The condition $M\models\sigma[\vec{a}]$ translates into $\langle\vec{a}\rangle\in\bar{M}[\bar{\sigma}]$; now, $\langle\vec{a}\rangle=\vec{a}$. Recall that $\bar{M}\restriction\boldsymbol{L}=\bar{M}\circ\Phi = |M|^{\#}$. Thus, also using (8), we have

$$\text{for all } M, N\models T,\ \vec{a}\in(\bar{M}\restriction\boldsymbol{L})[\mathcal{X}],\ \vec{b}\in(\bar{N}\restriction\boldsymbol{L})[\mathcal{X}],$$
$$\langle\vec{a}\rangle\in\bar{M}[\bar{\sigma}],\ (\bar{M}\restriction\boldsymbol{L}, \vec{a}) \approx (\bar{N}\restriction\boldsymbol{L}, \vec{b}) \implies \langle\vec{b}\rangle\in\bar{N}[\bar{\sigma}] \quad .$$

Since every $P \models \boldsymbol{C}$ is isomorphic to one of the form $\bar{M}$, with $M \models T$, we have the hypothesis of 5.(12). The conclusion gives θ in FOLDS over $\boldsymbol{L}$ such that $\bar{\sigma} =_{\Phi[\mathcal{X}]} \Phi[\mathcal{X}:\theta]$, which suffices.

The result of (20) can be paraphrased by saying that a first-order property of a diagram of categories, functors and natural transformations is invariant under equivalence iff the property is expressible in FOLDS with restricted equality as a property of the saturated anadiagram canonically associated with the diagram.

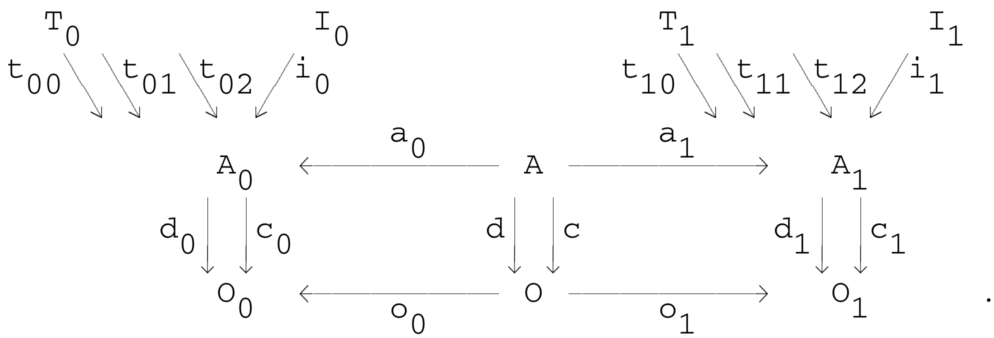
It is left to the reader to formulate stronger versions of (20), based on results of §5.

A *normal* theory for $\boldsymbol{I}$-diagrams is a theory T extending $\mathrm{T}_{\mathrm{diag}}[\boldsymbol{I}]$ such that if $M \models T$ and $D \simeq |M|$, then there is $N \models T$ such that $|N| = D$. For a restricted context $\mathcal{X}$, and formula σ of $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ with $\mathrm{Var}(\sigma) \subset \mathcal{X}$, we define the concepts " σ is preserved/reflected along equivalences of models of T " in the obvious way, in analogy to the case of a single category (see above). We have the following analog of (3).

(20') Let T be a normal theory of $\boldsymbol{I}$-diagrams of categories, functors and natural transformations. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$. Suppose that the first-order formula σ over $\mathrm{L}_{\mathrm{diag}}[\boldsymbol{I}]$ with free variables all in $\mathcal{X}$ is preserved and reflected along equivalences of models of T. Then there is a formula φ in FOLDS over $\mathbf{L}_{\mathrm{anadiag}}[\boldsymbol{I}]$ such that σ is equivalent to φ^{*} (defined above) in models of T.

Let us discuss the special case of $\boldsymbol{I} = (\,0 \xrightarrow{\langle 0,1\rangle} 1\,)$ consisting of two objects and an arrow between them; there are no 2-cells. The intended structures for $\mathrm{L}_{\mathrm{fun}} = \mathrm{L}_{\mathrm{diag}}[(0 \xrightarrow{\langle 0,1\rangle} 1)]$ are *functors*; more precisely, structures consisting of two categories connected by a functor. *Fibrations* are such structures. There are many first-order conditions on fibrations and on objects and morphisms in fibrations that are of interest. On the other hand, in [MR2], several elementary (first-order definable) classes of $\mathrm{L}_{\mathrm{fun}}$-structures were introduced as categorical formulations of modal logic; these "modal categories" are not in general fibrations.

Let me restate the basic concepts for $\mathrm{L}_{\mathrm{fun}}$. $\mathrm{L}_{\mathrm{fun}}$ is the following graph:



$\mathbf{L}_{\mathrm{anafun}} = \mathbf{L}_{\mathrm{anadiag}}[(0 \xrightarrow{\langle 0,1\rangle} 1)]$ is generated by $\mathrm{L}_{\mathrm{fun}}$, subject to appropriate equalities of composite arrows. A functor $F\colon \boldsymbol{X} \longrightarrow \boldsymbol{A}$ is regarded an $\mathrm{L}_{\mathrm{fun}}$-structure in such a way that the interpretation of the relations O and A are the graphs of the object-function and of the arrow-function of F , respectively.

Given functors $F\colon \boldsymbol{X} \longrightarrow \boldsymbol{A}$ and $G\colon \boldsymbol{Y} \longrightarrow \boldsymbol{B}$, an *equivalence* between them is a triple (E_0, E_1, e) as in

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{F} & \boldsymbol{A} \\ E_0\downarrow & \overset{e}{\swarrow} & \downarrow E_1 \\ \boldsymbol{Y} & \xrightarrow[G]{} & \boldsymbol{B} \end{array} \quad : \quad e\colon E_1F \xrightarrow{\cong} GE_0$$

in which E_0 and E_1 are equivalence functors. This notion of equivalence of functors can be motivated by saying that it is the combination of two simpler notions: one is the isomorphism of two parallel functors

$$\boldsymbol{X} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{A} \quad (\cong \downarrow e) ,$$

and the other is the relation between $\boldsymbol{X} \xrightarrow{F} \boldsymbol{A}$ and the composites $\boldsymbol{Y} \xrightarrow{E_0} \boldsymbol{X} \xrightarrow{F} \boldsymbol{A}$, $\boldsymbol{X} \xrightarrow{F} \boldsymbol{A} \xrightarrow{E_1} \boldsymbol{B}$ where E_0 , E_1 are equivalence functors. Since the second notion only involves changing a category to an equivalent one, the change affected on the functor should be an "inessential" one; the resulting composites should be "equivalent" to F ; they are, according to our definition. It is clear that the equivalence relation generated by the two special cases of "equivalence" is the full notion of "equivalence".

For $F:\boldsymbol{X}\to\boldsymbol{A}$ as an $\mathrm{L}_{\mathrm{fun}}$-structure, $F^{\#}$, the saturation of F, an $\mathbf{L}_{\mathrm{anafun}}$-structure, has, among others,

$$F^{\#}\mathrm{O} = \{ (X, A, \mu) : X\in\boldsymbol{X},\ A\in\boldsymbol{A},\ \mu: FX\xrightarrow{\cong}A\} ,$$

and

$$F^{\#}\mathrm{A} = \{ (X, A, \mu, Y, B, \nu, f, g) :$$

$$(X\xrightarrow{f}Y)\in\boldsymbol{X},\ (A\xrightarrow{g}B)\in\boldsymbol{A},\ \mu: FX\xrightarrow{\cong}A,\ \nu: FY\xrightarrow{\cong}B \text{ such that } \begin{array}{ccc} FX & \xrightarrow{\mu} & A \\ Ff\downarrow & \circ & \downarrow g \\ FY & \xrightarrow[\nu]{} & B \end{array} \} .$$

In the spirit of [M2] , within the notation for $F^{\#}\mathrm{A}$, the object A is also written as $F^{\#}_{\mu}(X)$, and $g = F^{\#}_{\mu,\ \nu}(f)$.

The various kinds of modal categories of [MR2] are each defined by a finite set of first-order axioms, and each kind of modal category is invariant under equivalence: if $F:\boldsymbol{X}\to\boldsymbol{A}$ belongs to the given kind, and $G:\boldsymbol{Y}\to\boldsymbol{B}$ is equivalent to $F:\boldsymbol{X}\to\boldsymbol{A}$, then so does $G:\boldsymbol{Y}\to\boldsymbol{B}$. It follows by our invariance theorem (15) that the axioms can be formulated in FOLDS, although *not as statements about the functor itself, but as statements about its saturation.* However, it is not necessary to use the invariance theorem (which is anyway proved in a non-constructive way) to obtain the individual FOLDS-statements; in each case, one can find them directly, rather easily. I will give an example of an axiom thus reformulated in FOLDS.

Suppose the functor $F:\boldsymbol{X}\to\boldsymbol{A}$ preserves monomorphisms, and consider the following condition on F:

(21) For any $X\in\boldsymbol{X}$, the induced map $F_X: \mathrm{S}_{\boldsymbol{X}}(X) \longrightarrow \mathrm{S}_{\boldsymbol{A}}(FX)$ of posets has a right adjoint (denoted $Y\mapsto\Box Y$, the necessity operator).

I want to show that the (21) can be equivalently written as a statement about $F^{\#}$. The simple observation is that if (21) holds, and $\mu : FX \xrightarrow{\cong} A$, then the map $\varphi=\varphi[\mu]:\mathrm{S}_{\mathbf{X}}(X)\longrightarrow \mathrm{S}_{\mathbf{A}}(A)$ defined by $\varphi([Z \overset{r}{\rightarrowtail} X]) = [FZ \overset{\mu\circ Fr}{\rightarrowtail} A]$ also has a right adjoint ($[Z \overset{r}{\rightarrowtail} X]$ is the subobject of X given by r); it is this latter, more general, statement that we can (almost) directly formulate in FOLDS about $F^{\#}$.

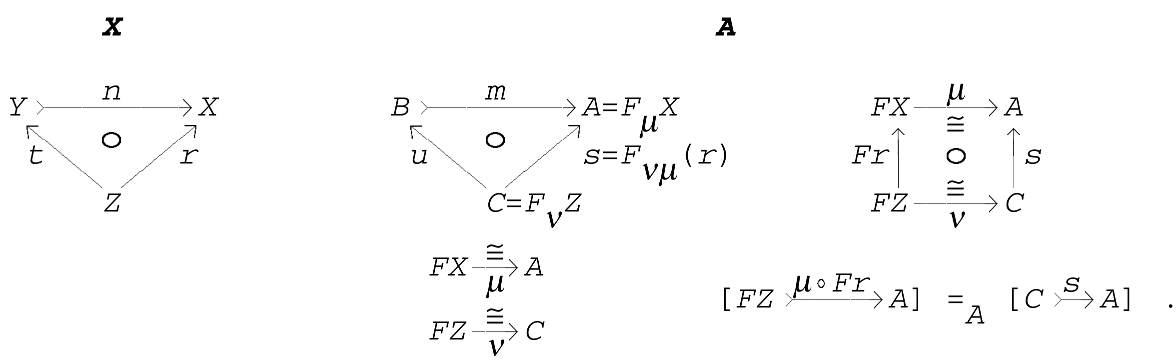
For variables $U, V:\mathrm{O}_0$, $u:\mathrm{A}_0(U, V)$, let $\mathrm{M}_0(U, V, u)$, abbreviated as $\mathrm{M}_0(u)$, and intended to say that u is a monomorphism, be the $\mathbf{L}_{\mathrm{anafun}}$-formula

$$\forall W:\mathrm{O}_0.\forall v, w:\mathrm{A}_0(W, U)(\exists z\in\mathrm{A}_0(W, V).\mathrm{T}_0(v, u, z)\wedge\mathrm{T}_0(w, u, z).\longrightarrow v=_{\mathrm{A}_0(W, U)}w) .$$

Changing all subscripts 0 to 1 , we get the formula $\mathrm{M}_1(u)$. Here is the sentence θ for which $F^{\#}\vDash\theta$ is equivalent to (21):

$$\forall X:\mathrm{O}_0\forall A:\mathrm{O}_1\forall\mu:\mathrm{O}(X, A)\forall B:\mathrm{O}_1\forall m:\mathrm{A}_1(B, A)\{\mathrm{M}_1(m)\longrightarrow$$
$$\exists Y:\mathrm{O}_0\exists n:\mathrm{A}_0(Y, X)[\mathrm{M}_0(n)\wedge\forall Z:\mathrm{O}_0\forall C:\mathrm{O}_1\forall\nu:\mathrm{O}(Z, C)\forall r:\mathrm{A}_0(Z, X)\forall s:\mathrm{A}_1(C, A)$$
$$(\mathrm{M}_0(r)\wedge\mathrm{M}_0(s)\wedge\mathrm{A}(\nu_d, \mu_c, r_{a_0}, s_{a_1})\longrightarrow$$
$$\exists t:\mathrm{A}_0(Z, Y).\mathrm{T}_0(Z, Y, X, t, n, r)\longleftrightarrow\exists u:\mathrm{A}_1(C, B).\mathrm{T}_1(C, B, A, u, m, s))]\} .$$

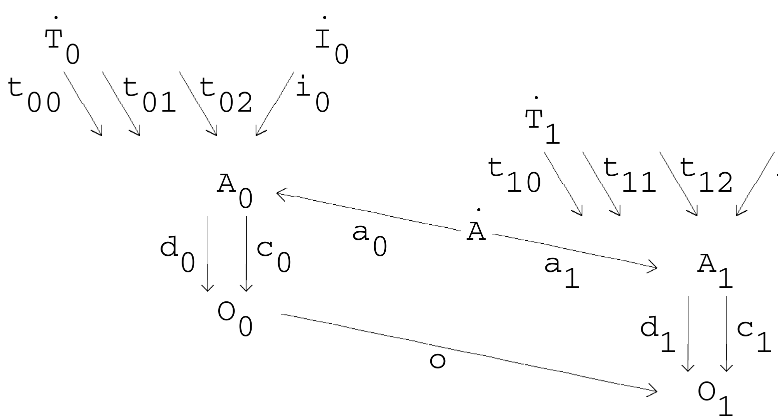
To help reading the sentence interpreted in $F^{\#}$, here is a display of the data involved:



Let us discuss fibrations. The first thing to say is that the concept of fibration is *not* invariant under equivalence of functors. An equivalence functor is, clearly, not necessarily a fibration; an identity functor is one, however; it follows that the concept of fibration is not invariant under equivalences of the form (E_0, Id, id) .

On the other hand, once we know that $F:\boldsymbol{X}\to\boldsymbol{A}$ and $G:\boldsymbol{Y}\to\boldsymbol{B}$ *are* fibrations, then the usually considered additional properties of F , and of diagrams in the fibration F , *are* inherited along arbitrary equivalences $F\xrightarrow{\simeq}G$. The reason is that any equivalence $F\xrightarrow{\simeq}G$ gives rise to a "strong" equivalence from F to G ; and the usually considered properties are invariant under the strong equivalences. In fact, the notion of strong equivalence is related to looking at a fibration as a structure for a new DS vocabulary $\mathbf{L}_{\mathtt{fib}}$. Let me explain.

Consider the following DSV $\mathbf{L}_{\mathtt{fib}}$:



;

here, besides the two obvious copies of $\mathbf{L}_{\mathtt{cat}}$, we have the equalities

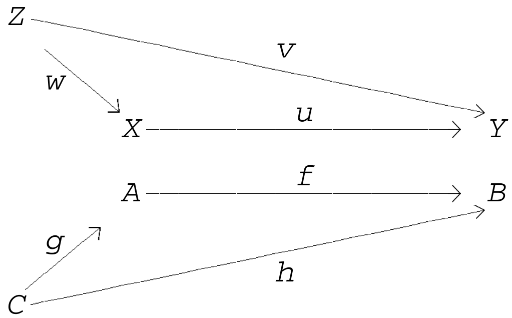
$$od_0a_0 = d_1a_1 \ , \ oc_0a_0 = c_1a_1 \ .$$

(The simpler version that has an arrow $A_0\to A_1$ in place of $A_0\longleftarrow A\longrightarrow A_1$ is not suitable; we need equality on A_1 to express fully the properties of the arrows of the base category; with the version indicated, A_1 would not be a top kind, therefore would not be eligible for carrying an equality predicate in the language.)

Among the $\mathbf{L}_{\mathtt{fib}}$-structures, we find the functors; given $F:\boldsymbol{X}\to\boldsymbol{A}$, it is understood as an $\mathbf{L}_{\mathtt{fib}}$-structure in the natural way in which the 0-copy of $\mathbf{L}_{\mathtt{cat}}$ is $\boldsymbol{X}$, the 1-copy $\boldsymbol{A}$, o is the object-function of F , and the relation A is the graph of the arrow-function of F . Note that whereas $\mathbf{L}_{\mathtt{fib}}$ is a simplification of $L_{\mathtt{fun}}$, its height is 4 , and that of $L_{\mathtt{fun}}$ is 3 . Here is an axiom in FOLDS over $\mathbf{L}_{\mathtt{fib}}$ that formulates the existence of (strongly) Cartesian arrows:

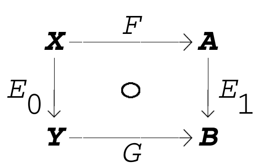
$$\forall A{:}\mathrm{O}_1 \forall B{:}\mathrm{O}_1 \forall f{:}\mathrm{A}_1(A,B) \forall Y{:}\mathrm{O}_0(B) \exists u{:}\mathrm{A}_0(A,B,X,Y)\{\dot{\mathrm{A}}(u,f) \wedge$$
$$\forall C{:}\mathrm{O}_1 \forall g{:}\mathrm{A}_1(C,A) \exists h{:}\mathrm{A}_1(C,B)[\dot{\mathrm{T}}_1(g,f,h) \wedge \forall Z{:}\mathrm{O}_0 \forall v{:}\mathrm{A}_0(C,A,Z,Y)(\dot{\mathrm{A}}(v,h) \rightarrow$$
$$\exists!\, w{:}\mathrm{A}_0(C,A,Z,X)(\dot{\mathrm{A}}(w,g) \wedge \dot{\mathrm{T}}_0(w,u,v)))]\} .$$

Here is a diagram to accompany the sentence:



We have employed the usual abbreviations in writing the atomic formulas; the unique existential quantifier $\exists!$ may be expanded in the expected way. Adding further axioms that are easily obtained, we get a sentence in FOLDS over $\mathbf{L}_{\mathrm{fib}}$ that axiomatizes the notion of fibration. This would not be possible to do over $\mathbf{L}_{\mathrm{anafun}}$.

Let us call functors $F{:}\boldsymbol{X}\rightarrow\boldsymbol{A}$ and $G{:}\boldsymbol{Y}\rightarrow\boldsymbol{B}$ *strongly equivalent*, $F\simeq_s G$, if there is an equivalence $(E_0, E_1, id){:}F\simeq G$ (in the previous sense), with an identity in the third component;



(22)

(23) For functors F and G, $F\simeq_s G$ iff $F\approx_{\mathbf{L}_{\mathrm{fib}}} G$. As a consequence, a first order property of objects and arrows in a prefibration (functor), in particular, in a fibration, is invariant under strong equivalence iff the property is expressible in FOLDS over $\mathbf{L}_{\mathrm{fib}}$.

I only outline the proof. Of course, the second statement is obtained *as a consequence of the first by* §5. Given $(\mathcal{R}, r, s) : F \xleftarrow[\mathbf{L}_{\text{fib}}]{} \!\!\!\rightarrow G$, for any $A \in F\mathrm{O}_1 = \mathrm{Ob}(\boldsymbol{A})$, let us pick $\bar{A} \in \mathcal{R}\mathrm{O}_1$ by the Axiom of Choice such that $r(\bar{A}) = A$, and put $E_1(A) = s(\bar{A})$. For $X \in \mathrm{Ob}(\boldsymbol{X})$, let $A = F(X)$; thus, $X \in F\mathrm{O}_0(A)$. By the very surjectivity of r, there is $\bar{X} \in \mathcal{R}\mathrm{O}_0(\bar{A})$ such that $r(\bar{X}) = X$; we let $E_0(X) = s(\bar{X})$. We have defined the object-functions of equivalence functors $E_1 : \boldsymbol{A} \to \boldsymbol{B}$, $E_0 : \boldsymbol{X} \to \boldsymbol{Y}$, and note (the main point) that, at least as far as the effect on objects is concerned, the diagram (22) commutes (and not just up to an isomorphism). The rest of the verification is left to the reader.

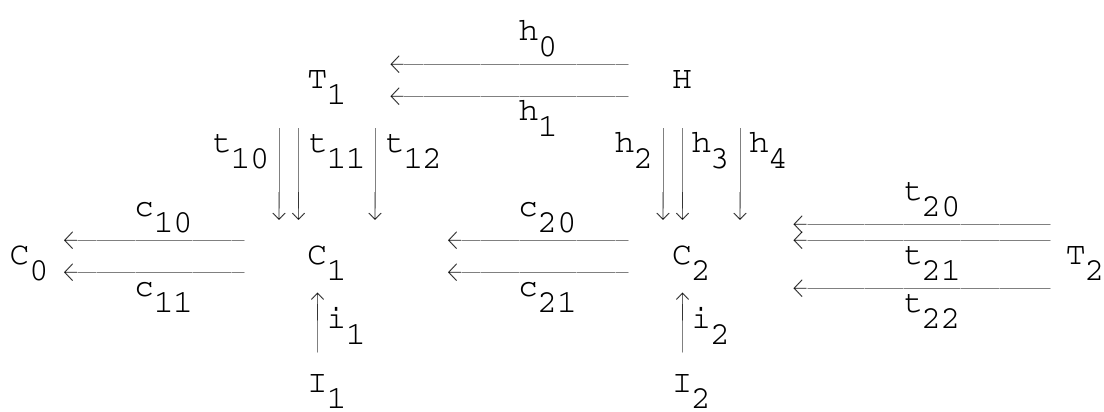
Note that the treatment of fibrations did not require a passage to an "anafunctor". The usually considered properties of fibrations are invariant under strong equivalence. On the other hand, there is a simple, and well-known, "transfer property" for morphisms of fibrations which ensures that for *fibrations* F and G, $F \simeq G$ iff $F \simeq_s G$; in fact if $(E_0, E_1, e) : F \simeq G$, there is $E'_0 : \boldsymbol{X} \to \boldsymbol{Y}$ such that $E'_0 \cong E_0$ and $(E'_0, E_1, id) : F \simeq G$.

§7. Equivalence of bicategories

For 2-categories and bicategories, see [M L], [Be], [S].

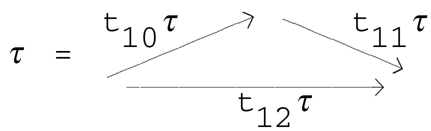
In this section, I discuss invariance of properties of bicategories, and of diagrams in bicategories, under biequivalence (however, I will call "biequivalence" "equivalence of bicategories"). To mention just two examples, the property of a bicategory having finite weighted (indexed) limits (see [S]) is a first-order property invariant under (bi)equivalence; but the property of a 2-category having finite 2-limits is not so invariant. The main result of this section (see the Corollary at the end) implies that the first-mentioned property can be expressed in FOLDS, although not quite in the language of the bicategory itself, but in a modification of it. In fact, the formulation of the said property in FOLDS can be done directly, quite easily.

One possible choice of a similarity type for 2-categories is the following graph $L_{2\text{-cat}}$:



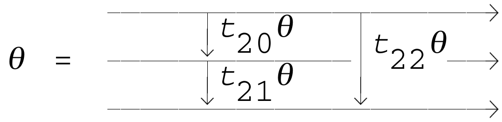
The following explains the meaning of these symbols in the case of a 2-category:

C_0 : (the set of all) objects (0-cells),
C_1 : arrows (1-cells),
C_2 : 2-cells;
c_{10} , c_{20} : domain,
c_{11} , c_{21} : codomain,
T_1 : commutative triangles



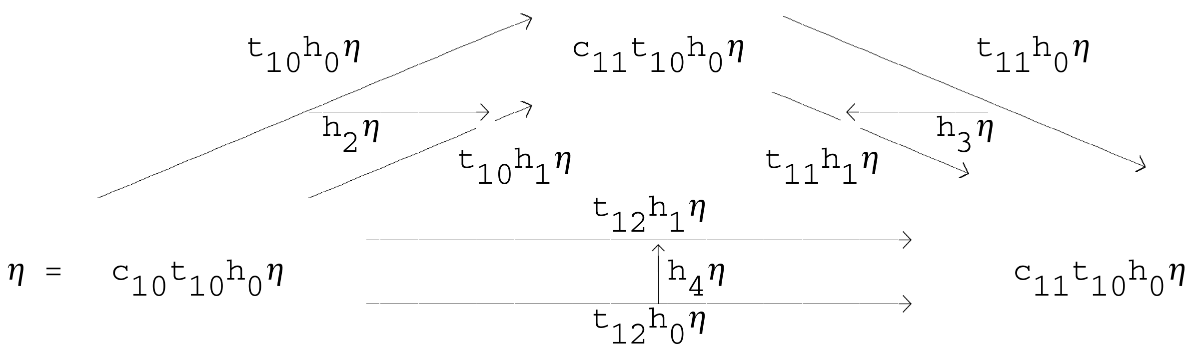
of 1-cells,

T_2 : commutative (for vertical composition) triangles



of 2-cells,

H : commutative (for horizontal composition) triangles



of 2-cells;

I_1 : identity 1-cells,

I_2 : identity 2-cells.

A 2-category is the same as a structure for $L_{2\text{-cat}}$ satisfying certain axioms $\Sigma_{2\text{-cat}}$ in multisorted first order logic with equality(ies) over $L_{2\text{-cat}}$.

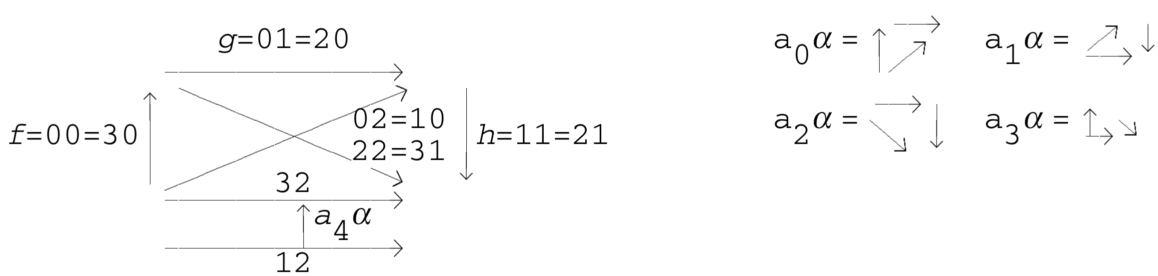
For the concept of bicategory we need, in addition, the sorts A , L and R , accommodating associativity isomorphisms, and left and right identity isomorphisms, respectively. More precisely, we introduce, besides these three new objects, the arrows

$$T_1 \overset{a_0,\, a_1,\, a_2,\, a_3}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow\!\!\longleftarrow} A \xrightarrow{a_4} C_2 \ ,$$

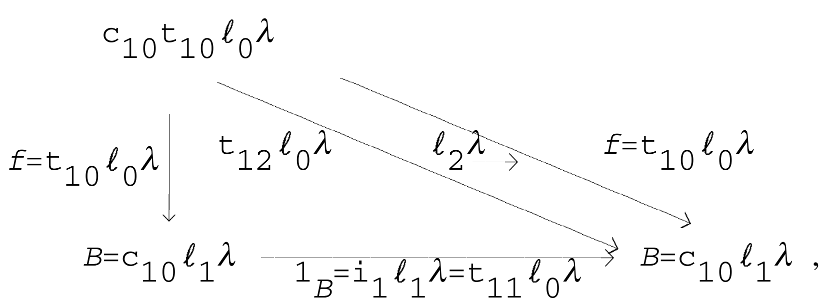
$$T_1 \xleftarrow{\ell_0} L \xrightarrow{\ell_1} I_1 ,\quad L \xrightarrow{\ell_2} C_2 ; \qquad T_1 \xleftarrow{r_0} R \xrightarrow{r_1} I_1 ,\quad R \xrightarrow{r_2} C_2 \ ;$$

with these additions to $L_{2\text{-cat}}$, we obtain L_{bicat}.

In a bicategory, the symbols of $L_{2\text{-cat}}$ are interpreted as expected (as in a 2-category). A stands for the set of 5-tuples $\alpha = (a_0\alpha, a_1\alpha, a_2\alpha, a_3\alpha, a_4\alpha)$ where the $a_i\alpha$ (i=0, 1, 2, 3) are commutative triangles of 1-cells (elements of T_1), and $a_4\alpha$ is a 2-cell, fitting together as in



with ij standing for $t_{1j}(a_i\alpha)$, and $a_4\alpha$ is the associativity isomorphism $\alpha_{f,\,g,\,h}: h(gf) \xrightarrow{\cong} (hg)f$. L is the set of triples $\lambda = (\ell_0\lambda, \ell_1\lambda, \ell_2\lambda)$ as in



and $\ell_2\lambda$ is the identity isomorphism $\lambda_f: 1_B \circ f \xrightarrow{\cong} f$. R is similar, *mutatis mutandis.*

Bicategories are $\mathrm{L}_{\mathrm{bicat}}$-structures satisfying a set Σ_{bicat} of axioms, in multisorted first-order logic with equality over $\mathrm{L}_{\mathrm{bicat}}$. Of course, 2-categories are those bicategories for which each $\alpha_{f,g,h}$, λ_f, ρ_f are identity 2-cells. We write $\mathrm{T}_{\mathrm{bicat}}$ for the theory $(\mathrm{L}_{\mathrm{bicat}}, \Sigma_{\mathrm{bicat}})$.

Now, we introduce the DSV $\mathbf{L}_{\mathrm{anabicat}}$. The underlying simple category is generated by the graph $\mathrm{L}_{\mathrm{bicat}}$, subject to the following equalities:

$$
\begin{aligned}
&c_{10}c_{20} = c_{10}c_{21},\ c_{11}c_{20} = c_{11}c_{21},\\
&c_{11}t_{10} = c_{10}t_{11},\ c_{10}t_{10} = c_{10}t_{12},\ c_{11}t_{11} = c_{11}t_{12},\\
&c_{10}i_1 = c_{11}i_1,\\
&c_{21}t_{20} = c_{20}t_{21},\ c_{20}t_{20} = c_{20}t_{22},\ c_{21}t_{21} = c_{21}t_{22},\\
&c_{20}i_2 = c_{21}i_2,\\
&t_{10}h_0 = c_{20}h_2,\ t_{10}h_1 = c_{21}h_2,\ t_{11}h_0 = c_{20}h_3,\ t_{11}h_1 = c_{21}h_3,\\
&t_{12}h_0 = c_{20}h_4,\ t_{12}h_1 = c_{21}h_4.
\end{aligned}
$$

$$
\begin{aligned}
&t_{10}a_0 = t_{10}a_3,\ t_{11}a_0 = t_{10}a_2,\ t_{12}a_2 = t_{11}a_3,\ t_{12}a_0 = t_{10}a_1,\\
&t_{11}a_1 = t_{11}a_2,\\
&c_{20}a_4 = t_{12}a_1,\ c_{21}a_4 = t_{12}a_3,\\
&i_1\ell_1 = t_{11}\ell_0,\ c_{20}\ell_2 = t_{12}\ell_0,\ c_{21}\ell_2 = t_{10}\ell_0,\\
&i_1r_1 = t_{10}r_0,\ c_{20}r_2 = t_{12}r_0,\ c_{21}r_2 = t_{11}r_0.
\end{aligned}
$$

The relations of $\mathbf{L}_{\mathrm{anabicat}}$ are exactly its maximal objects, that is, its level-3 objects, $\dot{\mathrm{I}}_2$, $\dot{\mathrm{T}}_2$, $\dot{\mathrm{H}}$, $\dot{\mathrm{A}}$, $\dot{\mathrm{L}}$ and $\dot{\mathrm{R}}$.

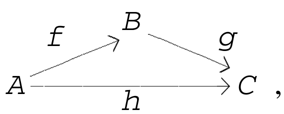
The equalities between composites arise naturally; they hold in a bicategory (as a $\mathrm{L}_{\mathrm{bicat}}$-structure); also, the relations of $\mathbf{L}_{\mathrm{anabicat}}$ are interpreted in a bicategory "relationally"; in brief, every bicategory is an $\mathbf{L}_{\mathrm{anabicat}}$-structure.

In [M2], the concepts of *anabicategory*, and *saturated anabicategory* were introduced. Although these concepts implicitly underlie all that follows, they will not be relied on explicitly.

An anabicategory is an $\mathbf{L}_{\mathrm{anabicat}}$-structure satisfying certain axioms $\Sigma_{\mathrm{anabicat}}$ in FOLDS (with restricted equality) over $\mathbf{L}_{\mathrm{anabicat}}$; a saturated anabicategory is one that satisfies a larger set $\Sigma_{\mathtt{sanabicat}}$ of axioms in FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ (these facts will be seen upon inspecting the definitions in [M2]). An anabicategory is like a bicategory, with the composition functors replaced by composition anafunctors.

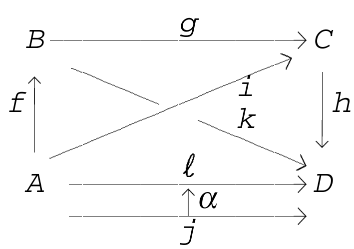
For the reader who has a copy of [M2], I now point out some details, which, however, are not needed later.

Let $\mathcal{A}$ be an anabicategory as in [M2]. In explaining in what way $\mathcal{A}$ is an $\mathbf{L}_{\mathtt{anabicat}}$-structure, we will write T_1 for $\mathcal{A}\mathrm{T}_1$, *etc*. For a diagram



(1)

$\mathrm{T}_1(f, g, h)$ (short for $\mathrm{T}_1(A, B, C, f, g, h)$) is the set $|\circ_{A, B, C}|((f, g), h)$, the set of specifications s for h being the composite of f and g , $h = g\circ_s f$ (see 3.1.(iv) in [M2]). For $f{:}A\to A \in \mathrm{C}_1$, $\mathrm{I}_1(A, f)$ is $|1_A|(*, f)$, the set of specifications i for f being the identity 1-cell on A , $f = 1_{A, i}$ (see 3.1.(iii) in [M2]). For



(2)

in $\mathcal{A}$, and

$$a\in\mathrm{T}_1(f, g, i)\ ,\ \ b\in\mathrm{T}_1(i, h, j)\ ,\ \ c\in\mathrm{T}_1(g, h, k)\ ,\ \ d\in\mathrm{T}_1(f, j, \ell)\ , \quad (3)$$

and $\alpha{:}j\to\ell$,we have

$$\mathrm{A}(a, b, c, d; \alpha) \iff \alpha = \alpha_{a, b, c, d}$$

(see 3.1.(vi) in [M2]). (According to our conventions in logic with dependent sorts, $\mathrm{A}(a, b, c, d; \alpha)$ is short for $\mathrm{A}(A, B, C, D; f, g, h, i, j, k, \ell; a, b, c, d; \alpha)$) .

Every bicategory (as an $\mathbf{L}_{\mathtt{anabicat}}$-structure) is an anabicategory, although not necessarily saturated.

Whereas the interpretation of $\mathtt{T}_1$ in a bicategory, the notion of "commutative triangle of 1-cells", is a *relation* on triangles of 1-cells (where a triangle of 1-cells is three objects and three arrows (1-cells) appropriately related via the domain/codomain functions), in an anabicategory, we have a sort of entity that may be called "specification for a commutative triangle of 1-cells". Such a specification does specify a unique triangle (via the maps $\mathtt{t}_{1i}$); however, the property "commutative" does not figure separately. You may say that a triangle is commutative if *there is* a specification for it to be commutative, but in the concept of anabicategory, we do not work with this notion, we only work with the specifications. In an anabicategory, the expression "commutative triangle (of 1-cells)" should always be interpreted as "specification for a commutative triangle".

Next, we define a translation of the language $\mathbf{L}_{\mathtt{anabicat}}$ into the theory $\mathtt{T}_{\mathtt{bicat}}$; that is, a $[\mathtt{T}_{\mathtt{bicat}}]$-bicat structure $I:\mathbf{L}_{\mathtt{anabicat}} \longrightarrow [\mathtt{T}_{\mathtt{bicat}}]$. Via this translation, every bicategory $\mathcal{A}$ gives rise to $\mathcal{A}^{\#} = \mathcal{A} \circ I$, an $\mathbf{L}_{\mathtt{anabicat}}$-structure. $\mathcal{A}^{\#}$ is in fact a saturated anabicategory; however, for the main result, we will not need this fact; we will use the actual construction of $\mathcal{A}^{\#}$ as an $\mathbf{L}_{\mathtt{anabicat}}$-structure only. (In [M2], $\mathcal{A}^{\#}$ was defined for the special case of a monoidal category (one-object bicategory) $\mathcal{A}$ only.) We define the passage $\mathcal{A} \mapsto \mathcal{A}^{\#}$; this will describe the said interpretation as well.

In $\mathcal{A}^{\#}$, the interpretation of the part

$$\begin{array}{ccccccc} C_0 & \overset{c_{10}}{\underset{c_{11}}{\leftleftarrows}} & C_1 & \overset{c_{20}}{\underset{c_{21}}{\leftleftarrows}} & C_2 & \overset{t_{20}}{\underset{t_{22}}{\overset{t_{21}}{\Lleftarrow}}} & T_2 \\ &&&& \uparrow && \\ &&&& I_2 && \end{array}$$

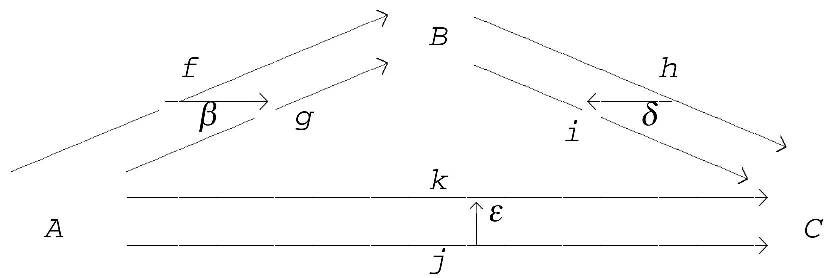
of $\mathbf{L}_{\mathtt{anabicat}}$ is the same as in $\mathcal{A}$.

Under (1) (0-cells and 1-cells in $\mathcal{A}$ as well as in $\mathcal{A}^{\#}$),

$$\mathcal{A}^{\#}\mathrm{T}_1(f, g, h) \underset{\mathrm{def}}{\equiv} \mathrm{Iso}^{\mathcal{A}}(gf, h)$$

= the set of all isomorphism 2-cells $gf \xrightarrow{\cong} h$.

If



and $s \in \mathcal{A}^{\#}\mathrm{T}_1(f, h, j)$, $t \in \mathcal{A}^{\#}\mathrm{T}_1(g, i, k)$, then

$$\mathcal{A}^{\#}\dot{\mathrm{H}}(s, t; \beta, \delta, \varepsilon) \underset{\mathrm{def}}{\Longleftrightarrow} \begin{array}{ccc} hf & \xrightarrow{s} & j \\ {\scriptstyle \delta\cdot\beta}\downarrow & \circ & \downarrow{\scriptstyle \varepsilon} \\ ig & \xrightarrow[t]{} & k \end{array} .$$

Under (2) and (3) in $\mathcal{A}^{\#}$,

$$\mathcal{A}^{\#}\dot{\mathrm{A}}(a, b, c, d; \alpha) \underset{\mathrm{def}}{\Longleftrightarrow} \begin{array}{ccc} h(gf) & \xrightarrow{\alpha_{f,g,h}} & (hg)f \\ \downarrow{\scriptstyle ha} & & {\scriptstyle cf}\downarrow \\ hi & \circ & kf \\ \downarrow{\scriptstyle b} & & {\scriptstyle d}\downarrow \\ j & \xrightarrow{\alpha} & \ell \end{array} \quad ; \qquad (4)$$

here a reference is made to the associativity isomorphism $\alpha_{f,\,g,\,h}$ given with $\mathcal{A}$.

For a 1-cell $f: A \to A$, $\mathcal{A}^{\#}\mathrm{I}_1(A; f) = \mathrm{Iso}(1_A, f)$.

For

$$
\begin{array}{ccc}
A & \xrightarrow{\quad h \quad} & B \\
{\scriptstyle f}\downarrow & \nearrow & \\
B & \xrightarrow[\quad g \quad]{} & B
\end{array} ,
$$

$a \in \mathcal{A}^{\#}\mathrm{T}_1(A, B, B; f, g, h)$, $i \in \mathrm{I}_1(B; g)$, $\lambda : \mathrm{C}_2(h, f)$,

$$
\mathcal{A}^{\#}\dot{\mathrm{L}}(a, i, \lambda) \quad \underset{\mathrm{def}}{\Longleftrightarrow} \quad
\begin{array}{ccc}
1_B f & \xrightarrow{\;if\;} & gf \\
{\scriptstyle \lambda_f}\downarrow & \circ & \downarrow{\scriptstyle a} \\
f & \xleftarrow[\;\lambda\;]{} & h
\end{array} ,
$$

where a reference is made to the identity isomorphism λ_f given with $\mathcal{A}$. The definition of $\mathcal{A}^{\#}\dot{\mathrm{R}}$ is a straightforward variant.

In a bicategory $\mathcal{A}$, a 1-cell $f : B \to A$ is an *equivalence* if there is $f' : A \to B$ such that $f \circ f' \cong 1_A$, $f' \circ f \cong 1_B$; this is equivalent to saying that for any $C \in \mathcal{A}$, the induced functor $f^* : \mathcal{A}(C, B) \longrightarrow \mathcal{A}(C, A)$ is an equivalence of categories.

We have the notion of *functor of bicategories*; this is just a different expression for "homomorphism of bicategories" (see [Be], [S]). A functor $F : \mathcal{X} \longrightarrow \mathcal{A}$ of bicategories is an *equivalence* (*of bicategories*) [instead of "biequivalence"], in notation $F : \mathcal{X} \xrightarrow{\simeq} \mathcal{A}$, if

(i) for every $A \in \mathcal{A}$, there is $X \in \mathcal{X}$ and an equivalence $f : FX \xrightarrow{\simeq} A$;

and

(ii) for $X, Y \in \mathcal{X}$, F induces an equivalence of categories $\mathcal{X}(X, Y) \longrightarrow \mathcal{A}(FX, FY)$.

See [S].

We say that the bicategories $\mathcal{X}$, $\mathcal{A}$ are *equivalent* [instead of "biequivalent"] if there is an equivalence $\mathcal{X} \xrightarrow{\simeq} \mathcal{A}$. Equivalence of bicategories is an equivalence relation (this requires the Axiom of Choice; the fact is well-known, but it also follows from (5) below).

Let $\boldsymbol{L} = \mathbf{L}_{\texttt{anabicat}}$.

(5) For any bicategories $\mathcal{X}$, $\mathcal{A}$, $\mathcal{X} \simeq \mathcal{A}$ iff $\mathcal{X} \approx_{\boldsymbol{L}} \mathcal{A}$.

Proof. (A) ("if") Let $(\mathcal{R}, r_0, r_1) : \mathcal{X}^{\#} \overset{\approx}{\underset{\boldsymbol{L}}{\longleftrightarrow}} \mathcal{A}^{\#}$. We construct $F:\mathcal{X} \xrightarrow{\simeq} \mathcal{A}$.

We write $\langle \varepsilon \rangle$ for $r_0(\varepsilon)$, and $[\varepsilon]$ for $r_1(\varepsilon)$. We will write $\mathcal{R}$ for $r_0^*\mathcal{X}^{\#} = r_1^*\mathcal{A}^{\#}$ too.

Given any $X \in \mathcal{X}C_0$, we pick (by Choice) $\bar{X} \in \mathcal{R}C_0$ such that $\langle \bar{X} \rangle = X$. We put $FX \underset{\text{def}}{=} [\bar{X}]$. For any $f:X \to Y$ in $\mathcal{X}$, pick (by Choice) $\bar{f} \in \mathcal{R}C_1(\bar{X}, \bar{Y})$ such that $\langle \bar{f} \rangle = f$, and for $X \underset{g}{\overset{f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \beta}}}} Y$, $\bar{\beta} \in C_2(\bar{f}, \bar{g})$ with $\langle \bar{\beta} \rangle = \beta$ ($\bar{\beta}$ is uniquely determined); define $Ff = [\bar{f}]$, $F\beta = [\bar{\beta}]$.

For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{X}$, $a \underset{\text{def}}{=} 1_{gf} \in \mathcal{X}^{\#}T_1(f, g, gf)$; let $\bar{a} \in \mathcal{R}T_1(\bar{f}, \bar{g}, \overline{gf})$ such that $\langle \bar{a} \rangle = a$; then $[\bar{a}] \in \mathcal{A}T_1(Ff, Fg, F(gf))$, that is, $[\bar{a}] : Fg \circ Ff \xrightarrow{\cong} F(gf)$. Therefore, we may define $F_{f,\, g} \underset{\text{def}}{=} [\bar{a}]$.

The coherence condition that the $F_{f,\, g}$ have to satisfy (the sense in which F preserves the associativity isomorphisms) reads as follows: given

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \xrightarrow{\ h\ } W ,$$

we have

$$
\begin{array}{ccc}
Fh(FgFf) & \xrightarrow{\alpha_{Ff,\,Fg,\,Fh}} & (FhFg)Ff \\
{\scriptstyle FhF_{f,\,g}}\downarrow & & \downarrow{\scriptstyle F_{g,\,h}Ff} \\
FhF(gf) & \circ ? & F(hg)Ff \\
{\scriptstyle F_{gf,\,h}}\downarrow & & \downarrow{\scriptstyle F_{f,\,hg}} \\
F(h(gf)) & \xrightarrow[F(\alpha_{f,\,g,\,h})]{} & F((hg)f)
\end{array} .
$$

Writing $a=1_{gf}$, $b=1_{h(gf)}$, $c=1_{hg}$, $d=1_{(hg)f}$, this amounts to the same as

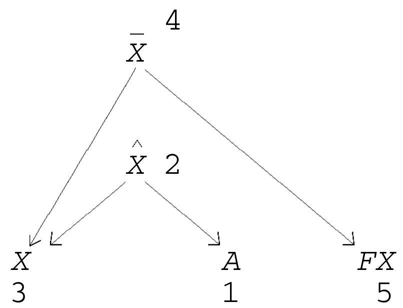
$$
\begin{array}{ccc}
[\bar{h}]([\bar{g}][\bar{f}]) & \xrightarrow{\alpha_{[\bar{f}],\,[\bar{g}],\,[\bar{h}]}} & ([\bar{h}][\bar{g}])[\bar{f}] \\
{\scriptstyle [\bar{h}][\bar{a}]}\downarrow & & \downarrow{\scriptstyle [\bar{c}][\bar{f}]} \\
[\bar{h}][\overline{gf}] & \circ ? & [\overline{hg}][\bar{f}] \\
{\scriptstyle [\bar{b}]}\downarrow & & \downarrow{\scriptstyle [\bar{d}]} \\
[\overline{h(gf)}] & \xrightarrow[{[\overline{\alpha_{f,\,g,\,h}}]}]{} & [\overline{hg}][\bar{f}]
\end{array} .
$$

But by (4), the last commutativity is equivalent to saying that $\mathcal{A}^{\#}\mathrm{A}([\bar{a}], [\bar{b}], [\bar{c}], [\bar{d}]; [\overline{\alpha_{f,\,g,\,h}}])$ holds. The latter is a consequence of $\mathcal{R}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \overline{\alpha_{f,\,g,\,h}})$, which in turn follows from $\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha_{f,\,g,\,h})$, which, finally, holds by (4) since a, b, c and d are identities.

The preservation by F of identity isomorphisms, and that of horizontal composition (see [MP], §4.1, (2)(v) and (2)(iv)) are similar, and use L, R and H, respectively.

The facts that F preserves identity 2-cells and vertical composition of 2-cells are immediate.

We **claim** that for any $A\in\mathcal{A}\mathrm{C}_0$, there is $X\in\mathcal{X}\mathrm{C}_0$ such that $FX\simeq A$. Given A, pick $\hat{X}\in\mathcal{R}\mathrm{C}_0$ with $[\hat{X}]=A$, and let $X=\langle\hat{X}\rangle$. (Picture:



.)

Consider $1_X \in \mathcal{X}C_1(X, X)$, and let $i \in \mathcal{R}C_1(\bar{X}, \hat{X})$, $j \in \mathcal{R}C_1(\hat{X}, \bar{X})$ such that $\langle i \rangle = \langle j \rangle = 1_X$. We have

$$[i]: FX \longrightarrow A, \quad [j]: A \longrightarrow FX$$

in $\mathcal{A}$. Let $f = 1_X \circ 1_X \in \mathcal{X}C_1(X, X)$, and $\bar{f} \in \mathcal{R}C_1(\bar{X}, \bar{X})$, $\hat{f} \in \mathcal{R}C_1(\hat{X}, \hat{X})$ such that $\langle \bar{f} \rangle = \langle \hat{f} \rangle = f$. Consider $1_f \in \mathcal{X}C_2(1_X \circ 1_X, f)$; then $1_f \in \mathcal{X}^{\#}T_1(X, X, X; 1_X, 1_X, f)$. Let $\iota \in T_1(\bar{X}, \hat{X}, \bar{X}; i, j, \bar{f})$, $\iota' \in T_1(\hat{X}, \bar{X}, \hat{X}; j, i, \hat{f})$ such that $\langle \iota \rangle = \langle \iota' \rangle = 1_f$. Then $[\iota] \in \mathcal{A}^{\#}T_1(FX, A, FX; [i], [j], [\bar{f}])$, and thus

$$[\iota]: [j] \circ [i] \xrightarrow{\cong} [\bar{f}]; \qquad (6)$$

similarly,

$$[\iota']: [i] \circ [j] \xrightarrow{\cong} [\hat{f}].$$

But, $\varphi \overset{=}{\underset{\text{def}}{}} \lambda_{1_X}: f \xrightarrow{\cong} 1_X$, that is, $\varphi \in \mathcal{X}^{\#}I_1(X, f)$. Thus, there is $\bar{\varphi} \in \mathcal{R}I_1(\bar{X}, \bar{f})$ (such that $\langle \bar{\varphi} \rangle = \varphi$). Then, $[\bar{\varphi}] \in \mathcal{A}^{\#}I_1(FX, [\bar{f}])$, *i.e.*, $[\bar{\varphi}]: 1_{FX} \xrightarrow{\cong} [\bar{f}]$. Combined with (6), we get $[j] \circ [i] \cong 1_{FX}$. Similarly, $[i] \circ [j] \cong 1_A$. The data $[i]$, $[j]$ provide an equivalence of FX and A as **claimed**.

Let us see that $F_{X, Y}: \mathcal{X}(X, Y) \longrightarrow \mathcal{A}(FX, FY)$ is an equivalence of categories. That it is a bijection on hom-sets is a consequence of the fact that $(\mathcal{R}, r_0, r_1)$ respects the equalities on C_2-sorts. To see essential surjectivity on objects, let $g: FX \to FY$, that is, $g \in \mathcal{A}^{\#}C_1([\bar{X}], [\bar{Y}])$. There is $\hat{f} \in \mathcal{R}C_1(\bar{X}, \bar{Y})$ such that $[\hat{f}] = g$; let $f = \langle \hat{f} \rangle$. We now have $\bar{f}, \hat{f}$ both in $\mathcal{R}C_1(\bar{X}, \bar{Y})$, and both "over" f. There are $i\in\mathcal{R}C_2(\bar{f}, \hat{f})$, $j\in\mathcal{R}C_2(\hat{f}, \bar{f})$, $\bar{\ell}\in\mathcal{R}C_2(\bar{f}, \bar{f})$, $\hat{\ell}\in\mathcal{R}C_2(\hat{f}, \hat{f})$ such that $\langle i\rangle=\langle j\rangle=\langle\bar{\ell}\rangle=\langle\hat{\ell}\rangle=1_f$. We have $\mathcal{X}^{\#}I_2(f; 1_f)$, hence, $\mathcal{R}I_2(\bar{f}; \bar{\ell})$ and $\mathcal{A}^{\#}I_2(Ff; [\bar{\ell}])$; that is, $[\bar{\ell}]=1_{Ff}$. Similarly, $[\hat{\ell}]=1_g$. Since $\mathcal{X}^{\#}T_2(f, f, f; 1_f, 1_f, 1_f)$, we have $\mathcal{R}T_2(\bar{f}, \hat{f}, \bar{f}; i, j, \bar{\ell})$ and $\mathcal{R}T_2(\hat{f}, \bar{f}, \hat{f}; j, i, \hat{\ell})$, and as a consequence, $\mathcal{A}^{\#}T_2(Ff, g, Ff; [i], [j], 1_{Ff})$ and $\mathcal{A}^{\#}T_2(g, Ff, g; [j], [i], 1_g)$; that is, $[j][i]=1_{Ff}$, $[i][j]=1_g$. This shows that $g\cong Ff$ as desired.

(B) ("only if") Let $F:\mathcal{X}\xrightarrow{\simeq}\mathcal{A}$, we construct $(\mathcal{R}, r_0, r_1):\mathcal{X}^{\#}\xleftarrow[\mathbf{L}]{}\!\!\rightarrow\mathcal{A}^{\#}$. We will again write $\langle\varepsilon\rangle$ for $r_0(\varepsilon)$, $[\varepsilon]$ for $r_1(\varepsilon)$.

We put $\mathcal{R}C_0 \overset{=}{\text{def}} \{(X, A, x): X\in\mathcal{X}C_0, A\in\mathcal{A}C_0, x \text{ is an equivalence } x:FX\xrightarrow{\simeq}A\}$; $\langle(X, A, x)\rangle \overset{=}{\text{def}} X$, $[(X, A, x)] \overset{=}{\text{def}} A$.

Let us introduce a helpful notation. For any object D of $\mathbf{L}_{\text{anabicat}}$, any $d_1\in\mathcal{X}D$ and $d_2\in\mathcal{A}D$, $\mathcal{R}D[d_1, d_2]$ stands for $\{d\in\mathcal{R}D: \langle d\rangle=d_1, [d]=d_2\}$, "the fiber of $\mathcal{R}D$ over (d_1, d_2) ". We extend this definition to any sort $\mathcal{R}D(e, e', \ldots)$ in $\mathcal{R}$, in place of $\mathcal{R}D$;

$$\mathcal{R}D(e, e', \ldots)[d_1, d_2] = \{d\in\mathcal{R}D(e, e', \ldots): \langle d\rangle=d_1, [d]=d_2\} ;$$

here, it is assumed that $d_1\in\mathcal{X}^{\#}D(\langle e\rangle, \langle e'\rangle, \ldots)$, $d_2\in\mathcal{A}^{\#}D([e], [e'], \ldots)$.

The definition of $\mathcal{R}C_0$ together with effect of r_1, r_2 on it, can be put, more succinctly, as

$$\mathcal{R}C_0[X, A] = \text{Equiv}(FX, A) = \{x: x:FX\xrightarrow{\simeq}A\} .$$

Continuing, we define, for $f:X\to Y$, $\bar{f}:A\to B$, $x=(X, A, x)$, $y=(Y, B, y)\in\mathcal{R}C_0$,

$$\mathcal{R}C_1(x, y)[f, \bar{f}] = \text{Iso}(y\circ Ff, \bar{f}\circ x) ,$$

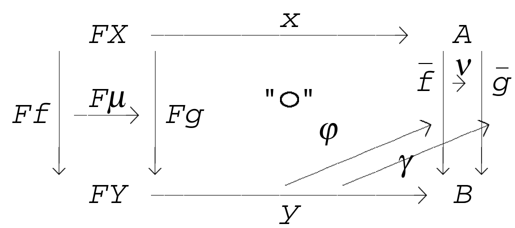
the set of all 2-cell-isomorphisms φ as in

$$
\begin{array}{ccc}
FX & \xrightarrow{\;x\;} & A \\
{\scriptstyle Ff}\downarrow & \overset{\cong}{\nearrow}\,\varphi & \downarrow{\scriptstyle \bar f} \\
FY & \xrightarrow[\;y\;]{} & B
\end{array} .
$$

$\mathcal{R}C_2$ is relational, meaning that its fibers are either $\{*\}$, or $\emptyset$. Instead of "$* \in \mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu]$", we just write "$\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu]$".

For $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{X}$, $A \underset{\bar g}{\overset{\bar f}{\rightrightarrows}} B$ in $\mathcal{A}$, x, y and φ as before, and

$\gamma \in \mathcal{R}C_1(x, y)[g, \bar g]$,

$\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu] \quad \underset{\text{def}}{\Longleftrightarrow}$ 

$$
\underset{\text{def}}{\Longleftrightarrow}\quad
\begin{array}{ccc}
y\circ Ff & \xrightarrow{\;y\circ F\mu\;} & y\circ Fg \\
{\scriptstyle \varphi}\downarrow & \circ & \downarrow{\scriptstyle \gamma} \\
\bar f\circ x & \xrightarrow[\;\nu\circ x\;]{} & \bar g\circ x
\end{array} .
$$

Using that x, y are equivalences, and that F is an equivalence of bicategories, we see that, for fixed x, y, φ, γ, the relation $\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu]$ of the variables μ, ν is a bijection

$$
\mu \mapsto \nu \;:\; \mathcal{X}C_2(f, g) \xrightarrow{\;\cong\;} \mathcal{A}C_2(\bar f, \bar g) .
$$

This implies that $(\mathcal{R}, r_0, r_1)$ preserves the equality relation $\dot{E}_{C_2}$. Also, with reference to

$X \to Y$ with 1-cells f, g, h and 2-cells $\mu: f \to g$, $\rho: g \to h$, $\xi: f \to h$; $A \to B$ with 1-cells $\bar f, \bar g, \bar h$ and 2-cells $\nu: \bar f \to \bar g$, $\sigma: \bar g \to \bar h$, $\zeta: \bar f \to \bar h$, and $\eta \in \mathcal{R}C_1[h, \bar h]$, we easily see that

$$
\mathcal{R}C_2(x, y; \varphi, \gamma)[\mu, \nu] ,\; \mathcal{R}C_2(y, z; \gamma, \eta)[\rho, \sigma] ,\; \rho\mu = \xi ,\; \sigma\nu = \zeta \;\Longrightarrow
$$

$$\mathcal{R}\mathrm{C}_2(x, z; \gamma, \eta)[\xi, \zeta] \ ,$$

from which it follows (by the above bijection $\mu \mapsto \nu$) that

$$\mathcal{R}\mathrm{C}_2(x, y; \varphi, \gamma)[\mu, \nu] \ , \ \mathcal{R}\mathrm{C}_2(y, z; \gamma, \eta)[\rho, \sigma] \ , \mathcal{R}\mathrm{C}_2(x, z; \gamma, \eta)[\xi, \zeta] \ \Longrightarrow \ \rho\mu = \xi \iff \sigma\nu = \zeta \ .$$

This means that $r_0^{-1}(\mathcal{X}^{\#}\mathrm{T}_2) = r_1^{-1}(\mathcal{A}^{\#}\mathrm{T}_2)$; that is, $(\mathcal{R}, r_0, r_1)$ preserves T_2 .

Given

$$(x: FX \xrightarrow{\simeq} A) \in \mathcal{R}\mathrm{C}_0[X, A] \ , \ f: X \to X \text{ in } \mathcal{X} \ , \ \bar{f}: A \to A \text{ in } \mathcal{A} \ , \varphi \in \mathcal{R}\mathrm{C}_1(x, x)[f, \bar{f}] \ ,$$

that is,

$$\begin{array}{ccc} FX & \xrightarrow{x} & A \\ {\scriptstyle Ff}\downarrow & \overset{\cong}{\nearrow}_{\varphi} & \downarrow{\scriptstyle \bar{f}} \\ FX & \xrightarrow[x]{} & A \end{array} \ ,$$

and $a: 1_X \xrightarrow{\cong} f$, $\bar{a}: 1_A \xrightarrow{\cong} \bar{f}$, we have

$$\mathcal{R}\mathrm{I}_1(\underset{\mathrm{c}_{10}\mathrm{i}_1}{x}, \varphi)[a, \bar{a}] \ \underset{\text{def}}{\iff} \ \begin{array}{ccc} xFf & \xrightarrow[\cong]{\varphi} & \bar{f}x \\ {\scriptstyle xFa}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \bar{a}x} \\ xF(1_X) & \circ & 1_A x \\ {\scriptstyle xF_X}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \lambda_x} \\ x1_{FX} & \xrightarrow[\rho_x]{\cong} & x \end{array} \ .$$

Given

(*) $(x: FX \xrightarrow{\simeq} A) \in \mathcal{R}\mathrm{C}_0[X, A]$, $(y: FY \xrightarrow{\simeq} B) \in \mathcal{R}\mathrm{C}_0[Y, B]$,
$(z: FZ \xrightarrow{\simeq} C) \in \mathcal{R}\mathrm{C}_0[Z, C]$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z,\quad X \xrightarrow{i} Z \text{ in } \mathcal{X}, \qquad A \xrightarrow{\bar f} B \xrightarrow{\bar g} C,\quad A \xrightarrow{\bar i} C \text{ in } \mathcal{A},$$

$$(a\colon gf \xrightarrow{\cong} i)\in \mathcal{X}^{\#}\mathrm{T}_1(f, g, i)\ ,\quad (\bar a\colon \bar g\bar f \xrightarrow{\cong} \bar i)\in \mathcal{A}^{\#}\mathrm{T}_1(\bar f, \bar g, \bar i)\ ,$$

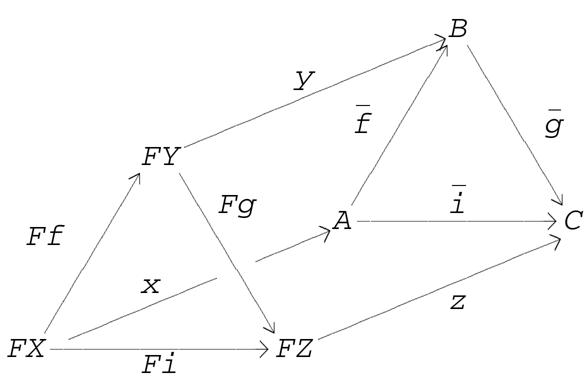
$$\varphi\in \mathcal{R}\mathrm{C}_1(x, y)[f, \bar f]\ ,\quad \gamma\in \mathcal{R}\mathrm{C}_1(y, z)[g, \bar g]\ ,\quad \iota\in \mathcal{R}\mathrm{C}_1(x, z)[i, \bar i]\ ,$$

we have

$$\mathcal{R}\mathrm{T}_1(\varphi, \gamma, \iota)[a, \bar a] \quad \underset{\text{def}}{\Longleftrightarrow}$$

$$\begin{array}{ccccc}
 & & (front) & & \\
(zFg)Ff \xleftarrow[\alpha]{\cong} z(FgFf) & & \xrightarrow{zFaF_{f,\,g}} & & zFi \\
(right)\ \gamma Ff \downarrow & & \circ & & \downarrow \iota\ (bottom) \\
(\bar g y)Ff \xleftarrow[\alpha]{\cong} \bar g(yFf) & & & & \bar i x \\
 & (left)\ \ \bar g\varphi \searrow & & \nearrow \bar a x\ \ (back) & \\
 & & \bar g(\bar f x) \xrightarrow[\alpha]{\cong} (\bar g\bar f)x & &
\end{array}$$

(we have referred to the following diagram of 1-cells, and its "faces":



).

The facts that $\dot{\mathrm{E}}_{\mathrm{I}_1}$, $\dot{\mathrm{E}}_{\mathrm{T}_1}$ are preserved are shown through the facts that the definitions of $\mathcal{R}\mathrm{I}_1$, $\mathcal{R}\mathrm{T}_1$ give bijections $a\mapsto \bar a$.

The proof that $(\mathcal{R}, r_0, r_1)$ so defined preserves $\dot{A}$ and $\dot{H}$ is put into Appendix D .

We have an "augmented" version of (5), similarly to §6. I will state this without proof; for the proof the details of the notion of anafunctor would be needed, together with a concept of cleavage; the proof is, in outline, quite similar to the proof of 5.(8).

Let $\boldsymbol{K}_0$ be the full subcategory of $\boldsymbol{L}=\mathbf{L}_{\mathtt{anabicat}}$ consisting of the objects C_0 , C_1 and C_2 . A *restricted context* is a context of $\boldsymbol{K}_0$. For a bicategory $\mathcal{A}$ and its saturation $\mathcal{A}^{\#}$, $\mathcal{A}\restriction\boldsymbol{K}_0=\mathcal{A}^{\#}\restriction\boldsymbol{K}_0$; $\mathcal{A}[\mathcal{X}]=\mathcal{A}^{\#}[\mathcal{X}]$ whenever $\mathcal{X}$ is restricted.

Let $\mathcal{X}$ be a restricted context. An *augmented bicategory* of type $\mathcal{X}$ is a pair $(\mathcal{A}, \vec{a})$ of a bicategory $\mathcal{A}$ and a tuple $\vec{a}\in\mathcal{A}[\mathcal{X}]$; symbols such as $(\mathcal{A}, \vec{a})$, $(\mathcal{X}, \vec{x})$ stand for augmented bicategories. The notation $E:(\mathcal{X}, \vec{x})\xrightarrow{\simeq}(\mathcal{A}, \vec{a})$ signifies that $E:\mathcal{X}\xrightarrow{\simeq}\mathcal{A}$ and $E(\vec{x})=\vec{a}$. The relations $\xrightarrow{\sim}$ and $\xleftrightarrow{\sim}$ are now defined in the same way as for $\boldsymbol{I}$-diagrams in §6. For bicategories, that is type-∅ augmented bicategories, the relations $\xrightarrow{\sim}$, $\xleftrightarrow{\sim}$ coincide with equivalence $\simeq$. Generalizing (5), we have

(7) For augmented bicategories $(\mathcal{X}, \vec{x})$, $(\mathcal{A}, \vec{a})$, $(\mathcal{X}^{\#}, \vec{x})\approx_{\boldsymbol{L}}(\mathcal{A}^{\#}, \vec{a})$ iff $(\mathcal{X}, \vec{x})\xleftrightarrow{\sim}(\mathcal{A}, \vec{a})$.

We can, analogously to §6, define a recursive translation $\theta\mapsto\theta^{*}$ from FOLDS formulas θ over $\boldsymbol{L}$ to formulas θ^{*} in ordinary multisorted logic over $\mathrm{L}_{\mathtt{bicat}}$ such that, if $\mathcal{X}=\mathtt{Var}(\theta)$ is a restricted context, then $\mathtt{Var}(\theta^{*})=\mathcal{X}$, and for any bicategory $\mathcal{A}$, $\vec{a}\in\mathcal{A}[\mathcal{X}]$, $\mathcal{A}^{\#}\models\theta[\vec{a}]$ iff $\mathcal{A}\models\theta^{*}[\vec{a}]$. We obtain the following analogs of 5.(20) and 5.(20').

(8)(a) Let T be a theory extending $\mathrm{T}_{\mathtt{bicat}}$. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathtt{anabicat}}$, σ an L_T-formula such that $\mathtt{Var}(\sigma)\subset\mathcal{X}$. The following two conditions (i), (ii) are equivalent.

(i) For any $M, N \models T$ and tuples $\vec{a} \in |M|[\mathcal{X}]$, $\vec{b} \in |N|[\mathcal{X}]$, $M \models \sigma[\vec{a}]$ and $(|M|, \vec{a}) \overset{\sim}{\longleftrightarrow} (|N|, \vec{b})$ imply $N \models \sigma[\vec{b}]$.

(ii) There is θ in FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ with $\mathtt{Var}(\theta) \subset \mathcal{X}$ such that for all $M \models T$ and tuples $\vec{a} \in |M|[\mathcal{X}]$, we have $M \models \sigma[\vec{a}]$ iff $M \models \theta^{*}[\vec{a}]$.

(**b**) In particular, if σ is a sentence over $\mathtt{L}_T$, and for any $M, N \models T$, $M \models \sigma$ and $|M| \simeq |N|$ imply $N \models \sigma$, then there is a sentence θ of FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ such that for any $M \models T$, $M \models \sigma$ iff $M \models \theta^{*}$.

(9) Let T be a normal theory of bicategories. Let $\mathcal{X}$ be a finite restricted context over $\mathbf{L}_{\mathtt{anabicat}}$. Suppose that the first-order formula σ over $\mathtt{L}_{\mathtt{bicat}}$ with free variables all in $\mathcal{X}$ is preserved and reflected along equivalences of models of T. Then there is a formula φ in FOLDS over $\mathbf{L}_{\mathtt{anabicat}}$ such that σ is equivalent to φ^{*} in models of T.

(8)(b) follows from (5) (proved in detail above) and §5. As was mentioned, the proofs of (8)(a) and (9) require a more detailed look at anabicategories, similarly to what we did in §5 on anadiagrams in the proof of (20)(a); this work is omitted here.

A paraphrase of (8) can be stated as follows. A first-order property of a bicategory, or of a diagram of 0-cells, 1-cells and 2-cells in a bicategory, is invariant under (bi)equivalence of bicategories if and only if it can be expressed in FOLDS as a statement about the saturation of the bicategory.

Appendix A: An alternative introduction of logic with dependent sorts.

The way we defined the basic concepts of FOLDS in §1 may look somewhat *ad hoc* because of the *a priori* role of the one-way (simple) categories as vocabularies. There is a more direct definition of FOLDS which does not start with assuming simple categories as vocabularies. The notion of "vocabulary" that arises naturally in the direct approach does, nevertheless, turn out to be equivalent to the one we started with in §1. More fully, the direct approach and the original approach turn out to be equivalent in all essential respects. This Appendix describes this state of affairs.

We first define the classes of entities called *kinds, sorts, variables*, *contexts* and *specializations*, and certain relation between these entities. Each kind, sort, variable, context and specialization has a certain *level*, which is a natural number; the definition of the said entities is by a simultaneous induction, proceeding by the level.

For the present purpose, we use the set-theoretic notion of *function* as a set of ordered pairs with the usual condition; the point is that we do not make the "categorical" specification of the codomain as part of the data for a function. Given functions s and t, $t\circ s$ is always defined and is a function; $\mathrm{dom}(t\circ s)=\{x\in\mathrm{dom}(s): s(x)\in\mathrm{dom}(t)\}$, and for $x\in\mathrm{dom}(t\circ s)$, $(t\circ s)(x)=t(s(x))$.

The *kinds of level* 0 are the entities of the form $\langle 0, \emptyset, a\rangle$, with a any set. We say that the kind $K=\langle 0, \emptyset, a\rangle$ *is of arity* $\emptyset$, and we write $K\overset{\circ}{\underset{\circ}{}}\emptyset$. The *sorts of level* 0 are the entities $\langle 1, K, \emptyset\rangle$, with K a kind of level 0; we put $\mathrm{Var}(K)=\emptyset$. A *variable of level* 0 is any entity of the form $\langle 2, X, a\rangle$ with X a sort of level 0, a any set; we say that the variable $x=\langle 2, X, a\rangle$ *is of sort* X, and we write $x:X$. (The definition ensures that every variable of level 0 has a unique sort of level 0.) A *context of level* 0 is a finite set of variables of level 0. A *specialization* of level 0 is a function s whose domain is a context of level 0, and for each $x\in\mathrm{dom}(s)$, $s(x)$ is a variable of the same sort as x.

Suppose n is a natural number, $n>0$, and we have defined what the *kinds, sorts, variables*, *contexts* and *specializations of level* k are, for each $k<n$, such that each context of level $<n$ is a finite set of variables of level $<n$, and each specialization of level $<n$ is a function whose domain and range are sets of variables of level $<n$. Suppose moreover that

we have defined the concept of a variable x being *of sort* X, for variables x and sorts X of level $<n$.

A *kind of level* n is an entity $\langle 0, \mathcal{Y}, a\rangle$, where $\mathcal{Y}$ is a context of level $n-1$, and a is an arbitrary set; we say that $\mathcal{Y}$ is *the arity* of $K=\langle 0, \mathcal{Y}, a\rangle$, and we write $K\overset{\circ}{\underset{\circ}{}}\mathcal{Y}$.

[Kinds are to form sorts (see below); kinds are incomplete sorts, with places for variables to fill; when these places are filled in a correct manner, then we have a sort. In our formulation, we did not introduce "places" as distinct from variables, although we could have done so; we used variables to denote "places"; this is the same as the "nameforms" in [K]. Our procedure may be compared to the one when, in ordinary first-order logic with several sorts, a relation symbol R is introduced in the form $R(x_0, x_1, \ldots x_{n-1})$, with distinct specific variables x_i of definite sorts; the arity of R then may be identified with the *set* $\mathcal{X}=\{x_0, x_1, \ldots x_{n-1}\}$; the atomic formula $R(y_0, y_1, \ldots y_{n-1})$ (y_i of the same sort as x_i) using R can then be identified with the pair (R, s) (= " $R(s)$ ") where s is the function with domain $\mathcal{X}$ for which $s(x_i)=y_i$.]

A *sort of level* n is any $X=\langle 1, K, s\rangle$, written more simply as $K(s)$, where K is a kind of level n, s is a specialization of level $n-1$, and $K\overset{\circ}{\underset{\circ}{}}\mathrm{dom}(s)$;
$\mathrm{Var}(X) \underset{\mathrm{def}}{=} \mathrm{range}(s)$.

For a sort X, a *variable of sort* X is any $x = \langle 2, X, a\rangle$; we write $x{:}X$.

A *context of level* n is any set of the form $\mathcal{Y}\cup\mathcal{X}$ where $\mathcal{Y}$ is a context of level $n-1$, $\mathcal{X}$ is a (non-empty, for having level exactly $=n$) finite set, and each $x\in\mathcal{X}$ is a variable of level n such that if $x{:}X$, then $\mathrm{Var}(X)\subset\mathcal{Y}$.

If $X=K(s)\,(=\langle 1, K, s\rangle)$, then $X|t$ denotes $K(t\circ s)\,(=\langle 1, K, t\circ s\rangle)$. [$X|t$ is the sort obtained "by substituting $t(x)$ simultaneously for each $x\in\mathrm{Var}(X)$ in X ".] t is *a specialization* (of level n) if t is a function whose domain is a context, and for every $x\in\mathrm{dom}(t)$, if $x{:}X$, then $X|t$ is a sort (of level $\le n$), and $t(x)$ is a variable (of level $\le n$) of sort $X|t$ (and there is at least one $x\in\mathrm{dom}(t)$ of level n).

The above may be put in a more compact manner, without talking about levels, as follows. We define classes

$$\mathtt{KIND}\ ,\ \mathtt{CONTEXT}\ ,\ \mathtt{SORT}\ ,\ \mathtt{SPEC}\ ,\ \mathtt{VARIABLE}$$

such that

$$\mathcal{X}\in\mathtt{CONTEXT} \implies \mathcal{X} \text{ is a finite subset of } \mathtt{VARIABLE}\ ,$$
$$s\in\mathtt{SPEC} \implies s \text{ is a function, } \mathtt{dom}(s) \text{ and } \mathtt{range}(s)\subset\mathtt{CONTEXT}\ ;$$

predicates

$$\mathring{\circ} \subset \mathtt{KIND}\times\mathtt{CONTEXT} \quad (\text{read } K\mathring{\circ}\mathcal{Y} \text{ as " } K \text{ is a kind of arity } \mathcal{Y}\text{ "})$$
$$: \subset \mathtt{VARIABLE}\times\mathtt{SORT} \quad (\text{read } x{:}X \text{ as " } x \text{ is a variable of sort } X\text{ "})$$

and the function

$$\mathtt{Var} : \mathtt{SORT}\longrightarrow\mathcal{P}_{\mathtt{fin}}(\mathtt{VARIABLE})\ ,$$

by the closure conditions:

1 $\mathcal{X}\in\mathtt{CONTEXT} \implies \langle 0,\mathcal{X},\mathtt{a}\rangle\in\mathtt{KIND}$ and $\langle 0,\mathcal{X},a\rangle\mathring{\circ}\mathcal{X}$;

2 $\varnothing\in\mathtt{CONTEXT}$;

3 $\mathcal{X}\in\mathtt{CONTEXT}\ ,\ X\in\mathtt{SORT}\ ,\ x{:}X\ ,\ \mathtt{Var}(X)\subset\mathcal{X} \implies \mathcal{X}\cup\{x\}\in\mathtt{CONTEXT}$;

4 $s\in\mathtt{SPEC}\ ,\ K\in\mathtt{KIND}\ ,\ K\mathring{\circ}\mathtt{dom}(s) \implies$
$\langle 1,K,s\rangle\in\mathtt{SORT}$ and $\mathtt{Var}(\langle 1,K,s\rangle)=\mathtt{range}(s)$;

5 $\varnothing\in\mathtt{SPEC}$;

6 $\mathcal{X}\in\mathtt{CONTEXT}\ ,\ s\in\mathtt{SPEC}\ ,\ X\in\mathtt{SORT}\ ,\ X|s\in\mathtt{SORT}\ ,$
$x{:}X\ ,\ x\notin\mathtt{dom}(s)\ ,\ \mathtt{Var}(X)\subset\mathcal{X}\ ,\ y{:}X|s \implies s\cup\{(x,y)\}\in\mathtt{SPEC}$;
$(\ \langle 1,K,s\rangle\,|\,t \underset{\mathrm{def}}{=} \langle 1,K,t\circ s\rangle\ )$

7 $X\in\mathtt{SORT} \implies \langle 2,X,a\rangle\in\mathtt{VARIABLE}$ and $\langle 2,X,a\rangle{:}X$.

By definition, the intended system (KIND, ...) is the *minimal* one satisfying the given closure conditions.

Let us give some examples. Let $\underline{\mathrm{O}}$, $\underline{\mathrm{A}}$, $\underline{\mathrm{A}}_1$, $\underline{U}$, $\underline{V}$, $\underline{u}$, $\underline{v}$ be arbitrary entities, $\underline{U}\neq\underline{V}$, $\underline{u}\neq\underline{v}$. Here are specific kinds, variables, sorts and contexts, introduced by the above rules; at the start of the line, the number of the clause used is shown:

2	$\emptyset\in\mathrm{CONTEXT}$,
1	$\bar{\mathrm{O}} \underset{\mathrm{def}}{=} \langle\emptyset,\emptyset,\underline{\mathrm{O}}\rangle \in \mathrm{KIND}$, $\bar{\mathrm{O}}\overset{\circ}{\underset{\circ}{:}}\emptyset$.
5	$s_0 \underset{\mathrm{def}}{=} (\emptyset:\emptyset\to\emptyset) \in \mathrm{SPEC}$
4	$\mathrm{O} \underset{\mathrm{def}}{=} \langle 1,\bar{\mathrm{O}},s_0\rangle \in \mathrm{SORT}$, $\mathrm{Var}(\mathrm{O}) = \emptyset$
7	$U \underset{\mathrm{def}}{=} \langle 2,\mathrm{O},\underline{U}\rangle \in \mathrm{VARIABLE}$, $U{:}\mathrm{O}$
	$V \underset{\mathrm{def}}{=} \langle 2,\mathrm{O},\underline{V}\rangle \in \mathrm{VARIABLE}$, $V{:}\mathrm{O}$
3 twice	$\{U, V\} \in \mathrm{CONTEXT}$
1	$\mathrm{A} \underset{\mathrm{def}}{=} \langle 0,\{U, V\},\underline{\mathrm{A}}\rangle \in \mathrm{KIND}$, $\mathrm{A}\overset{\circ}{\underset{\circ}{:}}\{U, V\}$
6 twice	$s_1 \underset{\mathrm{def}}{=} \mathrm{id}_{\{U, V\}}:\{U, V\}\to\{U, V\} \in \mathrm{SPEC}$
4	$\mathrm{A}(U, V) \underset{\mathrm{def}}{=} \langle 1,\mathrm{A},s_1\rangle \in \mathrm{SORT}$
7	$u \underset{\mathrm{def}}{=} \langle 2,\mathrm{A}(U, V),\underline{u}\rangle \in \mathrm{VARIABLE}$, $u{:}\mathrm{A}(U, V)$
	$v \underset{\mathrm{def}}{=} \langle 2,\mathrm{A}(U, V),\underline{v}\rangle \in \mathrm{VARIABLE}$, $v{:}\mathrm{A}(U, V)$
3...	$\{U, V, u, v\} \in \mathrm{CONTEXT}$
1	$\mathrm{A}_1 \underset{\mathrm{def}}{=} \langle 0,\{U, V, u, v\},\underline{\mathrm{A}}_1\rangle \in \mathrm{KIND}$

For a variable x , we have a unique sort X_x for which $x{:}\mathrm{X}_x$; $\mathrm{X}_x=\mathrm{K}_x(\mathrm{s}_x)$ for a uniquely determined kind K_x and specialization s_x . For a kind K , $\mathcal{X}_K$ is the context for which $K\overset{\circ}{\underset{\circ}{:}}\mathcal{X}_K$.

A *pre-vocabulary* is a set $\boldsymbol{K}$ of kinds such that $K\in\boldsymbol{K}$, $x\in\mathcal{X}_K$ imply that $\mathrm{K}_x\in\boldsymbol{K}$. (I am talking about *pre*-vocabularies because relations are not yet contemplated.)

We compare the present approach to the one in §1. Let $\boldsymbol{K}$ be a pre-vocabulary. We make $\boldsymbol{K}$ into a category with objects the elements of $\boldsymbol{K}$. Arrows of $\boldsymbol{K}$ are the identity arrows, and the $\mathrm{p}_x^K{:}K\to\mathrm{K}_x$, one for each pair $K\in\boldsymbol{K}$, $x\in\mathcal{X}_K$. Composition is defined thus. Given

$$K \xrightarrow{\mathrm{p}_x^K} \mathrm{K}_x \xrightarrow{\mathrm{p}_y^{\mathrm{K}_x}} \mathrm{K}_y \qquad (\, x\in\mathcal{X}_K ,\ y\in\mathcal{X}_{\mathrm{K}_x}\,) \ ,$$

$X_x = K_x(s_x)$, with $s_x : \mathcal{X}_{K_x} \twoheadrightarrow Var(X_x)$. $z \underset{def}{=} s_x(y) \in Var(X_x) \subset \mathcal{X}_K$; also, $K_z = K_y$; therefore, $K \xrightarrow{p_z^K} K_y$. We define $p_y^{K_x} \circ p_x^K = p_z^K$.

This composition is associative as is seen by using the equality $s(s_y(u)) = s_{s(y)}(u)$, which in turn is part of the definition of s being a specialization.

The category $\boldsymbol{K}$ so defined is clearly a simple category; the levels of kinds as given in the definition above are the same as their levels in $\boldsymbol{K}$.

Let us use $\boldsymbol{K}$ as a category of kinds in the way done in §1. I claim that the resulting notions of variable$_1$, sort$_1$, and context$_1$ are essentially the same as those of variable$_{\boldsymbol{K}}$, sort$_{\boldsymbol{K}}$ and context$_{\boldsymbol{K}}$ in the sense of the present Appendix, with the only kinds allowed the ones in $\boldsymbol{K}$. More precisely, we define, by a simultaneous recursion, functions

$$X \longmapsto \bar{X} : \mathrm{Sort}_{\boldsymbol{K}} \longrightarrow \mathrm{Sort}_1 \tag{1}$$

$$x \longmapsto \bar{x} : \mathrm{Variable}_{\boldsymbol{K}} \longrightarrow \mathrm{Variable}_1 ; \tag{2}$$

by putting $\overline{\langle 2, X, a\rangle} = \langle 2, \bar{X}, a\rangle$, and $\overline{\langle 1, K, s\rangle} = \langle 1, K, \langle x_p\rangle_{p \in K|\boldsymbol{K}}\rangle$, where $x_p = \overline{s(y)}$ for the unique y for which $p = p_y^K$. I leave it to the reader to check that (1) and (2) are bijections, and $\bar{x} : \bar{X} \Longleftrightarrow x : X$. Moreover, we have that the bijection (2) induces a bijection between $\mathrm{Context}_1$ and $\mathrm{Context}_{\boldsymbol{K}}$.

Let us return to the development started in this Appendix. A *relation-symbol* is an entity of the form $\langle 3, \mathcal{X}, a\rangle$ where $\mathcal{X}$ is a context; $\mathcal{X}$ is the *arity* of the relation-symbol $R = \langle 3, \mathcal{X}, a\rangle$; $R \overset{\circ}{\underset{\circ}{\,}} \mathcal{X}$. A *vocabulary* is a set $\boldsymbol{L}$ of kinds and relation-symbols such that the set $\boldsymbol{K}$ of kinds in $\boldsymbol{L}$ is a pre-vocabulary, and if R is a relation-symbol in $\boldsymbol{L}$, $R \overset{\circ}{\underset{\circ}{\,}} \mathcal{X}$, $x \in \mathcal{X}$, then $K_x \in \boldsymbol{K}$.

Our comparison above of pre-vocabularies and simple categories of §1 clearly extends to an essential bijection between vocabularies as defined here, and DSV's of §1.

An *atomic formula* (in logic without equality) is any $\langle 4, R, s\rangle$ where R is a relation-symbol, s is a specialization, and $R \overset{\circ}{\underset{\circ}{}} \mathtt{dom}(s)$.

I leave the rest of the development of FOLDS in the style of this Appendix, and its comparison to the main body of the paper, to the reader.

Appendix B: A fibrational theory of $\mathcal{L}$-equivalence

Consider fibrations $\mathcal{C}\!\downarrow\begin{smallmatrix}\mathbf{E}_{\mathcal{C}}\\ \\ \mathbf{B}_{\mathcal{C}}\end{smallmatrix}$, $\mathcal{D}\!\downarrow\begin{smallmatrix}\mathbf{E}_{\mathcal{D}}\\ \\ \mathbf{B}_{\mathcal{D}}\end{smallmatrix}$, and the category $\mathtt{Fib}[\mathcal{C},\mathcal{D}]$ of all maps

$$M=(M_1,M_2)\;:\;\mathcal{C}\to\mathcal{D}\;::\;\mathcal{C}\!\downarrow\begin{array}{ccc}\mathbf{E}_{\mathcal{C}} & \xrightarrow{M_2} & \mathbf{E}_{\mathcal{D}}\\ & \circ & \\ \mathbf{B}_{\mathcal{C}} & \xrightarrow[M_1]{} & \mathbf{B}_{\mathcal{D}}\end{array}\!\downarrow\mathcal{D} \qquad (1)$$

of fibrations; $\mathtt{Fib}[\mathcal{C},\mathcal{D}]$ is a full subcategory of $[\mathcal{C},\mathcal{D}]$; see [M3]. $\mathtt{Fib}[\mathcal{C},\mathcal{D}]$ is the total category of a fibration denoted $\mathtt{Fib}\langle\mathcal{C},\mathcal{D}\rangle$; its base-category is the functor-category $[\mathbf{B}_{\mathcal{C}},\mathbf{B}_{\mathcal{D}}]$, and the fiber over $U:\mathbf{B}_{\mathcal{C}}\to\mathbf{B}_{\mathcal{D}}$ has objects all the M as in (1) with the fixed $U=M_1$, and arrows as in $\langle\mathcal{C},\mathcal{D}\rangle$ defined in [M3]; the fiber of $\mathtt{Fib}\langle\mathcal{C},\mathcal{D}\rangle$ over U is a full subcategory of the fiber of $\langle\mathcal{C},\mathcal{D}\rangle$ over U. Given $(f:U\to V)\in[\mathbf{B}_{\mathcal{C}},\mathbf{B}_{\mathcal{D}}]$, and $N:\mathcal{C}\to\mathcal{D}$ over V, the Cartesian arrow $M=f^{*}(N)\xrightarrow{h=\theta_f}N$ is obtained by the stipulation that for all $A\in\mathbf{B}_{\mathcal{C}}$, $X\in\mathcal{C}^{A}$, $M(X)\xrightarrow{h_X}X$ is a Cartesian arrow over $f_A:U(A)\to V(A)$; the definition of M on arrows is the obvious one; see also below. The fact that M so defined is a map of fibrations is shown by the diagram:

$$\begin{array}{ccccccc}
MX & & \xrightarrow{M\theta_q} & & MY & & \\
 & \searrow^{h_X} & & & & \searrow^{h_Y} & \\
 & & NX & & \xrightarrow{N\theta_q} & & NY \\
UA & & \xrightarrow[Uq]{} & & UB & & \\
 & \searrow^{f_A} & & & & \searrow^{f_B} & \\
 & & VA & & \xrightarrow[Vq]{} & & VB\;.
\end{array}$$

Here, $\theta_q:X\to Y$ is a Cartesian arrow over $q:A\to B$; the issue is to show that $M\theta_q$ is Cartesian (over Uq). The definition of M on arrows makes $M\theta_q$ an arrow over Uq making the upper quadrangle commute (unique such $M\theta_q$ exists by h_Y being Cartesian). As a composite of Cartesian arrows, $(N\theta_q)\circ h_X$ is Cartesian; as a left factor of the last, $M\theta_q$ is Cartesian.

In what follows, the base categories $\mathbf{B}_{\mathcal{C}}$, $\mathbf{B}_{\mathcal{D}}$ will have finite limits. $\mathtt{Fiblex}\langle\mathcal{C},\mathcal{D}\rangle$ is the subfibration of $\mathrm{Fib}\langle\mathcal{C},\mathcal{D}\rangle$ with base-category $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$, a full subcategory of $[\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}}]$, with fibers unchanged from $\mathrm{Fib}\langle\mathcal{C},\mathcal{D}\rangle$.

Next, assume that $\mathcal{C}$ and $\mathcal{D}$ are $\wedge\exists$-fibrations. We have the *pre*fibration $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$, with base category $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$, and total category $\wedge\exists(\mathcal{C},\mathcal{D})$. The fiber over $U\in\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$ is the full subcategory of the fiber of $\mathrm{Fiblex}\langle\mathcal{C},\mathcal{D}\rangle$ over U with objects the maps of $\wedge\exists$-fibrations $M:\mathcal{C}\to\mathcal{D}$. $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$ is not a fibration; however, for certain maps $f:U\to V$, $f^*(N)$ calculated in $\mathrm{Fiblex}\langle\mathcal{C},\mathcal{D}\rangle$ does belong to $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$, as we proceed to point out (from which it will of course follow that over such f , Cartesian arrows do exist in $\wedge\exists\text{-}\langle\mathcal{C},\mathcal{D}\rangle$).

Assume that $\mathcal{D}$ is a $\wedge\exists$-fibration, with $\mathcal{Q}_{\mathcal{D}}=\mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$. Let us call $q\in\mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$ *surjective* if $\exists_q\mathbf{t}_A=\mathbf{t}_B$. If q is surjective, then for any $Y\in\mathcal{D}^B$, $\exists_q q^*Y=\exists_q(\mathbf{t}_A\wedge q^*Y)$ $=\exists_q\mathbf{t}_A\wedge Y=Y$ (where the second equality is Frobenius reciprocity). It is clear that a pullback of a surjective arrow is surjective, and the composite of two surjective arrows is surjective. It is also clear that if qr is surjective, then so is q .

Let us call a commutative square in $\boldsymbol{B}_{\mathcal{D}}$

$$\begin{array}{ccc} A & \xrightarrow{\ g\ } & B \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle b} \\ A' & \xrightarrow[\ g'\ ]{} & B' \end{array} \qquad\qquad (1')$$

a *quasi-pullback* if the canonical arrow $p:A'\to A\times_B B'=P$ is surjective.

Using the stated properties of surjective maps, we easily see that if in the quasi-pullback (1'), g is surjective, then so is g' .

Consider two adjoining squares and their composite:

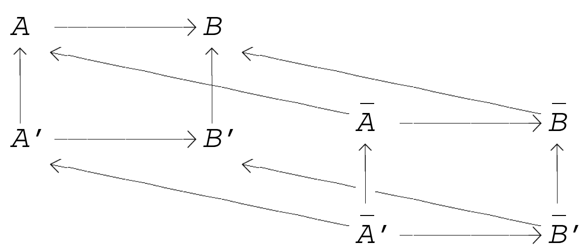
$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \uparrow & 1 & \uparrow & 2 & \uparrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & C \\ \uparrow & 3 & \uparrow \\ A' & \longrightarrow & C' \end{array} \qquad\qquad (1'')$$

(2) The "composite" of two quasi-pullbacks is again a quasi-pullback: if both 1 and 2 are quasi-pullbacks, then so is 3 .

The verification uses both the pullback and composition properties of surjective arrows noted above.

(3) In (1"), if 3 is a quasi-pullback, 2 is a pullback, and 1 commmutes, then 1 is a quasi-pullback.

(3') If in the commutative diagram

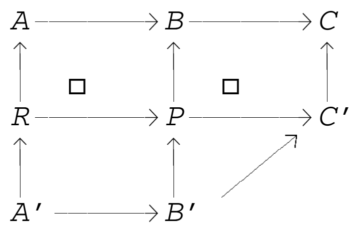


the two quadrangles $AA'\bar{A}\bar{A}'$ and $BB'\bar{B}\bar{B}'$ are pullbacks, and the square $AA'BB'$ is a quasi-pullback, then $\bar{A}\bar{A}'\bar{B}\bar{B}'$ is a quasi-pullback too.

This follows from (2) and (3).

(3") If in (1"), 3 is a quasi-pullback, and AB is surjective, then 2 is a quasi-pullback.

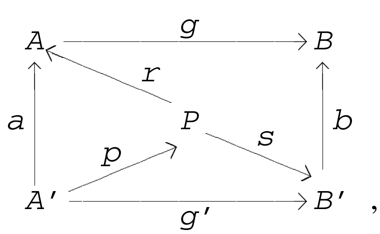
To see this, let $P=B\times_C C'$ for 2 , and $R=A\times_C C'$ for 3 . We have the commutative diagram



with two pullbacks as indicated. Since AB is surjective, so is RP. The assumption gives that $A'R$ is surjective. Now, the composite $A'P$ is surjective, and so is its left factor $B'P$, which is what we want.

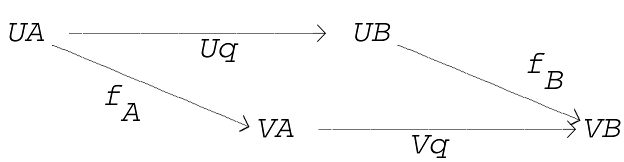
(4) The Beck-Chevalley condition for $\exists$ holds (not just with pullback squares, but also) with quasi-pullback squares.

Indeed, consider the diagram



and calculate: $\exists_{g'} a^* X = \exists_s \exists_p a^* X = \exists_s \exists_p q^* r^* X = \exists_s r^* X = b^* \exists_g X$; the third equality is the "quasi-pullback" property, the last ordinary B-C .

Let us continue to assume that $\mathcal{D}$ is a "full" $\wedge\exists$-fibration ($\mathcal{Q}_{\mathcal{D}}$ contains all arrows), let $\mathcal{C}$ be an arbitrary $\wedge\exists$-fibration, $(q: A \to B) \in \mathbf{B}_{\mathcal{C}}$. We call a map $(f: U \to V) \in \mathrm{Lex}(\mathbf{B}_{\mathcal{C}}, \mathbf{B}_{\mathcal{D}})$ *very surjective with respect to* q if the square



is a quasi-pullback. (The concept of "very surjective" is relative to the fibration $\mathcal{D}$, although it does not depend on the fibration $\mathcal{C}$ except for its base-category.)

(5) If f is very surjective with respect to an arrow q, then so it is with respect to any pullback of q; if f is very surjective with respect to a pair composable arrows, then so it is with respect to their composite.

This follows by (3) and (2).

We say that f is *very surjective* if it is very surjective with respect to every $q \in \mathcal{Q}_{\mathcal{C}}$; by (5), it is enough to require the condition for a "generating set" of q's .

(6) The composite of very surjective arrows (in $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}}, \boldsymbol{B}_{\mathcal{D}})$) is very surjective; the pullback of a very surjective arrow is very surjective.

This follows by using (2) and (3).

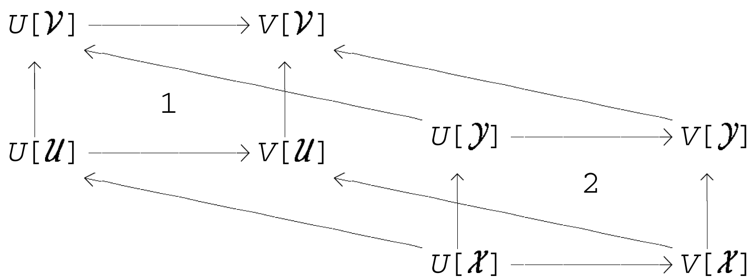
Let $\boldsymbol{K}$ be a simple category, $\boldsymbol{B} = \mathrm{Con}(\boldsymbol{K})^{\mathrm{op}}$; $\mathrm{Lex}(\boldsymbol{B}, \boldsymbol{B}_{\mathcal{D}})$ can be identified with $\mathrm{Fun}(\boldsymbol{K}, \boldsymbol{B}_{\mathcal{D}})$; this is the kind of base-category for the fibrations we are interested in. In §4, we made two different choices for the class $\mathcal{Q}$ of quantifiable arrows in $\boldsymbol{B}$. The choice for the purposes of the main body of §5 is $\mathcal{Q}^{\neq}$; this, in the version that is closed under composition, is simply the class of epimorphisms of $\boldsymbol{B}$. When we make the choice of $\mathcal{Q}^{=}$ for $\mathcal{Q}$, we get as the very surjective maps in the sense of this section the ones we called *normal* ones in §5; we leave it to the reader to verify this.

(6') Let $(f: U \to V) \in \mathrm{Fun}(\boldsymbol{K}, \boldsymbol{B}_{\mathcal{D}})$ be very surjective (with respect to $\mathcal{Q}^{\neq}$). For every finite context $\mathcal{X}$ over $\boldsymbol{K}$, $f_{[\mathcal{X}]}: U[\mathcal{X}] \to V[\mathcal{X}]$ is surjective. For any $K \in \boldsymbol{K}$, $f_K: U(K) \to V(K)$ is surjective.

The first assertion is shown by induction on the cardinality of $\mathcal{X}$. If $\mathcal{X}$ is of positive size, we can write $\mathcal{X}$ as $\mathcal{Y} \dot{\cup} \{x\}$ such that $\mathcal{Y}$ is a context too. By the paragraph after (4) in §4, for $K = \mathrm{K}_x$, we have a pushout diagram

$$\begin{array}{ccc} \mathcal{X}_K & \longrightarrow & \mathcal{X}^*_K \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

in $\mathrm{Con}(\boldsymbol{K})$, which, with $\mathcal{V}=\mathcal{X}_K$, $\mathcal{U}=\mathcal{X}^*_K$, gives rise to



to which (3') is applicable. The square 1 is a quasi-pullback (by f being very surjective), hence, so is 2 . Since by the induction hypothesis, $U[\mathcal{Y}]\to V[\mathcal{Y}]$ is surjective, so is $U[\mathcal{X}]\to V[\mathcal{X}]$.

The second assertion follows immediately from the first by the quasi-pullback

$$\begin{array}{ccc} U(K) & \xrightarrow{\pi^U_K} & U[K] \\ {\scriptstyle f_K}\downarrow & & \downarrow{\scriptstyle f_{[K]}} \\ V(K) & \xrightarrow[\pi^U_K]{} & V[K] \end{array} \quad ;$$

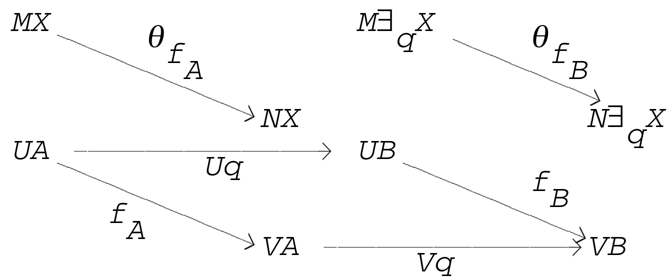
note that $U[K] = U[\mathcal{X}_K]$, *etc.*

Assume now that $\mathcal{C}$ and $\mathcal{D}$ are $\wedge\vee\exists$-fibrations, $\mathcal{D}$ a "full" one.

(7) If $f:U\to V$ is very surjective, and $N\in\wedge\vee\exists(\mathcal{C},\mathcal{D})$, the $M=f^*(N)$ calculated in $\mathrm{Fib}(\mathcal{C},\mathcal{D})$ is in fact in $\wedge\vee\exists(\mathcal{C},\mathcal{D})$.

First of all, using that for each $g \in \mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$, g^* is a morphism of lattices, we immediately see that M preserves the fiberwise operations.

Consider



$$M\exists_q X = f_B^* N\exists_q X = f_B^* \exists_{Nq} NX = \exists_{Mq} f_A^* NX = \exists_{Mq} MX \ ;$$

here, the first equality is the definition of M; the second the quality of N being a morphism of $\exists$-fibrations; the third f being very surjective; and the last again the definition of M.

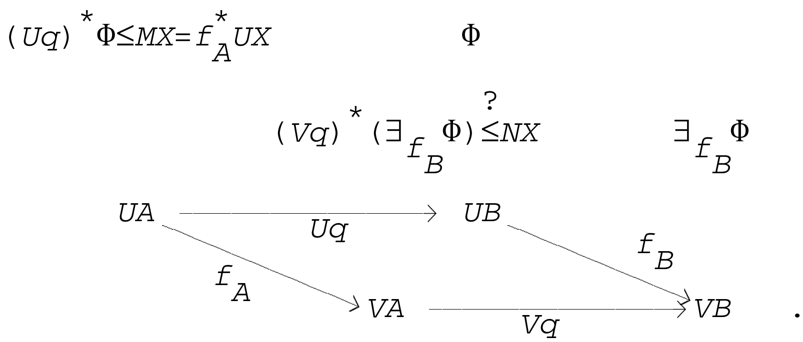
Now, assume in addition that both $\mathcal{C}$ and $\mathcal{D}$ are $\wedge\vee\to\exists\forall$-fibrations, again with $\mathcal{Q}_{\mathcal{D}} = \mathrm{Arr}(\boldsymbol{B}_{\mathcal{D}})$. I claim that

(8) If $f: U \to V$ is very surjective, then $N \in \wedge\vee\to\exists\forall(\mathcal{C}, \mathcal{D})$ implies that $M = f^*(N) \in \wedge\vee\to\exists\forall(\mathcal{C}, \mathcal{D})$.

The additional fiber-wise operation, Heyting implication, is dealt with as before. Let $(q: A \to B) \in \mathcal{Q}_{\mathcal{C}}$, $X \in \mathcal{C}^A$; we want to show that $M\forall_q X = \forall_{Mq} MX$; that is, for any $\Phi \in \mathcal{D}^{UB}$, $\Phi \leq_{UB} M\forall_q X \iff (Uq)^* \Phi \leq_{UA} MX$. The left-to-right implication is automatic. Assume

$$(Uq)^* \Phi \leq_{UA} MX \ , \tag{9}$$

and consider



As indicated, we consider the object $\exists_{f_B}\Phi$ over VB, and claim that the inequality marked ? is true.

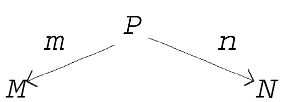
$$(Vq)^*(\exists_{f_B}\Phi) = \exists_{f_A}(Uq)^*\Phi \tag{10}$$

by the (generalized) B-C property for $\exists$ with quasi-pullbacks. (9) implies that

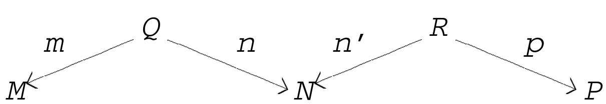
$$\exists_{f_A}(Uq)^*\Phi \le_{UA} \exists_{f_A} MX = \exists_{f_A} f_A^* NX \le NX \ . \tag{11}$$

(10) and (11) imply what we wanted. Now, from this, $\exists_{f_B}\Phi \le \forall_{Vq} NX = N(\forall_q X)$, and $\Phi \le f_B^* \exists_{f_B}\Phi \le f_B^* N(\forall_q X) = M(\forall_q X)$ as desired.

$M, N \in \wedge\vee\to\exists\forall(\mathcal{C}, \mathcal{D})$ are said to be *equivalent*, $M \sim N$, if there is a diagram



such that m, n are Cartesian in $\mathtt{Fiblex}(\mathcal{C}, \mathcal{D})$, and $m_1 : P_1 \to M_1$, $n_1 : P_1 \to N_1$ are very surjective. Equivalence is clearly reflexive and symmetric; it is transitive too; given



with the relevant properties, one forms the pullback

$$\begin{array}{ccccc} & & S_1 & & \\ & \overset{q_1}{\swarrow} & & \overset{r_1}{\searrow} & \\ Q_1 & & & & R_1 \\ & \underset{n_1}{\searrow} & & \underset{n'_1}{\swarrow} & \\ & & N_1 & & \end{array}$$

in $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$, and defines S as $(n''_1)^*(N_1)$, for $n''_1=n_1q_1=n'_1r_1$; let $n'':S\to N$ be the Cartesian arrow over n''_1 . Then n being Cartesian implies that there is a (unique) q over q_1 such that $nq=n''$; similarly for r over r_1 . Since n'' is Cartesian, so are q and r . Since q_1 , r_1 are pullbacks of very surjective arrows, they are very surjective. We conclude that mq and pr are Cartesian arrows over very surjective ones, which proves what we want.

Let us take $T=(\boldsymbol{L},\emptyset)$, the "empty theory" over the DSV $\boldsymbol{L}$, and let $\mathcal{C}=[T]$, a $\wedge\vee\to\exists\forall$-fibration with base-category $\boldsymbol{B}=(\mathrm{Con}[\boldsymbol{K}])^{\mathrm{op}}$ and class of quantifiable arrows $\mathcal{Q}=\mathcal{Q}^{\neq}$. Recall the canonical $i:\boldsymbol{K}\to\boldsymbol{B}$ induced by Yoneda. $\mathrm{Mod}_{\boldsymbol{C}}(\mathrm{T})=\mathrm{Str}_{\boldsymbol{C}}(\boldsymbol{L})$, and we have the fibration $\mathcal{E}:\mathrm{Mod}_{\boldsymbol{C}}(T)\to\boldsymbol{C}^{\boldsymbol{K}}$ as explained in §5. We also have the fibration

$$\mathcal{D}=\mathrm{Fiblex}\langle\mathcal{C},\mathcal{P}(\boldsymbol{C})\rangle : \mathrm{Fiblex}[\mathcal{C},\mathcal{P}(\boldsymbol{C})]\longrightarrow\mathrm{Lex}(\boldsymbol{B},\boldsymbol{C}) .$$

We have a "forgetful" morphism $()^{-}:\mathcal{D}\to\mathcal{E}$; $()^{-}_{1}$ is the equivalence

$$U\mapsto U\circ i : \mathrm{Lex}(\boldsymbol{B},\boldsymbol{C})\xrightarrow{\simeq}\boldsymbol{C}^{\boldsymbol{K}};$$

and $()^{-}_{2}$ is defined as $P\mapsto P^{-}$ was defined in §4 (see (5)) for the special case when $P\in\mathrm{Mod}_{\mathcal{P}(\boldsymbol{C})}(\mathcal{C})\subset\mathrm{Fiblex}[\mathcal{C},\mathcal{P}(\boldsymbol{C})]$. It is easy to verify that $()^{-}$ is a morphism of fibrations.

We have the quasi-inverse

$$U\mapsto[U] : \boldsymbol{C}^{\boldsymbol{K}}\xrightarrow{\simeq}\mathrm{Lex}(\boldsymbol{B},\boldsymbol{C}) \qquad (12)$$

specified so that $[U]([\mathcal{X}]) = U[\mathcal{X}]$; we have the canonical isomorphism $j_U\colon [U]^{-} \cong U$ natural in U. $(\,)^{-}\colon \mathcal{D}\to\mathcal{E}$ restricts to an equivalence

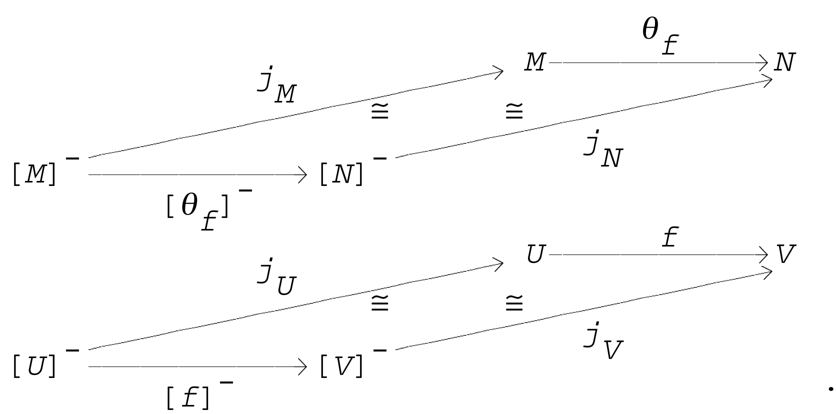
$$(\,)^{-} : \mathrm{Mod}^{\mathrm{iso}}_{\mathcal{P}(\boldsymbol{C})}(\mathcal{C}) \longrightarrow \mathrm{Mod}^{\mathrm{iso}}_{\boldsymbol{C}}(T) \quad , \qquad (13)$$

whose quasi-inverse is

$$M \mapsto [M] : \mathrm{Mod}^{\mathrm{iso}}_{\boldsymbol{C}}(T) \longrightarrow \mathrm{Mod}^{\mathrm{iso}}_{\mathcal{P}(\boldsymbol{C})}(\mathcal{C}) \subset \mathrm{Fiblex}[\mathcal{C},\mathcal{P}(\boldsymbol{C})]$$

constructed in §4 , with the canonical isomorphism $j_M\colon [M]^{-} \cong M$ natural in M. These are connected to (12) by $[M]_1 = [M\restriction\boldsymbol{K}]$, $(j_M)_1 = j_{M\restriction\boldsymbol{K}}$.

Let us deduce (1)(b) of §5 from (8); let's use the notation and hypotheses of 5.(1)(b). Consider the following diagram in the fibration $\mathcal{E}$:



The two quadrangles commute, by the naturality of j . It follows that $[\theta_f]^{-}\colon [M]^{-} \longrightarrow [N]^{-}$ is Cartesian over $[f]^{-}\colon [U]^{-} \longrightarrow [V]^{-}$. Consider the Cartesian arrow $\theta_{[f]}\colon [f]^{*}[N] \longrightarrow [N]$ over $[f]\colon [U] \longrightarrow [V]$ in $\mathcal{D}$. Since $(\,)^{-}$ is a morphism of fibrations,

$$(\theta_{[f]})^{-} : ([f]^{*}[N])^{-} \longrightarrow [N]^{-}$$

is Cartesian over the same $[f]^{-}\colon [U]^{-} \longrightarrow [V]^{-}$. It follows that there is an isomorphism

$([f]^*[N])^- \xrightarrow{\cong} M$ over $1_{[U]^-}$. But then, since (13) is full and faithful, it follows that $[f]^*[N] = M$. Hence,

$$M[\mathcal{X}:\varphi] = ([f]^*[N])[\mathcal{X}:\varphi] = f^*_{\mathcal{X}}([N][\mathcal{X}:\varphi]) = f^*_{\mathcal{X}}(N[\mathcal{X}:\varphi]) ,$$

where the second equality is the description of Cartesian arrows in $\mathcal{D}$, the last is the definition of $[N]$; and this is what was to be proved.

Continuing in this manner, we see that, for $M, N \in \mathrm{Mod}_{\mathcal{C}}(T)$, $M \sim_{\mathcal{L}} N$ in the sense of §4 iff $[M] \sim [N]$ in the sense of this Appendix.

Appendix C: More on $\boldsymbol{L}$-equivalence and equality.

Ordinary multisorted first-order logic without equality and without operation symbols (only relations are allowed) is a special case of FOLDS as follows. Let L be a multisorted, purely relational vocabulary. We associate a DSV $\boldsymbol{L}$ with L. The kinds of $\boldsymbol{L}$ are the sorts of L; the relations of $\boldsymbol{L}$ are the relation symbols of L. For R is sorted " $R\subset\prod_{i<n}X_i$ ", we have proper arrows $p_i^R:R\to X_i$ $(i<n)$. This completes the description of $\boldsymbol{L}$. Clearly, the $\boldsymbol{L}$-structures are essentially the same as the L-structures.

$\boldsymbol{L}$ just constructed is a very simple DSV; its category of kinds has height 1.

Now, a natural notion of "isomorphism" for L-structures "without equality" is the ordinary notion of isomorphism modified by dropping single-valuedness and 1-1-ness. Let M, N be L-structures. By definition, $h:M\underset{L}{\overset{\sim}{\longleftrightarrow}}N$ means a family of relations $h_X:MX\longrightarrow\!\!\!\!\!|\,\,NX$ ($X\in\mathrm{Sort}(L)$) such that $\mathrm{dom}(h_X)=MX$, $\mathrm{range}(h_X)=NX$, and for any " $R\subset\prod_{i<n}X_i$ " in L, $\vec{a}=\langle a_i\rangle_{i<n}\in\prod_{i<n}MX_i$, $\vec{b}=\langle b_i\rangle_{i<n}\in\prod_{i<n}NX_i$, we have that $a_i h_{X_i} b_i$ for all $i<n$ (briefly, $\vec{a}h\vec{b}$) implies that $\vec{a}\in MR\iff\vec{b}\in NR$. It is pretty clear that $h:M\underset{L}{\overset{\sim}{\longleftrightarrow}}N$ preserves the meaning of L-formulas *without equality*: $\vec{a}h\vec{b}\Longrightarrow(M\models\varphi[\vec{a}]\iff N\models\varphi[\vec{a}])$; this would hold good for infinitary logic, and other extended notions of "formula". It is also clear that if for each sort X of L, there is a relation " $\mathrm{E}_X\subset X\times X$ " whose interpretation in both M and N is ordinary equality on X, then $h:M\underset{L}{\overset{\sim}{\longleftrightarrow}}N$ is the same as an ordinary isomorphism $M\xrightarrow{\cong}N$.

The last-mentioned notion of "relational isomorphism" coincides with the relational version of $\boldsymbol{L}$-equivalence, for $\boldsymbol{L}$ the DSV constructed for L as above, defined as follows. For a general DSV $\boldsymbol{L}$, we call the $\boldsymbol{L}$-equivalence $(W,m,n):M\underset{\boldsymbol{L}}{\longleftrightarrow}N$ *relational* if m and n are jointly monomorphic; we indicate the said quality by the letter r in $(W,m,n):M\underset{\boldsymbol{L}}{\overset{\mathrm{r}}{\longleftrightarrow}}N$. This means that for every kind K in $\boldsymbol{L}$, the pair (m_K,n_K) of functions is jointly monomorphic, that is, the span $MK\xleftarrow{m_K}WK\xrightarrow{n_K}NK$ is a relation.

For simplicity, we deal with Set-valued structures in what follows. Suppose $(W, m, n): M \overset{r}{\underset{\mathbf{L}}{\leftrightarrow}} N$. For each kind K, define the relation $\rho_K \subset MK \times NK$ by $a\rho_K b \iff \exists c \in WK.\, m_K c = a \wedge n_K c = b$. For $\mathcal{X}$ a finite context, $\vec{a} = \langle a_x \rangle_{x \in \mathcal{X}} \in M[\mathcal{X}]$, $\vec{b} = \langle b_x \rangle_{x \in \mathcal{X}} \in N[\mathcal{X}]$, we write $\vec{a}\rho_{\mathcal{X}}\vec{b} \iff \exists \vec{c} \in W[K].\, m\vec{c} = \vec{a} \wedge n\vec{c} = \vec{b}$. It turns out however that $\vec{a}\rho_{\mathcal{X}}\vec{b} \iff \forall x \in \mathcal{X}.\, a_x \rho_{K_x} b_x$. Indeed, the left-to-right direction is obvious. Conversely, let $c_x \in WK_x$ such that $m_{K_x} c_x = a_x \wedge n_{K_x} c_x = b_x$. I claim that $\vec{c} = \langle c_x \rangle_{x \in \mathcal{X}} \in W[K]$. For this, we need that if $y \in \mathcal{X}$, $p \in K_y | \mathbf{K}$, then

$$c_{x_{y,p}} = (Wp)(c_y) \ . \tag{1}$$

But $m(Wp)(c_y) = (Mp)(mc_y) = (Mp)(a_y) = a_{x_{y,p}}$, and similarly $n(Wp)(c_y) = b_{x_{y,p}}$; since $c = c_{x_{y,p}} \in WK_{x_{y,p}}$ is uniquely determined by the property $m(c) = a_{x_{y,p}}$ & $n(c) = b_{x_{y,p}}$, (1) follows.

As a consequence, a relational equivalence can be described in terms of the relations ρ_K as follows. A *relational equivalence* $\rho: M \overset{r}{\underset{\mathbf{L}}{\leftrightarrow}} N$ is a family $\rho = \langle \rho_K \rangle_{K \in \mathbf{K}}$ of relations $\rho_K \subset MK \times NK$ such that, with

$$\vec{a}\rho_{\mathcal{X}}\vec{b} \underset{\text{def}}{\iff} \forall x \in \mathcal{X}.\, a_x \rho_{K_x} b_x , \tag{2}$$

the following hold:

(3) For any $p: K \to K_p$, $a \in MK$, $b \in NK$

$$a\rho_K b \implies (Mp)(a)\rho_{K_p}(Np)(b) \ .$$

(4) For any $K \in \mathbf{K}$, $\vec{a} \in M[K] = M[\mathcal{X}_K]$, $\vec{b} \in N[K] = N[\mathcal{X}_K]$,

$$\vec{a}\rho_{\mathcal{X}_K}\vec{b} \ \& \ a \in MK(\vec{a}) \implies \exists b \in NK(\vec{b}).\, \vec{a}a\rho_{\mathcal{X}_K^*}\vec{b}b \, .$$

$$\vec{a}\rho_{\mathcal{X}_K}\vec{b} \ \& \ b \in NK(\vec{b}) \implies \exists a \in MK(\vec{a}).\, \vec{a}a\rho_{\mathcal{X}_K^*}\vec{b}b \ .$$

(5) For any relation R in $\mathbf{L}$, and $\vec{a}\in M[R]=M[\mathcal{X}_R]$, $\vec{b}\in N[R]=N[\mathcal{X}_R]$,

$$\vec{a}\rho_{\mathcal{X}_R}\vec{b} \implies (\vec{a}\in MR \iff \vec{b}\in NR) .$$

(the notations $\mathcal{X}_K$, $\mathcal{X}_K^*$, $\mathcal{X}_R$ are from §4; $\vec{a}a$ denotes $\langle d_x\rangle_{x\in\mathcal{X}_K^*}\in M[\mathcal{X}_K^*]$ for which $d_x=a_x$ when $x\in\mathcal{X}_K$, and $d_{x_K}=a$).

By what we said above, every $(W, m, n):M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$ gives rise to a $\rho:M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$ ((3) is naturality, (4) is the very surjective condition, (5) is the preservation of relations). Conversely, given $\rho:M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$, putting $WK=\{\langle K, a, b\rangle: a\rho_K b\}$, $m_K(\langle K, a, b\rangle)=a$, $n_K(\langle K, a, b\rangle)=b$ gives $(W, m, n):M\overset{r}{\underset{\mathbf{L}}{\leftrightarrow}}N$.

We can make some steps towards Infinitary First Order Logic with Dependent Types. (We refer to [Ba] as a basic reference on infinitary logic and back-and-forth systems.) Let us fix the DSV $\mathbf{L}$ as before. The syntax of the logic $\mathbf{L}_{\infty,\omega}$ of FOLDS over $\mathbf{L}$ with arbitrary (set) size conjunction and disjunction, and finite quantification should be obvious; as usual, we only allow formulas that have finitely many free variables. To fix ideas, we consider logic without equality. $M\equiv_{\mathbf{L}_{\infty,\omega}}N$ means that M and N satisfy the same $\mathbf{L}_{\infty,\omega}$-sentences without equality. We have the following "back-and-forth" characterization of the relation $\equiv_{\mathbf{L}_{\infty,\omega}}$. A *weak relational* $\mathbf{L}$*-equivalence* $\rho:M\overset{r}{\underset{\mathbf{L}_{\infty,\omega}}{\longleftrightarrow}}N$ is a system $\rho=\langle\rho_{\mathcal{X}}\rangle_{\mathcal{X}}$ of relations $\rho_{\mathcal{X}}\subset M[\mathcal{X}]\times N[\mathcal{X}]$, indexed by all finite contexts, satisfying the following conditions (6)-(9):

(6) for any specialization $s:\mathcal{X}\longrightarrow\mathcal{Y}$, $\vec{a}\in M[\mathcal{Y}]$, $\vec{b}\in N[\mathcal{Y}]$,

$$\vec{a}\rho_{\mathcal{Y}}\vec{b} \implies (\vec{a}\circ s)\rho_{\mathcal{X}}(\vec{b}\circ s) ;$$

here, if $\vec{a}=\langle a_y\rangle_{y\in\mathcal{Y}}$, then $\vec{a}\circ s=\langle a_{s(x)}\rangle_{x\in\mathcal{X}}$.

(7) $\emptyset\rho_{\emptyset}\emptyset$ holds.

(8) For any finite contexts $\mathcal{X}$, $\mathcal{X}\dot{\cup}\{x\}$, $\vec{a}\in M[\mathcal{X}]$, $\vec{b}\in N[\mathcal{X}]$,

$$\vec{a}\rho_{\mathcal{X}}\vec{b} \ \& \ a\in MK(\vec{a}) \implies \exists b\in NK(\vec{b}).\vec{a}a\rho_{\mathcal{X}\dot{\cup}\{x\}}\vec{b}b \ ,$$
$$\vec{a}\rho_{\mathcal{X}}\vec{b} \ \& \ b\in NK(\vec{b}) \implies \exists a\in MK(\vec{a}).\vec{a}a\rho_{\mathcal{X}\dot{\cup}\{x\}}\vec{b}b \ .$$

(9) = (5)

We say that M and N are *weakly* $\boldsymbol{L}$*-equivalent*, $M\sim_{\boldsymbol{L},\mathrm{w}}N$, if there is $\rho:M\xleftarrow[\boldsymbol{L}_{\infty,\omega}]{\mathrm{r}}\!\!\rightarrow N$.

Given $\rho:M\xleftarrow[\boldsymbol{L}]{\mathrm{r}}\!\!\rightarrow N$, then, with making the definitions as in (2), we also have $\rho:M\xleftarrow[\boldsymbol{L}_{\infty,\omega}]{\mathrm{r}}\!\!\rightarrow N$. The reader will see that in the case of ordinary multisorted logic, the definition of weak relational $\boldsymbol{L}$-equivalence reduces to the well-known concept of "back-and-forth system" that figures in the characterization of ∞,ω-equivalence. Thus, the following generalizes that characterization.

(10)(a) For $\boldsymbol{L}$-structures M and N, $M\equiv_{\boldsymbol{L}_{\infty,\omega}}N$ iff $M\sim_{\boldsymbol{L},\mathrm{w}}N$.

(b) For countable $\boldsymbol{L}$-structures M and N, $M\equiv_{\boldsymbol{L}_{\infty,\omega}}N$ iff $M\sim_{\boldsymbol{L}}N$.

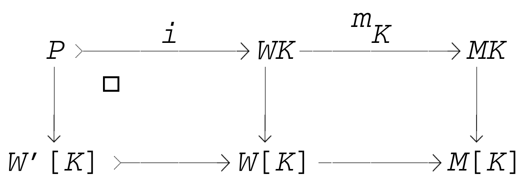
(c) For any countable $\boldsymbol{L}$, and countable $\boldsymbol{L}$-structure M, there is a ("Scott"-)sentence σ_M of $\boldsymbol{L}_{\omega_1,\omega}$ such that $N\equiv_{\boldsymbol{L}_{\infty,\omega}}M$ iff $N\models\sigma_M$.

The proofs are routine variants of those of the classical cases.

There is a simple categorical restatement of the notion of weak $\boldsymbol{L}$-equivalence. Consider $\boldsymbol{B}=(\mathrm{Set}^{\boldsymbol{K}})^{\mathrm{op}}_{\mathrm{fin}}$ as before. An $\boldsymbol{L}$*-pseudo-structure* P is a functor $\boldsymbol{B}\to\mathrm{Set}$, together with a subset $P(R)\subset P([\mathcal{X}])$ for each relation R of $\boldsymbol{L}$. A morphism of $\boldsymbol{L}$-pseudo-structures is a natural transformation of functors $\boldsymbol{B}\to\mathrm{Set}$ preserving each R in the obvious sense. Each $\boldsymbol{L}$-structure M can be regarded as a pseudo-structure, since any functor $\boldsymbol{K}\to\mathrm{Set}$ has a canonical extension $\boldsymbol{B}\to\mathrm{Set}$ which is in fact finite-limit preserving. Let $\mathrm{PStr}(\boldsymbol{L})$ be the category of pseudo-structures. We have a forgetful functor $\mathcal{E}':\mathrm{PStr}(\boldsymbol{L})\to\mathrm{Set}^{\boldsymbol{B}}$; $\mathcal{E}'$ can be seen to be a fibration. Now, a (not-necessarily-relational) *weak* $\boldsymbol{L}$*-equivalence* $(W,m,n):M\xleftarrow[\boldsymbol{L},\mathrm{w}]{}\!\!\rightarrow N$ is, by definition, a functor $W\in\mathrm{Set}^{\boldsymbol{B}}$, together with arrows $m:W\to\mathcal{E}'M$, $n:W\to\mathcal{E}'N$ such that m, n are very surjective with respect to all epimorphisms in $\boldsymbol{B}$ (according to the definition before B.(5), with $\mathrm{Lex}(\boldsymbol{B}_{\mathcal{C}},\boldsymbol{B}_{\mathcal{D}})$ replaced by $\mathtt{Set}^{\boldsymbol{B}}$), and there is a pseudo-structure P, with Cartesian arrows $\theta_m : P \to M$, $\theta_n : P \to N$ over m and n, respectively. We write $M \sim_{\boldsymbol{L},\,\mathrm{w}} N$ for: there exists $(W, m, n) : M \xleftrightarrow[\boldsymbol{L},\,\mathrm{w}]{} N$.

It is not hard to show that $M \sim_{\boldsymbol{L},\,\mathrm{w}} N$ iff there is a weak relational $\boldsymbol{L}$-equivalence $\rho : M \xleftrightarrow[\boldsymbol{L}_{\infty,\,\omega}]{\mathrm{r}} N$; the proof is similar to the proof below concerning non-weak relational equivalences.

We return to ordinary (non-weak) equivalences. When M and N are $\mathtt{Set}$-valued $\boldsymbol{L}$-structures, with any $(W, m, n) : M \xleftrightarrow[\boldsymbol{L}]{} N$, there is a relational $(W', m', n') : M \xleftrightarrow[\boldsymbol{L}]{\mathrm{r}} N$; in fact, W' can be chosen as a subfunctor of W, with m' and n' being restrictions of m and n, respectively. To define $W'K \subset WK$, we use recursion on the level of K. Fix K. The induction hypothesis gives us the inclusion $W'[K] \rightarrowtail W[K]$. Consider the pullback $P = WK \times_{W[K]} W'[K]$ as in



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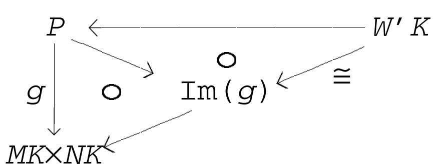
with i an inclusion; look at $g = \langle m_K i, n_K i \rangle : P \longrightarrow MK \times NK$, and, using the Axiom of Choice, split $h : P \twoheadrightarrow \mathrm{Im}(g)$ by an inclusion $k : W'K \rightarrowtail P$ as in



;

we have defined $W'K$. Inspection shows that W' is appropriate.

For not necessarily $\mathtt{Set}$-valued $\boldsymbol{L}$-structures M, N, let us write $M \sim_{\boldsymbol{L},\,\mathrm{r}} N$ for: there exists $(W, m, n) : M \xleftrightarrow[\boldsymbol{L}]{\mathrm{r}} N$.

What we saw says that the concept $M \sim_{\boldsymbol{L}} N$ remains unchanged, at least for $\mathtt{Set}$-valued models, if we ignore all but the relational $\boldsymbol{L}$-equivalences:

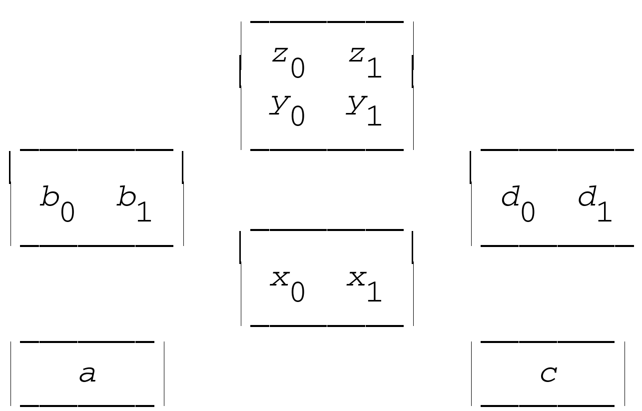
$$M \sim_{\boldsymbol{L}} N \iff M \sim_{\boldsymbol{L},\,\mathrm{r}} N. \tag{11}$$

However, the more general notion $(M, \vec{a}) \sim_{\boldsymbol{L}} (N, \vec{b})$ goes wrong under the same alteration. For one thing, the need for not-necessarily-relational $\boldsymbol{L}$-equivalences is natural if we look at the proof of 5.(4). Given $\mathcal{X}$ and the tuples $\vec{a} \in M[\mathcal{X}]$, $\vec{b} \in N[\mathcal{X}]$ as there, the desired $\boldsymbol{L}$-equivalence $(W, m, n) : (M, \vec{a}) \xleftrightarrow[\boldsymbol{L}]{} (N, \vec{b})$ is constructed so as to continue the mappings $x \mapsto a_x$, $x \mapsto b_x$; if the latter two mappings are not jointly monomorphic, the resulting $\boldsymbol{L}$-equivalence will not be relational. On the other hand, the entry of non-relational $\boldsymbol{L}$-equivalences is not just a characteristic of the proof of 5.(4); it is in fact unavoidable.

Consider the following example of a DSV, called $\boldsymbol{L}$:

$$\begin{array}{ccc} E_1 & \underset{e_{11}}{\overset{e_{10}}{\rightrightarrows}} & K_1 \\ & & \big\downarrow p \\ E_0 & \underset{e_{01}}{\overset{e_{00}}{\rightrightarrows}} & K_0 \end{array} \qquad pe_{10} = pe_{11} \quad .$$

A *standard* structure M for $\boldsymbol{L}$ is one for which, for $b_0, b_1 \in M[K_1]$, that is, $(Mp)b_0 = (Mp)b_1$, we have $b_0 (ME_1) b_1 \iff b_0 = b_1$, and also, ME_0 is ordinary equality on MK_0. Consider the following example for an $\boldsymbol{L}$-equivalence $(W, m, n) : M \xleftrightarrow[\boldsymbol{L}]{} N$, for certain M and N:



.

Here, $MK_0 = \{a\}$, $MK_1 = \{b_0, b_1\}$, $NK_0 = \{c\}$, $MK_1 = \{d_0, d_1\}$, $WK_0 = \{x_0, x_1\}$,

$WK_1=\{y_0, y_1, z_0, z_1\}$, $y_0, z_0 \overset{Wp}{\longmapsto} x_0$, $y_1, z_1 \overset{Wp}{\longmapsto} x_1$, and

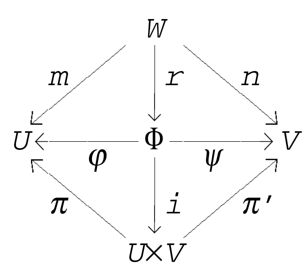
$$y_0 \overset{m}{\longmapsto} b_0 ,\ y_1 \overset{m}{\longmapsto} b_1 ,\ z_0 \overset{m}{\longmapsto} b_1 ,\ z_1 \overset{m}{\longmapsto} b_0 ,$$

$$y_0 \overset{n}{\longmapsto} d_0 ,\ y_1 \overset{n}{\longmapsto} d_0 ,\ z_0 \overset{n}{\longmapsto} d_1 ,\ z_1 \overset{n}{\longmapsto} d_1 ;$$

E_0 and E_1 are interpreted in M and N as equality.

This shows that, for the context $\mathcal{X}=\{x_0, x_1 : K_0; y_0 : K_1(x_0); y_1 : K_1(x_1)\}$, and for $\vec{a}=\langle a/x_0, a/x_1, b_0/y_0, b_1/y_1\rangle$, $\vec{c}=\langle c/x_0, c/x_1, d_0/y_0, d_0/y_1\rangle$, we have $(M, \vec{a}) \sim_{\boldsymbol{L}} (N, \vec{c})$. On the other hand, there is no relational equivalence $(W', m', n') : (M, \vec{a}) \underset{\boldsymbol{L}}{\longleftrightarrow} (N, \vec{c})$. In any such, $W'K_0$ is a singleton $\{x\}$; $x \overset{m'}{\longmapsto} a$, $x \overset{n'}{\longmapsto} c$; we have some $u_0, u_1 \in W'K_1(x)$ such that $u_0 \overset{m'}{\longmapsto} b_0$, $u_1 \overset{m'}{\longmapsto} b_1$; and the preservation of E_1 implies $n'(u_0) \neq n'(u_1)$, contradiction.

This example also dispels the possible belief that an $\boldsymbol{L}$-equivalence $(W, m, n) : M \underset{\boldsymbol{L}}{\longleftrightarrow} N$ can always be reduced to a relational one by taking the image of (W, m, n) . Let $U=M\restriction\boldsymbol{K}$, $V=N\restriction\boldsymbol{K}$, and consider



(12)

where r and i form the surjective/injective factorization of $\langle m, n\rangle : W \to U\times V$. In other words, when $i : \Phi \to U\times V$ is an inclusion, for any $K\in\boldsymbol{K}$, the relation $\Phi K \subset MK\times NK$ is given by $a(\Phi K)b \iff \exists c\in WK.\, mc=a \,\&\, nc=b$. When applied in our example, (Φ, φ, ψ) so defined does not preserve E_1 .

I now turn to some remarks on equality.

Let $\boldsymbol{L}$ be an arbitrary DSV. Let us augment $\boldsymbol{L}$ to $\boldsymbol{L}^{\mathrm{G}}$, another DSV, by adding a relation

$\dot{\mathrm{G}}_K$ to $\mathbf{L}$ for every $K\in\mathrm{Kind}(\mathbf{L})$, with proper arrows $g_{K0}:\dot{G}_K\to K$, $g_{K1}:\dot{G}_K\to K$, together with all composites $pg_{Ki}:\dot{G}_K\to K_p$, $p\in K|\mathbf{L}$ $(i=0, 1)$. We do *not* identify pg_{K0} with pg_{K1}. For an $\mathbf{L}^{\mathrm{G}}$-structure M,

$$M[\dot{\mathrm{G}}_K] = \{(\vec{a}, a, \vec{b}, b) : \vec{a}, \vec{b}\in M[K], a\in MK(\vec{a}), b\in MK(\vec{b})\} .$$

The letter G is used because we are dealing with *global* equality as opposed to fiberwise equality (see below). A *standard* $\mathbf{L}^{\mathrm{G}}$-structure M is one in which, for $\vec{a}, \vec{b}\in M[K]$, $a\in MK(\vec{a})$, $b\in MK(\vec{b})$, $(\vec{a}, a, \vec{b}, b)\in M(\dot{\mathrm{G}}_K)$ iff $a=b$; more briefly, $M(\dot{\mathrm{G}}_K)$ as a subset of $MK\times MK$ is $\{(a, a): M(K)\}$. Any $\mathbf{L}$-structure can be made into a standard $\mathbf{L}^{\mathrm{G}}$-structure in exactly one way. When an $\mathbf{L}$-structure is used as an $\mathbf{L}^{\mathrm{G}}$-structure, we mean the corresponding standard $\mathbf{L}^{\mathrm{G}}$-structure.

The effect of adding global equalities is that all $\mathbf{L}$-equivalences can be canonically replaced by relational ones, by taking the image of the given one. If $(W, m, n): M\xleftrightarrow[\mathbf{L}^{\mathrm{G}}]{} N$, then for (Φ, φ, ψ) defined above, we have $(\Phi, \varphi, \psi): M\xleftrightarrow[\mathbf{L}^{\mathrm{G}}]{r} N$.

To see this, first we show that the arrow r in (12) is very surjective; that is, for any $K\in\mathbf{K}$, the diagram

$$\begin{array}{ccc} W(K) & \xrightarrow{\;r_K\;} & \Phi(K) \\ \downarrow & & \downarrow \\ W[K] & \xrightarrow[\;r_{[K]}\;]{} & \Phi(K) \end{array} \tag{13}$$

is a quasi-pullback. Assume $\vec{a}\in M[K]$, $\vec{b}\in N[K]$, $a\in MK(\vec{a})$, $b\in NK(\vec{b})$ such that $(\vec{a}, \vec{b})\in\Phi[K]$, $(a, b)\in\Phi K(\vec{a}, \vec{b})$, and $\vec{c}\in W[K]$ with $m\vec{c}=\vec{a}$, $n\vec{c}=\vec{b}$ (that is, $r_{[K]}(\vec{c})=(\vec{a}, \vec{b})$); we want $c\in WK(\vec{c})$ such that $mc=a$ and $nc=b$. By the definition of Φ, there is $d\in WK$ with $md=a$, $nd=b$. By the very surjectivity of n, there is $c\in WK(\vec{c})$

such that $nc{=}b$. But by the presence of the relation G_K, $md(M\mathrm{G}_K)mc$ iff $nd(N\mathrm{G}_K)nc$; that is, $md{=}mc$ iff $nd{=}nc$; which says that $mc{=}a$ as desired.

By B.(6'), the induced map $r_{[\mathcal{X}]}:W[\mathcal{X}]\longrightarrow\Phi[\mathcal{X}]$ is surjective.

Now, looking at

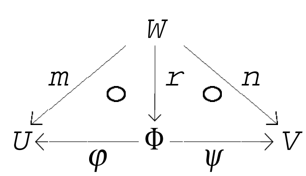
$$\begin{array}{ccccc} W[K] & \xrightarrow{r_{[K]}} & \Phi[K] & \xrightarrow{\varphi_{[K]}} & M[K] \\ \uparrow & & \uparrow & & \uparrow \\ W(K) & \xrightarrow[r_K]{} & \Phi(K) & \xrightarrow[\varphi_K]{} & M(K) \end{array} \qquad \begin{array}{ccc} W[K] & \xrightarrow{m_{[K]}} & M[K] \\ \uparrow & & \uparrow \\ W(K) & \xrightarrow[m_K]{} & M(K) \end{array}$$

we see that B.(3") is applicable to yield that φ is very surjective.

Given a relation $R\in\mathrm{Rel}(\boldsymbol{L}^{\mathrm{G}})$, if $(\vec{a},\vec{b})\in\Phi[R]$, then by $r_{[R]}:W[R]\longrightarrow\Phi[R]$ being surjective, there is $\vec{c}\in W[R]$ with $r_{[R]}(\vec{c})=(\vec{a},\vec{b})$, that is, $m\vec{c}{=}\vec{a}$, $n\vec{c}{=}\vec{b}$, and thus $\vec{a}\in MR$ iff $\vec{b}\in NR$. This completes showing that $(\Phi,\varphi,\psi):M\overset{r}{\underset{\boldsymbol{L}^{\mathrm{G}}}{\longleftrightarrow}}N$.

We have shown something more general (and more technical), which is independent of equality. This is that

(14) If $(W,m,n):M\underset{\boldsymbol{L}}{\longleftrightarrow}N$ and we have



such that r is very surjective, then $(\Phi,\varphi,\psi):M\underset{\boldsymbol{L}}{\longleftrightarrow}N$;

the relational quality of (φ,ψ) is not relevant to this.

Clearly, a relational equivalence preserving global equalities on all kinds is nothing but an isomorphism. We have shown that $M\sim_{\boldsymbol{L}^{\mathrm{G}}}N$ implies that $M\cong N$, and $(M,\vec{a})\sim_{\boldsymbol{L}^{\mathrm{G}}}(N,\vec{b})$ implies $(M,\vec{a})\cong(N,\vec{b})$. But, all formulas in multisorted logic over $|\boldsymbol{L}|$ are preserved by isomorphism. By the invariance theorem 5.(12), we conclude the following.

(15) For any context $\mathcal{X}$ over $\boldsymbol{L}$, and any formula σ of multisorted logic over $|\boldsymbol{L}|$ with $\mathrm{Var}(\sigma)\subset\mathcal{X}$ [remember, a variable $x{:}X$ of FOLDS counts as a variable of sort K_X in multisorted logic], there is a FOLDS formula θ over $\boldsymbol{L}^{\mathrm{G}}$ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that σ and θ^* are logically equivalent (over $\mathcal{X}$):

$\models\forall\mathcal{X}(\sigma\longleftrightarrow\theta^*)$; or in other words, $M[\mathcal{X}:\sigma]=M[\mathcal{X}:\theta^*]$ for any $\boldsymbol{L}$-structure M. (We apply 5.(12) to $I:\boldsymbol{L}^{\mathrm{G}}\to[(|\boldsymbol{L}|,\Sigma[\boldsymbol{L}])]$; for $\Sigma[\boldsymbol{L}]$, see §1. I is essentially the identity except that all the G_K's are interpreted as equality. In $M[\mathcal{X}:\theta^*]$, M is understood as a standard $\boldsymbol{L}^{\mathrm{G}}$-structure.)

Notice the small point that in the statement of (15), we are not allowed to start with a $|\boldsymbol{L}|$-formula σ with arbitrary free variables; the free variables have to form a context. E.g., in the case of the language of categories, a formula with a single arrow-variable cannot (of course) have an equivalent in FOLDS *with the same free variables*; we have to add the "domain and the codomain of the arrow-variable" as free variables.

Let us hasten to add that it is possible to show (15) directly, by a rather simple structural induction on the formula σ.

We have an instance of what we may call *expressive completeness of* FOLDS: full first-order logic over $|\boldsymbol{L}|$ can be expressed in $\boldsymbol{L}^{\mathrm{G}}$. This is accompanied by a mode of deductive completeness. We will give a deductive system for entailments over $\boldsymbol{L}^{\mathrm{G}}$, extending the standard system for $\boldsymbol{L}^{\mathrm{G}}$ for logic without equality by specific rules related to the G-predicates, which is complete for semantics restricted to standard $\boldsymbol{L}^{\mathrm{G}}$-structures, that is, semantics of true equality.

The set $\mathrm{G}_K|\boldsymbol{L}$, the arity of the relation G_K, is the set

$$\{p\mathrm{g}_{K0}:\ p\in K|\boldsymbol{L}\}\cup\{\mathrm{g}_{K0}\}\cup\{p\mathrm{g}_{K1}:\ p\in K|\boldsymbol{L}\}\cup\{\mathrm{g}_{K1}\}$$

Accordingly, we will write atomic formulas $\mathrm{G}_K(\vec{z})$, $\vec{z}$ indexed by $\mathrm{G}_K|\boldsymbol{L}$, in the form $\mathrm{G}_K(\vec{x}, x, \vec{y}, y)$; here, $\vec{x}=\langle x_{p\mathrm{g}_{K0}}\rangle_{p\in K|\boldsymbol{L}}$, $x:K(\vec{x})$, $\vec{y}=\langle y_{p\mathrm{g}_{K0}}\rangle_{p\in K|\boldsymbol{L}}$, $y:K(\vec{y})$.

Here are some other pieces of notation. For any object A of $\boldsymbol{L}$ (kind or relation), and tuples $\vec{x}=\langle x_p\rangle_{p\in A|\boldsymbol{L}}$, $\vec{y}=\langle y_p\rangle_{p\in A|\boldsymbol{L}}$ for which $A(\vec{x})$, $A(\vec{y})$ (types or atomic formulas) are well-formed, $\vec{x}\mathrm{G}_{[A]}\vec{y}$ denotes the formula

$$\bigwedge_{p\in A|\boldsymbol{L}}\mathrm{G}_{\mathrm{K}_p}(\langle x_{qp}\rangle_{q\in \mathrm{K}_p|\boldsymbol{L}}, x_p, \langle y_{qp}\rangle_{q\in \mathrm{K}_p|\boldsymbol{L}}, y_p) .$$

When $\vec{x}=\langle x_p\rangle_{p\in K|\boldsymbol{L}}$, $\vec{x}^p \overline{\overline{\mathrm{def}}} \langle x_{qp}\rangle_{q\in \mathrm{K}_p|\boldsymbol{L}}$.

V. Global-equality axioms.

$$(\mathrm{G}_1)\qquad \frac{}{\mathbf{t} \xRightarrow[\mathcal{X}]{} \mathrm{G}_K(\vec{x}, x, \vec{x}, x)}$$

$$(\mathrm{G}_2)\qquad \frac{}{\mathrm{G}_K(\vec{x}, x, \vec{y}, y) \xRightarrow[\mathcal{X}]{} \mathrm{G}_K(\vec{y}, y, \vec{x}, x)}$$

$$(\mathrm{G}_3)\qquad \frac{}{\mathrm{G}_K(\vec{x}, x, \vec{y}, y) \wedge \mathrm{G}_K(\vec{y}, y, \vec{z}, z) \xRightarrow[\mathcal{X}]{} \mathrm{G}_K(\vec{x}, x, \vec{z}, z)}$$

$$(\mathrm{G}_4)\qquad \frac{}{\mathrm{G}_K(\vec{x}, x, \vec{y}, y) \xRightarrow[\mathcal{X}]{} \mathrm{G}_K(\vec{y}^p, y_p, \vec{x}^p, x_p)}\qquad (\ p\in K|\boldsymbol{L}\ )$$

$$(\mathrm{G}_5)\qquad \frac{}{\vec{x}\mathrm{G}_{[K]}\vec{y} \xRightarrow[\mathcal{X}]{} \exists y:K(\vec{y}).\mathrm{G}_K(\vec{x}, x, \vec{y}, y)}\qquad (\ x:K(\vec{x})\ )$$

$$(\mathrm{G}_6) \qquad \frac{}{\vec{x}\mathrm{G}_{[R]}\vec{y} \underset{\mathcal{X}}{\Longrightarrow} R(\vec{x}) \longleftrightarrow R(\vec{y})}$$

The proof of the said completeness is done in the traditional manner; we use completeness for logic without equality over $\boldsymbol{L}^{\mathrm{G}}$ for the theory whose axioms are the (conclusion-)entailments in the equality rules. Given any structure M for $\boldsymbol{L}^{\mathrm{G}}$ satisfying the equality axioms, we construct a standard $\boldsymbol{L}^{\mathrm{G}}$-structure $M/\sim$ which is $\boldsymbol{L}^{\mathrm{G}}$-elementary equivalent to M. For a kind K, let $\sim_K$ be the relation on the set MK defined by $a\sim_K b \Longleftrightarrow M\mathrm{G}_K([a], a, [b], b)$ holds; here $[a]=\langle (Mp)(a)\rangle_{p\in K|\boldsymbol{L}}$, and similarly for $[b]$. By (G_1), (G_2) and (G_3), each $\sim_K$ is an equivalence relation; let us write $a/\sim$ for the equivalence class containing a. (G_4) implies that if $f:K\to K'$, $a_i\in MK$, $a'_i=(Mf)(a_i)\in MK'$, then $a_1\sim_K a_2 \Rightarrow a'_1\sim_{K'} a'_2$. Let $U=M\restriction\boldsymbol{K}$. We define $U/\sim:\boldsymbol{K}\to\mathrm{Set}$ by $(U/\sim)(K)=(UK)/\sim$ $(\underset{\mathrm{def}}{=}\{a/\sim: a\in UK\})$, and $((U/\sim)(f))(a/\sim)=((Uf)(a))/\sim$, which is well-defined.

For $\vec{a}=\langle a_p\rangle_{p\in R|\boldsymbol{K}}\in M[R]$, we put $\vec{a}/\sim\ = \langle a_p/\sim\rangle_{p\in R|\boldsymbol{K}}\in (M/\sim)R$.

We define $M/\sim$ by $(M/\sim)\restriction\boldsymbol{K}=U/\sim$, and

$$(M/\sim)R(\vec{a}/\sim) \underset{\mathrm{def}}{\Longleftrightarrow} MR(\vec{a}) ;$$

by (G_6), this is well-defined; we have completed the definition of $M/\sim$.

For any finite context $\mathcal{X}$, we have $(M/\sim)[\mathcal{X}] = (M[\mathcal{X}])/\sim$ $(\underset{\mathrm{def}}{=}\{\vec{a}/\sim: \vec{a}\in M[\mathcal{X}]\})$. Moreover, when $\vec{a}\in M[K]$, then $(M/\sim)K(\vec{a}/\sim) = MK(\vec{a})/\sim$ $(\underset{\mathrm{def}}{=}\{a/\sim: a\in MK(\vec{a})\}$. This is not automatic; it requires (G_5). Finally, we show, by structural induction, that for any θ over $\boldsymbol{L}^{\mathrm{G}}$ with $\mathrm{Var}(\theta)\subset\mathcal{X}$, and $\vec{a}\in M[\mathcal{X}]$,

$$M/\sim\ \models \theta[\vec{a}/\sim] \quad\Longleftrightarrow\quad M\models \theta[\vec{a}] \quad .$$

Having the construction $M\mapsto M/\sim$ with the properties shown, the proof of the standard completeness for $\boldsymbol{L}^{\mathrm{G}}$ can be completed in the expected manner.

In place of global equality, it seems natural to consider *fiberwise equality* for FOLDS. Let, for any DSV $\boldsymbol{L}$, $\boldsymbol{L}^{\mathrm{E}}$ denote the DSV obtained by adding to $\boldsymbol{L}$ a new relation E_K for every kind K, with $\mathrm{E}_K \overset{\mathrm{e}_{K0}}{\underset{\mathrm{e}_{K1}}{\rightrightarrows}} K$ and $p\mathrm{e}_{K0}=p\mathrm{e}_{K1}$ $(p\in K|\boldsymbol{K})$ as for maximal kinds in $\boldsymbol{L}^{\mathrm{eq}}$. A *standard* $\boldsymbol{L}^{\mathrm{E}}$-structure is one in which each E_K is interpreted as equality; to give a standard $\boldsymbol{L}^{\mathrm{E}}$-structure is the same as to give an $\boldsymbol{L}$-structure. In what follows, M and N are $\boldsymbol{L}$-structures; when they figure as $\boldsymbol{L}^{\mathrm{E}}$-structures, they mean the corresponding standard ones.

Suppose $\rho: M \underset{\boldsymbol{L}^{\mathrm{E}}}{\overset{\mathrm{r}}{\leftrightarrow}} N$. I claim that each $\rho_K \subset MK\times NK$ is the graph of a bijection $MK\to NK$. By (6'), $\mathrm{dom}(\rho_K)=MK$, $\mathrm{codom}(\rho_K)=NK$. Thus, it remains to show that

$$a_i\in MK,\ b_i\in NK,\ a_i\rho_K b_i\ (i=1,2) \implies a_1=a_2 \iff b_1=b_2 \tag{16}$$

We show this by induction on the level of K. Assume the hypotheses of (16). Let $a_i\in MK(\vec{a}^i)$, $b_i\in NK(\vec{b}^i)$. Then, if $\vec{a}^i=\langle a^i_p\rangle_{p\in K|\boldsymbol{K}}$, $\vec{b}^i=\langle b^i_p\rangle_{p\in K|\boldsymbol{K}}$, then $a^i_p\rho_{K_p} b^i_p$ (by (3)).

Assume (*e.g.*) $a_1=a_2$. Then $\vec{a}^1=\vec{a}^2 \underset{\mathrm{def}}{=} \vec{a}$, that is, $a^1_p=a^2_p$ for all $p\in K|\boldsymbol{K}$. By the induction hypothesis, (16) applied to K_p, we have $b^1_p=b^2_p$, that is, $\vec{b}^1=\vec{b}^2 \underset{\mathrm{def}}{=} \vec{b}$. We have $a_1, a_2\in MK(\vec{a})$, $b_1, b_2\in NK(\vec{b})$, and $\vec{a}a_i\rho_{\mathcal{X}^*_K}\vec{b}b_i$. Therefore, by (6), $M\mathrm{E}_K(\vec{a}, a_1, a_2) \iff N\mathrm{E}_K(\vec{b}, b_1, b_2)$; that is, $a_1=a_2 \iff b_1=b_2$ as desired.

Given that each ρ_K is a bijection, clearly, ρ is an isomorphism $\rho: M\xrightarrow{\cong} N$ (of $\boldsymbol{L}$-structures). We conclude

$$M \sim_{\boldsymbol{L}^{\mathrm{E}},\mathrm{r}} N \implies M\cong N \tag{17}$$

(the above argument did not depend essentially on the fact that we dealt with Set-valued structures)

Applying 5.(12), we obtain

(18) For every *sentence* σ in multisorted logic (with equality) over $|\mathbf{L}|$ there is a sentence $\bar{\sigma}$ of FOLDS over $\mathbf{L}^{\mathrm{E}}$ such that for every $\mathbf{L}$-structure M, $M\models\sigma \Longleftrightarrow M\models\bar{\sigma}$ (here, in the first instance, M figures as an $|\mathbf{L}|$-structure; in the second instance as a standard $\mathbf{L}^{\mathrm{E}}$-structure).

Proof. Consider the interpretation $I:\mathbf{L}^{\mathrm{E}}\to[T]$, where $T=(|\mathbf{L}|,\Sigma_{\mathbf{L}})$, extending the "identity" interpretation $\mathbf{L}\to[T]$, and interpreting each E_K as equality. We apply 5.(12) to I, $\mathcal{X}=\emptyset$ and σ. Suppose $M, N\models T$ are Set-valued models (!),

$$M\restriction\mathbf{L}^{\mathrm{E}}\sim_{\mathbf{L}^{\mathrm{E}}}N\restriction\mathbf{L}^{\mathrm{E}} \qquad (19)$$

and $M\models\sigma$. M and N are $\mathbf{L}$-structures, and $M\restriction\mathbf{L}^{\mathrm{E}}$, $N\restriction\mathbf{L}^{\mathrm{E}}$ are the corresponding standard $\mathbf{L}^{\mathrm{E}}$-structures. By (19) and (11), it follows that $M\cong N$. Since "everything" is invariant under isomorphism, $N\models\sigma$. Thus, the hypothesis of 5.(12) holds. The conclusion is exactly what we want.

Note that the result of (18) cannot be generalized to formulas with free variables in place of sentences. That is, the statement of (15), with $\mathbf{L}^{\mathrm{E}}$ replacing $\mathbf{L}^{\mathrm{G}}$ is not true. This is shown by the example that we gave above; in that example, $\mathbf{L}=\mathbf{L}_0^{\mathrm{E}}$ for $\mathbf{L}_0$ consisting of K_0, K_1 and p (and no relations). With $\mathcal{X}=\{x_0, x_1, y_0, y_1\}$ as in the example, *if* for the formula $\sigma\equiv y_0=y_1$ (whose free variables are in $\mathcal{X}$) there were θ in FOLDS over $\mathbf{L}$ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that, for every $\mathbf{L}_0$-structure M (also counted as a standard $\mathbf{L}$-structure) and

$$\vec{a}=\langle a_0, a_1\in MK_0; b_0\in MK_1(a_0); b_1\in MK_1(a_1)\rangle,$$

$$M\models\sigma(\vec{a}) \Longleftrightarrow b_0=b_1 \overset{?}{\Longleftrightarrow} M\models\theta(\vec{a})$$

then for every equivalence $(W, m, n):(M, \vec{a})\xleftrightarrow[\mathbf{L}]{}(N, \vec{c})$, where

$$\vec{c}=\langle c_0,\ c_1\in NK_0;\ d_0\in NK_1(c_0);\ d_1\in NK_1(c_1)\rangle\ ,$$

since it would preserve θ, we would have

$$b_0=b_1 \iff d_0=d_1\ ;$$

but the example shows that this conclusion is false.

(18) can be used to give another proof of 6.(3), the Freyd-Blanc characterization result, at least for $\mathcal{X}=\emptyset$; this proof is a variant of what is contained in [FS].

Let T be a normal theory of categories with additional structure. Assume σ is an L_T-sentence such that for $M, N\models T$, $|M|\simeq|N|$ implies that $M\models\sigma$ iff $N\models\sigma$. In particular,

for $M, N\models T$, $|M|\cong|N|$ implies that $M\models\sigma$ iff $N\models\sigma$.

By ordinary model theory (a version of Beth definability), it follows that there is a sentence τ in multisorted logic over $|\boldsymbol{L}_{\mathrm{cat}}|$ such that for models of T, σ and τ are equivalent. By (18), there is a sentence ψ in FOLDS over $\boldsymbol{L}^{\mathrm{E}}_{\mathrm{cat}}$ which is equivalent to σ in all $\boldsymbol{L}_{\mathrm{cat}}$-structures (also counted as standard $\boldsymbol{L}^{\mathrm{E}}_{\mathrm{cat}}$-structures). There are two E-predicates in ψ, E_{O} and E_{A}. Replace each occurrence $\mathrm{E}_{\mathrm{O}}(X, Y)$ of E_{O} by the formula

$$\text{"}X\cong Y\text{"} \equiv \exists f\in\mathrm{A}(X, Y).\exists g\in\mathrm{A}(Y, X).\exists h\in\mathrm{A}(X, X).\exists i\in\mathrm{A}(Y, Y)$$
$$(\mathrm{I}(h)\wedge\mathrm{I}(i)\wedge\mathrm{T}(f, g, h)\wedge\mathrm{T}(g, f, i))\ ;$$

call the result θ. Notice that θ is a FOLDS formula of $\mathrm{L}^{\mathrm{eq}}_{\mathrm{cat}}$ (it has only the allowable equality predicates in $\mathrm{L}^{\mathrm{eq}}_{\mathrm{cat}}$). I claim that for all $M\models T$,

$$M\models\sigma \iff M\models\theta\ .$$

Let $M\models T$. $|M|$ is a category; let $|M|_{\mathrm{s}}$ be its skeleton. Since $|M|\simeq|M|_{\mathrm{s}}$, by the normality of T, there is $N\models T$ such that $|N| = |M|_{\mathrm{s}}$. Now

$M \models \sigma \iff N \models \sigma$ since $|M| \simeq |N|$, and $M, N \models T$

$\iff N \models \tau$ since $N \models T$

$\iff |N| \models \tau$

$\iff |N| \models \psi$

$\iff |N| \models \theta$ since $|N|$ is skeletal (that is, for objects X, Y, $X=Y$ iff $X \cong Y$)

$\iff |M| \models \theta$ since $|M| \simeq |N|$, and θ is a FOLDS formula with equality over $\boldsymbol{L}_{\mathrm{cat}}$

$\iff M \models \theta$.

This method of proof is also applicable to the "higher" cases. Let us consider the case of bicategories; let us show that if a sentence σ in multisorted logic over $\mathrm{L}_{\mathrm{bicat}} = |\boldsymbol{L}_{\mathrm{anabicat}}|$ is invariant under equivalence of bicategories, then σ is equivalent in bicategories to θ^{*} for a FOLDS sentence θ over $\boldsymbol{L}_{\mathrm{anabicat}}$; θ^{*} is the translate of θ such that $\mathcal{A} \models \theta^{*} \iff \mathcal{A}^{\#} \models \theta$.

A bicategory $\mathcal{A}$ is *skeletal* if any two equivalent objects are equal, and any two isomorphic parallel 1-cells are equal. For any bicategory $\mathcal{A}$, there is a skeletal one, $\mathcal{A}_{\mathrm{s}}$, which is (bi)equivalent to $\mathcal{A}$.

The first step is to use Beth definability to the interpretation $\Phi : \boldsymbol{L}_{\mathrm{anabicat}} \to [\mathrm{T}_{\mathrm{bicat}}]$. Since $\mathcal{A}^{\#} \cong \mathcal{B}^{\#}$ implies that $\mathcal{A} \simeq \mathcal{B}$, it follows that there is a sentence τ in multisorted logic over $|\boldsymbol{L}_{\mathrm{anabicat}}|$ such that for every bicategory $\mathcal{A}$, $\mathcal{A} \models \sigma \iff \mathcal{A}^{\#} \models \tau$. By (18), we can find a sentence ψ in FOLDS over $\boldsymbol{L}^{\mathrm{E}}_{\mathrm{anabicat}}$ such that, in particular, $\mathcal{A}^{\#} \models \tau \iff \mathcal{A}^{\#} \models \psi$. Now, transform ψ in the following way. Each occurrence $\mathrm{E}_{\mathrm{C}_0}(X, Y)$ of $\mathrm{E}_{\mathrm{C}_0}$ is replaced by the formula

" $X \simeq Y$ " $\equiv$ ...

and each occurrence $\mathrm{E}_{\mathrm{C}_1}(X \overset{f}{\underset{g}{\rightrightarrows}} Y)$ of $\mathrm{E}_{\mathrm{C}_1}$ is replaced by the formula

" $f \cong g$ " $\equiv$ ...

The resulting sentence θ is in $\boldsymbol{L}^{\text{eq}}_{\text{anabicat}}$. I claim that for any bicategory $\mathcal{A}$, $\mathcal{A}\vDash\sigma \Longleftrightarrow \mathcal{A}\vDash\theta^*$. Indeed,

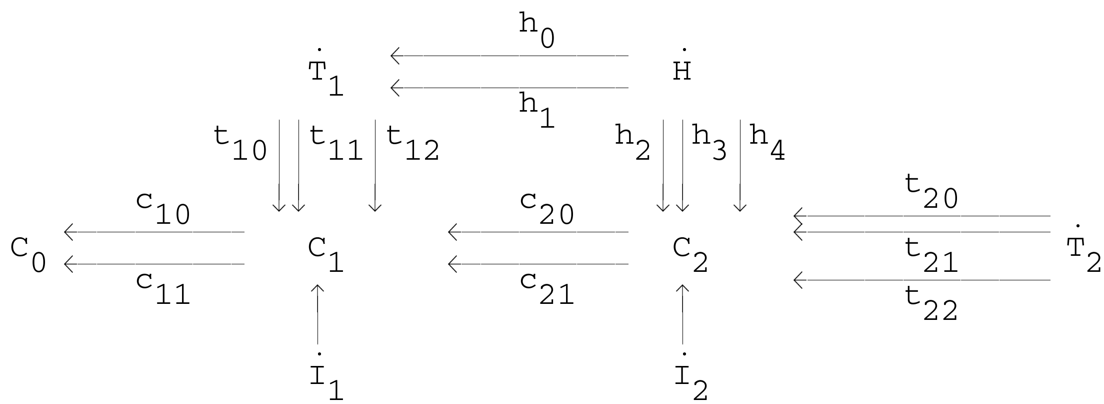
$$\mathcal{A}\vDash\sigma \Longleftrightarrow \mathcal{A}_{\text{s}}\vDash\sigma \Longleftrightarrow (\mathcal{A}_{\text{s}})^{\#}\vDash\tau \Longleftrightarrow (\mathcal{A}_{\text{s}})^{\#}\vDash\psi \Longleftrightarrow (\mathcal{A}_{\text{s}})^{\#}\vDash\theta \Longleftrightarrow \mathcal{A}^{\#}\vDash\theta \Longleftrightarrow \mathcal{A}\vDash\theta^* \;;$$

the next-to-last last biconditional holds because $\mathcal{A}_{\text{s}}\simeq\mathcal{A}$, of which $(\mathcal{A}_{\text{s}})^{\#}\sim_{\boldsymbol{L}^{\text{eq}}}\mathcal{A}^{\#}$ ($\boldsymbol{L}=\boldsymbol{L}_{\text{anabicat}}$) is a consequence, and because θ is a FOLDS sentence over $\boldsymbol{L}^{\text{eq}}$.

This proof replaces the general invariance theorem 5.(12) by Beth definability, and a special case of that invariance theorem, (18). It falls somewhat short of the results of §7, partly because we have confined the situation to an empty context $\mathcal{X}$. Also, this approach is not available in constructive category theory; the existence of the skeleton (already in the classical case of mere categories) depends on the Axiom of Choice. As we will see in Appendix E, the main theory of equivalence of §5 has a constructive version involving intuitionistic logic. Modifying the notions of equivalence to notions of "anaequivalence" (using, and building on, [M2]), we obtain versions of the results of sections 6 and 7 for constructive category theory.

Appendix D: Calculations for §7.

D1. Define the *generalized* DS vocabulary $\boldsymbol{L}_{\texttt{2-cat}}$ as the full subcategory of $\boldsymbol{L}_{\texttt{anabicat}}$ on the objects of $\texttt{L}_{\texttt{2-cat}}$, with relations $\dot{\texttt{I}}_1$, $\dot{\texttt{I}}_2$, $\dot{\texttt{T}}_1$, $\dot{\texttt{H}}$, $\dot{\texttt{T}}_2$; it is *generalized* since a non-maximal object, $\texttt{T}_1$, is also made into a relation. Accordingly, an $\boldsymbol{L}_{\texttt{2-cat}}$-structure is a functor from $\boldsymbol{L}_{\texttt{2-cat}}$ in which the listed relations (including $\texttt{T}_1$) are interpreted relationally. This is the picture for $\boldsymbol{L}_{\texttt{2-cat}}$:



A *2-category-sketch* (2-cat-sketch) is, by definition, a structure of type $\boldsymbol{L}_{\texttt{2-cat}}$; maps of 2-cat-sketches are natural transformations of functors. For a 2-cat-sketch S, $|S|$ is its underlying 2-graph, its reduct to

$$\texttt{C}_0 \;\underset{\texttt{c}_{11}}{\overset{\texttt{c}_{10}}{\leftleftarrows}}\; \texttt{C}_1 \;\underset{\texttt{c}_{21}}{\overset{\texttt{c}_{20}}{\leftleftarrows}}\; \texttt{C}_2 \;.$$

Any bicategory has an underlying 2-cat-sketch. We will look at maps $S \longrightarrow \mathcal{A}$, $S \in \texttt{2-catSk}$, $\mathcal{A}$ a bicategory.

Let $S \underset{N}{\overset{M}{\rightrightarrows}} \mathcal{A}$. A *transformation* $\tau : M \to N$ is given by

(i) $\tau_X : MX \to NX$ for each $X \in S(\texttt{C}_0)$;

(ii) for each $(f : X \to Y) \in S(\texttt{C}_1)$, $\tau_f : Nf \circ \tau_X \overset{\cong}{\Longrightarrow} \tau_Y \circ Mf$ as in

$$
\begin{array}{ccc}
MX & \xrightarrow{\tau_X} & NX \\
{\scriptstyle Mf}\downarrow & \overset{\tau_f}{\swarrow}\ \cong & \downarrow{\scriptstyle Nf} \\
MY & \xrightarrow[\tau_Y]{} & NY
\end{array}
$$

such that

(a) for any $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ with $\varphi\downarrow$ in S,

$$
\begin{array}{ccc}
Nf\circ\tau_X & \xrightarrow{\tau_f} & \tau_Y\circ Mf \\
{\scriptstyle N\varphi\circ\tau_X}\downarrow & \circ & \downarrow{\scriptstyle \tau_Y\circ M\varphi} \\
Ng\circ\tau_X & \xrightarrow[\tau_g]{} & \tau_Y\circ Mg
\end{array}
;
$$

(b) $(f: X \to Y)\in S(I_1) \Longrightarrow \tau_f = 1_{\tau_X}$; and

(c) for every $A \xrightarrow{f} B \xrightarrow{g} C$, $A \xrightarrow{h} C \in S(T_1)$ (note that $MgMf = Mh$, $NgNf = Nh$),

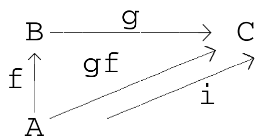
$$
\begin{array}{ccccc}
MA & \xrightarrow{Mf} & MB & \xrightarrow{Mg} & MC \\
{\scriptstyle\tau_A}\downarrow & \overset{\tau_f}{\swarrow} & {\scriptstyle\tau_B}\downarrow & \overset{\tau_g}{\swarrow} & {\scriptstyle\tau_C}\downarrow \\
NA & \xrightarrow[Nf]{} & NB & \xrightarrow[Ng]{} & NC
\end{array}
\;=\;
\begin{array}{ccc}
MA & \xrightarrow{Mh} & MC \\
{\scriptstyle\tau_A}\downarrow & \overset{\tau_h}{\swarrow} & {\scriptstyle\tau_C}\downarrow \\
NA & \xrightarrow[Nh]{} & NC
\end{array}
,
$$

that is,

$$
\begin{array}{ccccc}
(\tau_C Mg)Mf & \xrightarrow{\tau_g Mf} & (Ng\tau_B)Mf & \xleftarrow{\alpha} & Ng(\tau_B Mf) \\
{\scriptstyle\alpha}\uparrow & & \circ & & \downarrow{\scriptstyle\tau_f Ng} \\
\tau_C(MgMf) & \xrightarrow[\tau_h]{} & (NgNf)\tau_A & \xleftarrow[\alpha]{} & Ng(Nf\tau_A)
\end{array}
.
$$

Given $S \underset{N}{\overset{M}{\rightrightarrows}} \mathcal{X}$ with $\downarrow\tau$ and $\Phi: T \longrightarrow S$, we have $T \underset{N\Phi}{\overset{M\Phi}{\rightrightarrows}} \mathcal{X}$ with $\downarrow\tau\Phi$ for which $(\tau\Phi)_f = \tau_{\Phi f}$ for $f \in T(C_1)$.

D2. Going back to the definition of $\mathcal{R}\mathrm{T}_1$ in part (B) of the proof of 7.5, and using the notation there, that definition can be put as follows. Consider the 2-cat-sketch S_0:

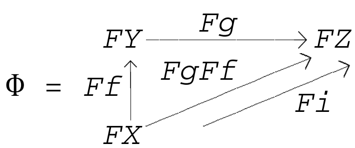


$a\colon gf \xrightarrow{\cong} i$

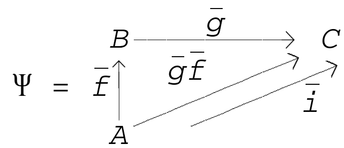
$(\ \mathrm{S}_0(\mathrm{T}_1)=\{(f,g,gf)\}\ ,$
$\mathrm{S}_0(\mathrm{C}_2)=\{a,a^{-1},1_i,1_{gf}\}\ )$

,

and the two diagrams $S_0 \underset{\Psi}{\overset{\Phi}{\rightrightarrows}} \mathcal{A}$ defined as



;

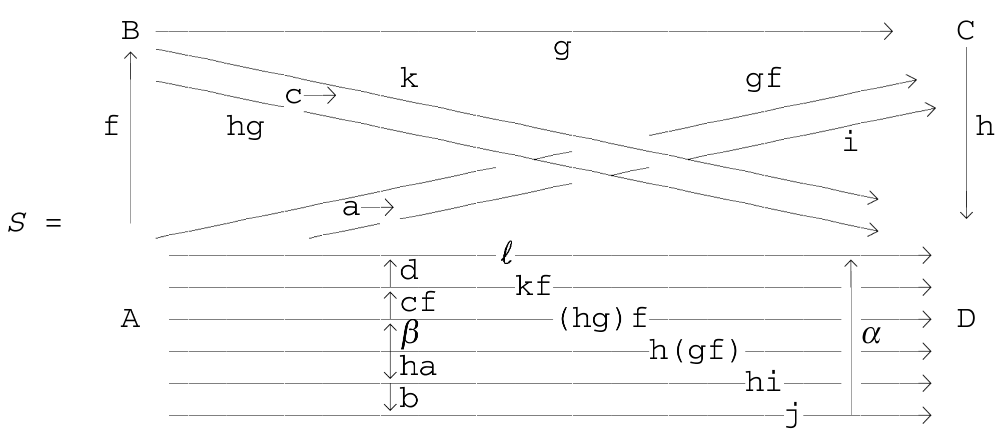


.

$\Phi a = FaF_{f,\,g}\colon FgFf \xrightarrow{\cong} Fi$ $\qquad$ $\Psi a = \bar{a}\colon \bar{g}\bar{f} \xrightarrow{\cong} \bar{i}$

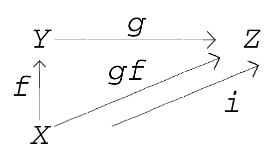
Then $\mathcal{R}\mathrm{T}_1(\varphi,\gamma,\eta)[a,\bar{a}]$ iff $x, y, z, \varphi, \gamma, \iota$ are the components of a map $\Phi \to \Psi$.

D3. In what follows, we will consider the following 2-cat-sketch S and various of its parts (subsketches):



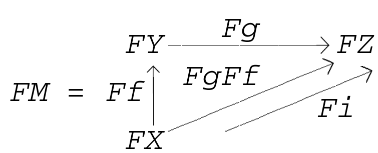
$S(T_1)$ has six elements, (f, g, gf), $(gf, h, h(gf))$, (i, h, hi), (g, h, hg), $(f, hg, h(gf))$, (f, k, kf); the notations showing composition are purely symbolic. The horizontal compositions cf and ha signify the presence of elements "1_f" and "1_h"of $S(I_2)$, and two corresponding elements of $S(H)$. $S(I_1)=\emptyset$. There are further 2-cells and elements of $S(I_2)$ and $S(T_2)$ to the effect that a, b, c, d, α and β are isomorphisms, and α is the composite $d(cf)\beta(ha)^{-1}b^{-1}$.

In case of a general 2-cat-sketch S, for a sketch-map $M: S \to \mathcal{X}$ and a functor $F: \mathcal{X} \to \mathcal{A}$ of bicategories, the composite FM cannot be defined (think of a sketch in which a 1-cell is a composite in two different ways); in the case of our S however, since S is sufficiently "free", a useful sense can be ascribed to FM. First of all, for S_0 from D2, for $M: S_0 \to \mathcal{X}$, $F: \mathcal{X} \to \mathcal{A}$, FM is defined as Φ was above: for



$$a: gf \xrightarrow{\cong} i$$

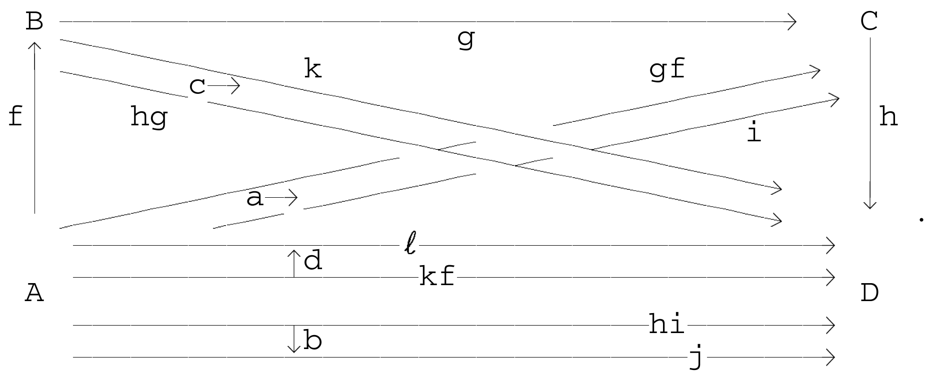
as M, we put FM to be



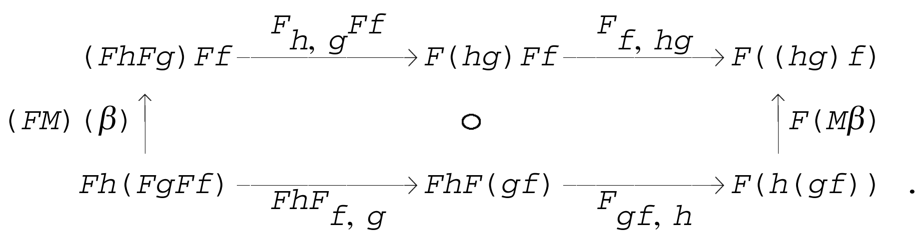
.

$$Fa = FaF_{f,g}: FgFf \xrightarrow{\cong} Fi$$

Now, there are four mappings of the form $S_0 \to S$, corresponding to the four items $a: gf \xrightarrow{\cong} i$, $b: hi \xrightarrow{\cong} j$, $c: hg \xrightarrow{\cong} k$, $d: kf \xrightarrow{\cong} \ell$. We define, for any $M: S \to \mathcal{X}$ and $F: \mathcal{X} \to \mathcal{A}$, $FM: S \to \mathcal{A}$ as follows. First, we make sure that for any of the four maps $\sigma: S_0 \to S$, $(FM)\sigma = F(M\sigma)$. This requirement determines FM as far as its restriction to the subsketch

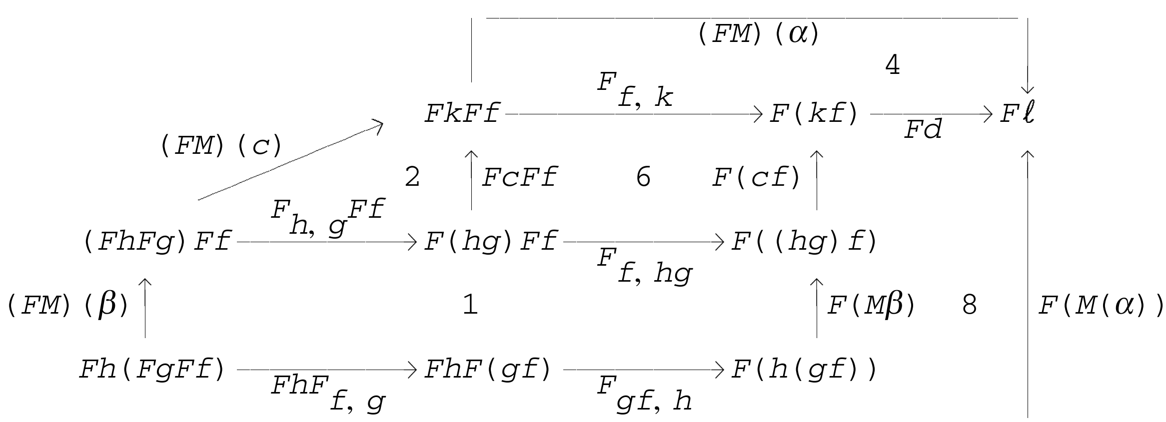


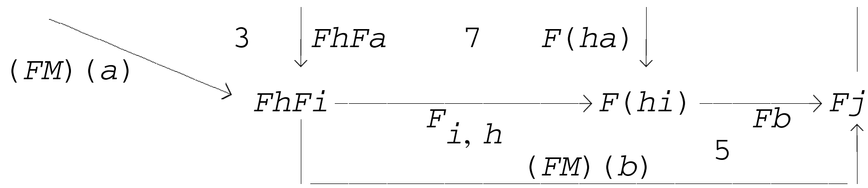
is concerned. But then the effect of FM is uniquely determined on the items $h(gf)$, $(hg)f$, cf, ha . Next, we define $(FM)(\beta)$ so that the following diagram commutes; we wrote f, g, h for Mf , Mg , Mh :



Finally, the effect of FM on α in S is now uniquely determined. It is worth noting that if $M\beta = \alpha_{f,\ g,\ h}$, then $(FM)(\beta) = \alpha_{Ff,\ Fg,\ Fh}$ ($f{=}Mf$, etc.); the reason is that F "preserves" α (see above).

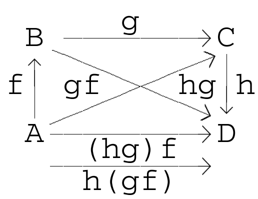
I **claim** that, for $FM{:}\,S\to\mathcal{A}$ so defined, $(FM)(\alpha) = F(M(\alpha))$. This is demonstrated by the following commutative diagram:





Here, the cell 1 commutes by the definition of $(FM)(\beta)$; 2, 3, 4, 5 commute by the definition of FM on the 2-cells a, b, c, d; 6 and 7 by the naturality of $F_{\_,\_}$; and 8 by the fact that $M\alpha$ is the appropriate composite. The assertion is the commutativity of the outside perimeter of the diagram.

D4. Let S_1 be the following subsketch of S :



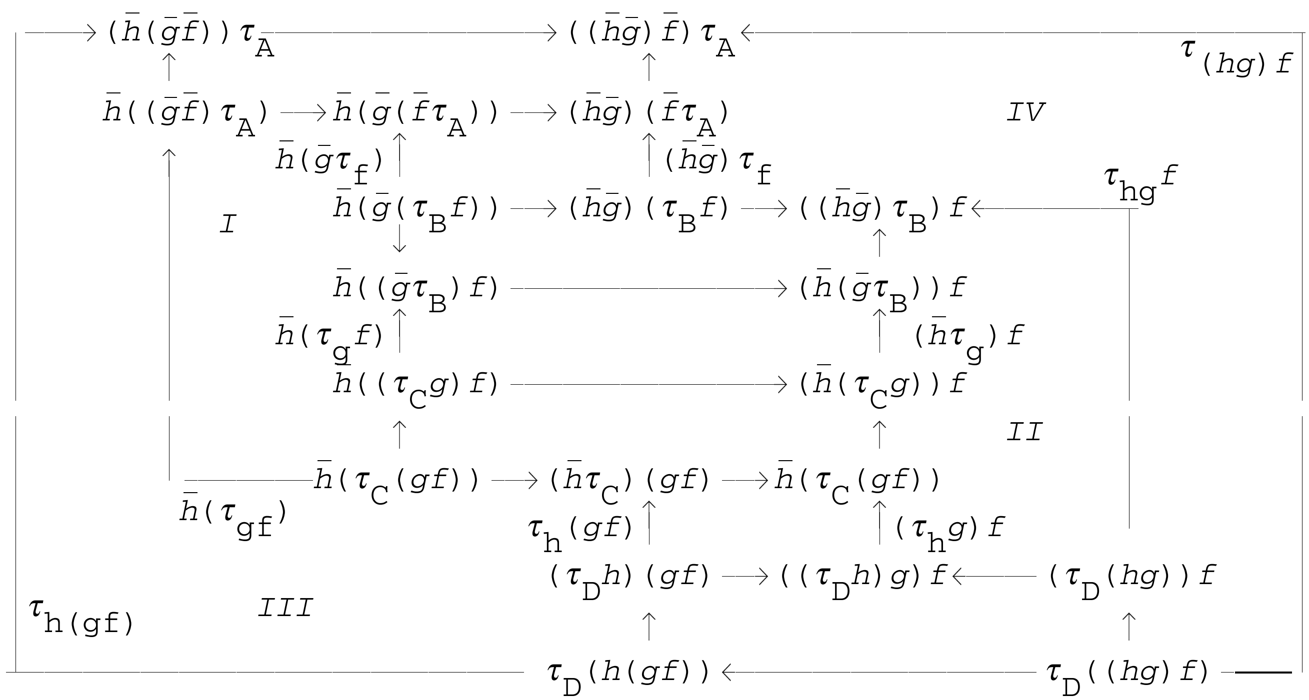
($S_1(t)=\emptyset$ for all $t\in L_{2\text{-cat}}$, except for $t=C_0, C_1, T_1$) , and let S_2 be the sketch (subsketch of S) obtained by adding the 2-cell $\alpha: h(gf) \longrightarrow (hg)f$ to S_1 . Suppose we have $M, N: S_2 \to \mathcal{A}$ such that $M\alpha = \alpha_{Mf,\,Mg,\,Mh}$ and $N\alpha = \alpha_{Nf,\,Ng,\,Nh}$ (associativity isomorphisms), and, also writing M for $M{\restriction}S_1$, we have

$$S_1 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A} \qquad (1)$$

Then τ is a map with respect to S_2 , that is,

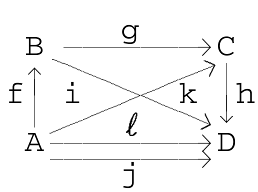
$$S_2 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A} \ .$$

This fact expresses the naturality of the associativity isomorphism in a sense that is considerably stronger than the one required in the definition of bicategory. The proof of the assertion is contained in the diagram



in which t is written for Mt, $\bar{t}$ for Nt, for all relevant values of t, and all unmarked arrows are instances of associativity isomorphisms, possibly horizontally composed with a 1-cell. The issue is the commutativity of the outside quadrangle. The four cells marked I, II, III and IV commute by the definition of τ being a map as in (1). The commutativity of the pentagons are the associativity coherence axioms for bicategory; the commutativity of the small quadrangles are instances of the (ordinary) naturality of the associativity isomorphism. Since all cells commute, the outside commutes as a consequence, and this is what we want.

D5. Now, start with the part (subsketch) S_3



of S ($S_3(t)=\emptyset$ for all $t\in L_{2\text{-cat}}$, except for $t=C_0, C_1$), and a map

$$S_3 \;\underset{N}{\overset{M}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow\sigma}}}}\; \mathcal{A}\ . \tag{2}$$

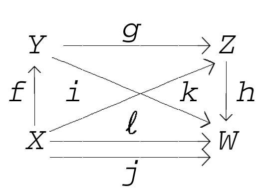
It is clear that if we have any $T \underset{Q}{\overset{P}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow\theta}}}} \mathcal{A}$, and T' is the sketch obtained by adding a new element " $gf=h$ " to $T(\mathrm{T}_1)$, where f and g are already in T, but h is new, then P, Q and θ uniquely extend to $T' \underset{Q}{\overset{P}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow\theta}}}} \mathcal{A}$. Now, let S_4 be the part of S which is S without the 2-cells ($S_4(t)=S(t)$ for $t=\mathrm{C}_0,\mathrm{C}_1,\mathrm{T}_1$ and $S_4(t)=\emptyset$ otherwise). Applying the above remark four times, we have, a unique extension

$$S_4 \;\underset{N}{\overset{M}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow\sigma}}}}\; \mathcal{A}$$

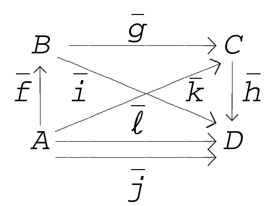
of (2).

D6. Suppose T is a sketch, T' is a subsketch of T missing only some 2-cells and T_2-elements of T, and that T is generated by T' in the sense that T is the least subsketch T'' of T such that T'' contains T' and every time when $(\rho, \sigma.\theta)\in T(\mathrm{T}_2)$, $\rho, \sigma\in T''(\mathrm{C}_2)$, then $\theta\in T''(\mathrm{C}_2)$, and every time when $(\rho, \sigma.\theta)\in T(\mathrm{H})$, $\rho, \sigma\in T''(\mathrm{C}_2)$, then $\theta\in T''(\mathrm{C}_2)$. Then every transformation $T' \underset{N}{\overset{M}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow\tau}}}} \mathcal{A}$ is also one as in $T \underset{N}{\overset{M}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow\tau}}}} \mathcal{A}$. This is immediate.

D7. Let us turn to the proof that $\mathcal{R}$ preserves A . What we need to show is this. Assume that we have



in $\mathcal{X}$,



in $\mathcal{A}$,

the items listed under (*) in §7, the further items

$$(w\colon FW \xrightarrow{\simeq} D) \in \mathcal{R}\mathrm{C}_0[W, D] ,$$
$$(b\colon hi \xrightarrow{\cong} j) \in \mathcal{X}^{\#}\mathrm{T}_1(i, h, j) , \quad (\bar{b}\colon \bar{h}\bar{i} \xrightarrow{\cong} \bar{j}) \in \mathcal{A}^{\#}\mathrm{T}_1(\bar{i}, \bar{h}, \bar{j}) ,$$
$$(c\colon hg \xrightarrow{\cong} k) \in \mathcal{X}^{\#}\mathrm{T}_1(g, h, k) , \quad (\bar{c}\colon \bar{h}\bar{g} \xrightarrow{\cong} \bar{k}) \in \mathcal{X}^{\#}\mathrm{T}_1(\bar{g}, \bar{h}, \bar{k}) ,$$
$$(d\colon kf \xrightarrow{\cong} \ell) \in \mathcal{X}^{\#}\mathrm{T}_1(f, k, \ell) , \quad (\bar{d}\colon \bar{k}\bar{f} \xrightarrow{\cong} \bar{\ell}) \in \mathcal{X}^{\#}\mathrm{T}_1(\bar{f}, \bar{k}, \bar{\ell}) ,$$
$$\eta \in \mathcal{R}\mathrm{C}_1(z, w)[h, \bar{h}] , \quad \psi \in \mathcal{R}\mathrm{C}_1(x, w)[j, \bar{j}] , \quad \kappa \in \mathcal{R}\mathrm{C}_1(y, w)[k, \bar{k}] ,$$
$$\lambda \in \mathcal{R}\mathrm{C}_1(x, w)[\ell, \bar{\ell}] ;$$

and assume that

$$\mathcal{R}\mathrm{T}_1(\varphi, \gamma, \iota)[a, \bar{a}] , \quad \mathcal{R}\mathrm{T}_1(\iota, \eta, \psi)[b, \bar{b}] ,$$
$$\mathcal{R}\mathrm{T}_1(\gamma, \eta, \kappa)[c, \bar{c}] , \quad \mathcal{R}\mathrm{T}_1(\varphi, \psi, \lambda)[d, \bar{d}]$$

hold. Under these conditions, we want that if $\mathcal{R}\mathrm{C}_2(x, w; \psi, \lambda)[\alpha, \bar{\alpha}]$, then

$$\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha) \iff \mathcal{A}^{\#}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \bar{\alpha}) .$$

I claim that it suffices to show that

$$\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha) \text{ and } \mathcal{A}^{\#}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \bar{\alpha}) \text{ imply } \mathcal{R}\mathrm{C}_2(x, w; \psi, \lambda)[\alpha, \bar{\alpha}] .$$

We use that for the given a, b, c, d , there is a unique α such that $\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha)$ (see (4) in §7), and similarly for $\bar{a}, \bar{b}, \bar{c}, \bar{d}$; and we use that for the given $x, w; \psi, \lambda$, the relation $\mathcal{R}\mathrm{C}_2(x, w; \psi, \lambda)[\alpha, \bar{\alpha}]$ of the variables $\alpha, \bar{\alpha}$ establishes a bijection $\alpha \mapsto \bar{\alpha} : \mathcal{X}\mathrm{C}_2(j, \ell) \xrightarrow{\cong} \mathcal{A}\mathrm{C}_2(\bar{j}, \bar{\ell})$. The claim now is easily seen.

Thus, we assume $\mathcal{X}^{\#}\mathrm{A}(a, b, c, d; \alpha)$ and $\mathcal{A}^{\#}\mathrm{A}(\bar{a}, \bar{b}, \bar{c}, \bar{d}; \bar{\alpha})$.

Recall the sketch S . The data give us diagrams $M_0 : S \longrightarrow \mathcal{X}$, $N : S \longrightarrow \mathcal{A}$; the effect of

M_0 , N are given by the notation, except that $M_0\beta = \alpha_{f,\,g,\,i}$ (associativity iso in $\mathcal{X}$) and $N\beta = \alpha_{\bar{f},\,\bar{g},\,\bar{i}}$ (associativity iso in $\mathcal{A}$). Composing M_0 with F, we get $M{=}FM_0 : S \longrightarrow \mathcal{A}$ (see D3). Consider the restrictions $M: S_3 \to \mathcal{A}$, $N: S_3 \to \mathcal{A}$. The data $x,\ y,\ z,\ w,\ \varphi,\gamma,\iota,\eta,\psi,\kappa,\lambda$ supply the components of a map

$$S_3 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A} \ .$$

By D5, we have a unique extension of τ , also denoted by τ , as in

$$S_4 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A} \ .$$

Let S_5 be the subsketch of S that consists of S_4 , and the 2-cells $a,\ b,\ c,\ d$. The assumptions and D2 (applied to the four maps $S_0 \to S$) tell us that we have

$$S_5 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A} \ .$$

Now, add also β back to S_5 , getting S_6 . Since by D3,

$$M\beta = (FM_0)(\beta) = \alpha_{Ff,\ Fg,\ Fh} = \alpha_{Mf,\ Mg,\ Mh} \ ,$$

D4 says that we have

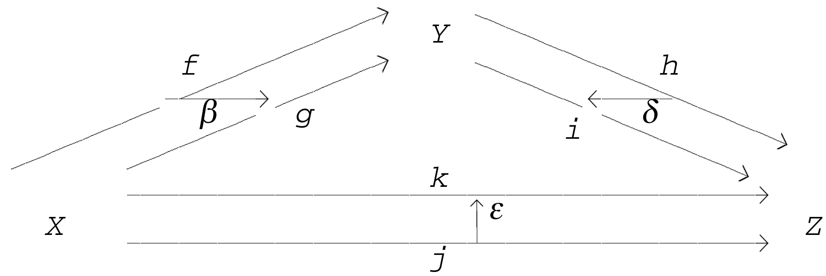
$$S_6 \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A} \ ,$$

and finally D6 says that

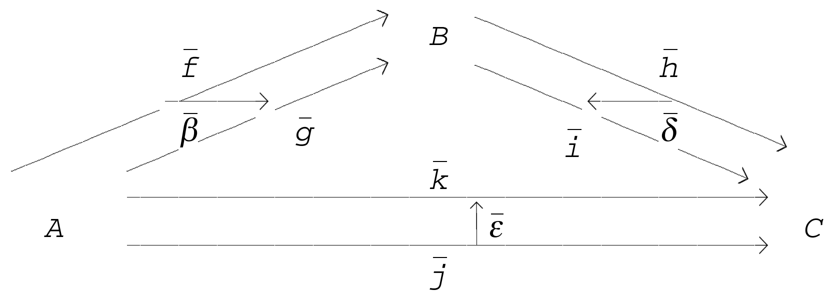
$$S \underset{N}{\overset{M}{\underset{\downarrow\tau}{\rightrightarrows}}} \mathcal{A} \ .$$

The fact that τ is natural with respect to α is the desired fact $\mathcal{R}C_2(x,\ w;\ \psi,\ \lambda)[\alpha,\ \bar{\alpha}]$, since, by D3, $M\alpha = F(M_0\alpha)$.

D8. The proof that $(\mathcal{R}, r_0, r_1)$ preserves H is similar, and simpler. Now, the situation is this. We have



in $\mathcal{X}$, and



in $\mathcal{A}$; we have

$$(x:FX\xrightarrow{\simeq}A)\in\mathcal{R}\mathrm{C}_0[X,A]\ ,\quad (y:FY\xrightarrow{\simeq}B)\in\mathcal{R}\mathrm{C}_0[Y,B]\ ,\quad (z:FZ\xrightarrow{\simeq}C)\in\mathcal{R}\mathrm{C}_0[Z,C]\ ,$$
$$\varphi\in\mathcal{R}\mathrm{C}_1(x,y)[f,\bar f]\ ,\quad \eta\in\mathcal{R}\mathrm{C}_1(y,z)[h,\bar h]\ ,\quad \psi\in\mathcal{R}\mathrm{C}_1(x,z)[j,\bar j]\ ,$$
$$\gamma\in\mathcal{R}\mathrm{C}_1(x,y)[g,\bar g]\ ,\quad \iota\in\mathcal{R}\mathrm{C}_1(y,z)[i,\bar i]\ ,\quad \kappa\in\mathcal{R}\mathrm{C}_1(x,z)[k,\bar k]\ ,$$
$$s\in\mathcal{X}^{\#}\mathrm{T}_1(f,h,j)\ ,\quad t\in\mathcal{X}^{\#}\mathrm{T}_1(g,i,k)\ ,\quad \bar s\in\mathcal{A}^{\#}\mathrm{T}_1(\bar f,\bar h,\bar j)\ ,\quad \bar t\in\mathcal{A}^{\#}\mathrm{T}_1(\bar g,\bar i,\bar k)$$

such that

$$\mathcal{R}\mathrm{C}_2(x,y;\varphi,\gamma)[\beta,\bar\beta]\ ,\quad \mathcal{R}\mathrm{C}_2(y,z;\eta,\iota)[\delta,\bar\delta]\ , \tag{3}$$
$$\mathcal{R}\mathrm{T}_1(\varphi,\eta,\psi)[s,\bar s]\ \text{ and }\ \mathcal{R}\mathrm{T}_1(\gamma,\iota,\kappa)[t,\bar t]\ . \tag{4}$$

Under these conditions, we want that

$$\mathcal{R}C_2(x, z; \psi, \kappa)[\varepsilon, \bar{\varepsilon}] \Longrightarrow (\mathcal{X}^{\#}H(s, t; \beta, \delta, \varepsilon) \Longleftrightarrow \mathcal{A}^{\#}H(\bar{s}, \bar{t}; \bar{\beta}, \bar{\delta}, \bar{\varepsilon}))\ .$$

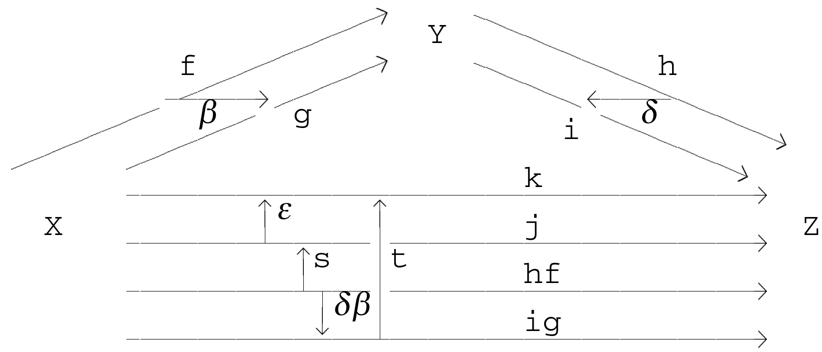
Again, it suffices to show that

$$\mathcal{X}^{\#}H(s, t; \beta, \delta, \varepsilon) \text{ and } \mathcal{A}^{\#}H(\bar{s}, \bar{t}; \bar{\beta}, \bar{\delta}, \bar{\varepsilon}) \qquad (5)$$

imply $\mathcal{R}C_2(x, z; \psi, \kappa)[\varepsilon, \bar{\varepsilon}]$. (6)

Assume (5). Consider the 2-cat-sketch

$T =$

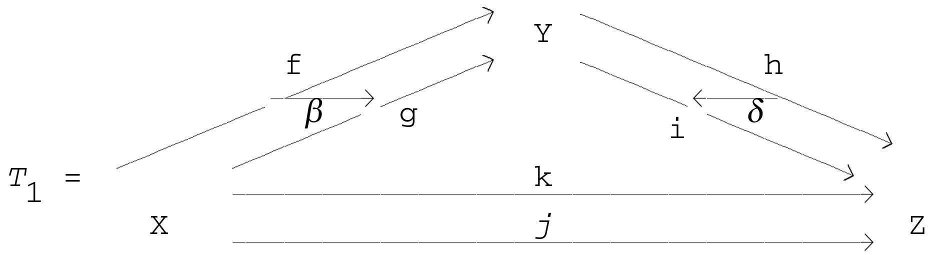


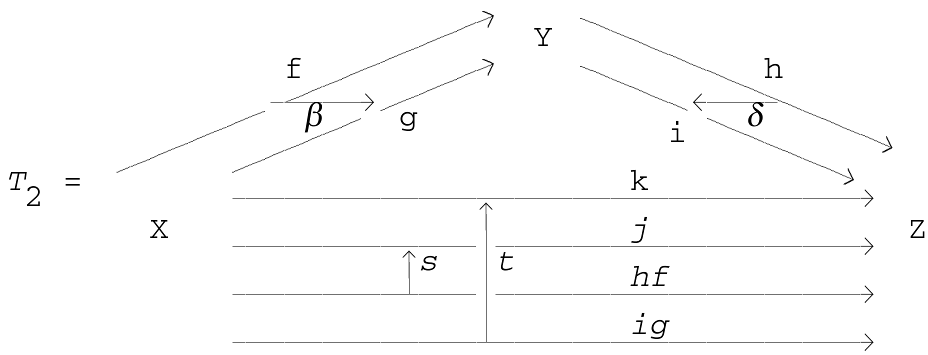
We have (f, h, hf), $(g, i, ig) \in T(T_1)$, $(\beta, \delta, \delta\beta) \in T(H)$, and $\begin{array}{ccc} hf & \xrightarrow{s} & j \\ \delta\beta \downarrow & \circ & \downarrow \varepsilon \\ ig & \xrightarrow[t]{} & k \end{array}$ (the latter by an (unmarked) 2-cell σ , and (s, ε, σ), $(\delta\beta, t, \sigma) \in T(T_2)$).

The conditions in (5) ensure that the data we have give rise to morphisms $M_0 : T \to \mathcal{X}$, $N: T \to \mathcal{A}$. As in the case of the sketch S , we can form the composite $M = FM_0 : T \to \mathcal{A}$; we have $M(s) = Fs \circ F_{f,\, h}$, $M(t) = Ft \circ F_{g,\, i}$; the commutativity of the diagram

$$\begin{array}{ccc} FhFf & \xrightarrow{F\delta F\beta} & FiFg \\ F_{f,\, h} \downarrow & \circ & \downarrow F_{g,\, i} \\ F(hf) & \xrightarrow{F(\delta\beta)} & F(ig) \\ Fs \downarrow & \circ & \downarrow Ft \\ F(j) & \xrightarrow[F\varepsilon]{} & F(k) \end{array}$$

ensures that M is indeed $M\colon T \to \mathcal{A}$. Consider the following subsketches of T:





The data x, y, z, φ, γ, η, ι, ψ, κ give, via the relation (3), a map

$$T_1 \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ ,$$

which, by (4) and D2, uniquely extends to

$$T_2 \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ .$$

By D6, this extends to

$$T \begin{array}{c} \xrightarrow{\quad M \quad} \\ \downarrow \tau \\ \xrightarrow[\quad N \quad]{} \end{array} \mathcal{A} \ .$$

The naturality of τ with respect to ε is the desired relation (6).

Appendix E: More on equivalence and interpolation

In this section, S and T are small Heyting categories, $\mathbf{L}$ is a DSV , $\mathbf{K}$ its category of kinds, and $F:\mathbf{L}\to S$, $G:\mathbf{L}\to T$ are S- , resp. T-valued $\mathbf{L}$-structures. $\mathrm{Mod}(S)$ denotes the category of coherent functors $S\to\mathrm{Set}$, a full subcategory of Set^S ; similarly for $\mathrm{Mod}(T)$.

Primarily, we have in mind T (also, S) obtained as the Lindenbaum-Tarski category $[T_0]$ of a theory T_0 in intuitionistic logic. We will be looking at *Kripke-models of* T ; that is, Heyting functors $\Phi:T\to Set^{\mathbf{C}}$, with various exponent categories $\mathbf{C}$; we write $\Phi\models T$ for " Φ is a Kripke model of T ". " σ is a sentence of T ", " $\Phi\models\sigma$ " and other unexplained notation have the meanings analogous to the ones used in §5.

We have the following intuitionistic version of the interpolation theorem 5.(7)(a).

(1) Assume that σ,τ are sentences of T , and for all Kripke models Φ, $\Psi\models T$,

$$\Phi\models\sigma \ \ \& \ \ \Phi\restriction\mathbf{L}\sim_{\mathbf{L}}\Psi\restriction\mathbf{L} \ \Longrightarrow \ \Psi\models\tau \ .$$

Then there is an $\mathbf{L}$-sentence θ in logic with dependent sorts without equality such that for all $\Phi\models T$,

$$\Phi\models\sigma \Longrightarrow \Phi\restriction\mathbf{L}\models\theta \qquad \text{and} \qquad \Phi\restriction\mathbf{L}\models\theta \Longrightarrow \Phi\models\tau \ .$$

In (5) below, we will reformulate (and strengthen) the theorem in a purely syntactical fashion, by removing references to Kripke semantics.

We will imitate [M4] in the proof of (1).

When $I:T\to Q$ is a Heyting functor , and $F:\mathbf{L}\to T$, we have an obvious composite $IF:\mathbf{L}\to Q$.

Recall that for $\mathbf{L} \overset{H}{\underset{I}{\rightrightarrows}} Q$, $\alpha : H \underset{\mathbf{L}}{\leftrightarrow} I$ (called an $\mathbf{L}$-*equivalence*) is $\alpha = (A, \alpha_0, \alpha_1)$, with $A : \mathbf{K} \to Q$ and $\alpha_0 : A \to H{\restriction}\mathbf{K}$, $\alpha_1 : A \to I{\restriction}\mathbf{K}$ with suitable properties. Given also $J : Q \to R$, we have the composite $J\alpha \underset{\text{def}}{\equiv} (JA, J\alpha_0, J\alpha_1) : JH \underset{\mathbf{L}}{\leftrightarrow} JI$; the requisite properties are easily checked.

Consider data as in

$$\begin{array}{ccc} S & \xrightarrow{\quad H \quad} & Q \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I} \\ \mathbf{L} & \xrightarrow[\quad G \quad]{} & T \end{array} \qquad\qquad \alpha : HF \underset{\mathbf{L}}{\overset{\cong}{\longleftrightarrow}} IG \tag{2}$$

with H, I Heyting functors. Fixing the items $\mathbf{L}$, S, T, F, G, and for Q a Heyting category, let C_Q be the groupoid whose objects are triples (H, I, α) as in (2), and whose arrows $(H, I, \alpha) \xrightarrow{\cong} (H', I', \alpha')$ (where $\alpha = (A, \alpha_0, \alpha_1)$, $\alpha' = (A', \alpha'_0, \alpha'_1)$) are triples $(\varphi : H \xrightarrow{\cong} H', \psi : I \xrightarrow{\cong} I', \gamma : A \xrightarrow{\cong} A')$ of natural isomorphisms such that

$$\begin{array}{ccc} HF & \xrightarrow{\quad \varphi F \quad} & H'K \\ {\scriptstyle \alpha_0}\uparrow & \circ & \uparrow{\scriptstyle \alpha'_0} \\ A & \xrightarrow{\quad \gamma \quad} & A' \\ {\scriptstyle \alpha_1}\downarrow & \circ & \downarrow{\scriptstyle \alpha'_1} \\ IG & \xrightarrow[\quad \psi G \quad]{} & I'G \end{array} \tag{2'}$$

Composition in C_Q is defined in the obvious way. We may write $(Q; H, I, \alpha)$ for (H, I, α) to emphasize Q.

Given an object $\Gamma = (Q; H, I, \alpha)$ of C_Q, and $L : Q \to R$, a Heyting functor, we have the composite object $L\Gamma = (R; LH, LI, L\alpha)$ (with $L\alpha$ described above) of C_R. Moreover, we have the functor

$$\Gamma^*_R = \Gamma^* \ : \ \mathrm{Hom}(Q, R) \longrightarrow \mathrm{C}_R$$

where $\mathrm{Hom}(Q, R)$ is the category (groupoid) of Heyting functors $Q \to R$ with isomorphisms as arrows; the object-function of γ^* is $L \mapsto L\gamma$ as described, the arrow-function being similarly defined by composition.

There are $Q{=}S{+}_{\boldsymbol{L}}T$, a Heyting category, and $\Gamma\in\mathrm{C}_Q$, given by the data

$$\begin{array}{ccc} S & \xrightarrow{\;I_0\;} & S{+}_{\boldsymbol{L}}T \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I_1} \\ \boldsymbol{L} & \xrightarrow[\;G\;]{} & T \end{array} \qquad \alpha\colon I_0F \xleftrightarrow[\boldsymbol{L}]{\cong} I_1G \qquad , \tag{3}$$

such that $(Q;\Gamma)$ enjoys the universal property that for any Heyting category R, Γ_R^* is a *surjective* (on objects) equivalence of categories (groupoids).

The description of $Q=S{+}_{\boldsymbol{L}}T$ is as follows. Q is the Lindenbaum-Tarski category $[Q_0]$ of a theory Q_0 in intuitionistic logic. L_{Q_0} consists of $\mathrm{L}_S\bigsqcup\mathrm{L}_T$, the disjoint union of the underlying graphs of S and T, together with new objects $\mathrm{A}K$, one for each $K\in\boldsymbol{K}$, arrows $\mathrm{A}p\colon\mathrm{A}K\to\mathrm{A}K_p$, one for each $K\in\boldsymbol{K}$ and $p\in K|\boldsymbol{K}$, and arrows $\alpha_{0K}\colon\mathrm{A}K\to FK$, $\alpha_{1K}\colon\mathrm{A}K\to GK$. The axioms of Σ_{Q_0} are those of S and T (formulated for the symbols that are the images of the original symbols of S and T in $\mathrm{L}_S\bigsqcup\mathrm{L}_T$), together with axioms amounting to the assertion that $(A,\alpha_0,\alpha_1)=(\mathrm{A}K,\alpha_{0K},\alpha_{1K})_{K\in\boldsymbol{K}}$ is an $\boldsymbol{L}$-equivalence between the S-model and the T-model involved. The object $\Gamma\in\mathrm{C}_Q$ is the evident one. Kripke-models of $S{+}_{\boldsymbol{L}}T$ are essentially the same as triples $(M\models S,\ N\models T,\ \alpha\colon M\xleftrightarrow[\boldsymbol{L}]{}N)$; this fact is essentially the universal property of $(S{+}_{\boldsymbol{L}}T,\ \gamma)$ with respect to R a presheaf category $\mathrm{Set}^{\boldsymbol{C}}$.

We call (3) the $\boldsymbol{L}$*-pushout of* $(F\colon\boldsymbol{L}\to S,\ G\colon\boldsymbol{L}\to T)$.

Next, we introduce some auxiliary concepts.

Suppose that in

$$\begin{array}{ccc} S & \xrightarrow{\;H\;} & Q \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I} \\ \boldsymbol{L} & \xrightarrow[\;G\;]{} & T \end{array}$$

Q is a coherent category, H and I are coherent functors (however, S and T are still the same Heyting categories as before). Let $A:\mathbf{K}\to Q$, $\alpha_0:A\to HF\restriction\mathbf{K}$, $\alpha_1:A\to IG\restriction\mathbf{K}$. We write $\alpha=(A,\alpha_0,\alpha_1):H\xrightarrow[*]{}I$ if the following holds:

(3') for every finite $\mathbf{K}$-context $\mathcal{X}$, and any $\mathbf{L}$-formula θ of FOLDS,

$(\alpha_0)^*_{[\mathcal{X}]}(F[\mathcal{X}:\theta])\leq_{A[\mathcal{X}]}(\alpha_1)^*_{[\mathcal{X}]}(G[\mathcal{X}:\theta])$.

This refers to the arrows $HF[\mathcal{X}]\xleftarrow{(\alpha_0)[\mathcal{X}]}A[\mathcal{X}]\xrightarrow{(\alpha_1)[\mathcal{X}]}IG[\mathcal{X}]$ induced by α_0 and α_1. We write $(A,\alpha_0,\alpha_1):H\xleftrightarrow[*]{}I$ if both $(A,\alpha_0,\alpha_1):H\xrightarrow[*]{}I$ and $(A,\alpha_1,\alpha_0):I\xrightarrow[*]{}H$; of course, this just means an equality in place of $\leq_{A[\mathcal{X}]}$ in (3'). Finally, we write $(A,\alpha_0,\alpha_1):H\xleftrightarrow[\#]{}I$ if $\alpha=(A,\alpha_0,\alpha_1):H\xleftrightarrow[*]{}I$ and α_0 and α_1 are very surjective.

Notice that if $(A,\alpha_0,\alpha_1):H\xleftrightarrow[\#]{}I$, then $\alpha=(A,\alpha_0,\alpha_1):H\xleftrightarrow[\mathbf{L}]{}I$; the latter involves preserving atomic $\mathbf{L}$-formulas only.

Let us explain the meaning of the last-mentioned concepts when $Q=\mathrm{Set}$, and $H=M\in\mathrm{Mod}(S)$, $I=N\in\mathrm{Mod}(T)$.

With $\mathcal{X}$ and φ as above, let $\vec{a}=\langle a_x\rangle_{x\in\mathcal{X}}\in(M\restriction\mathbf{K})[\mathcal{X}]$. We write $M\models_{\mathrm{w}}\varphi[\vec{a}]$ for $\langle\vec{a}\rangle\in M(F[\mathcal{X}:\varphi])$ ($\subset M(F[\mathcal{X}])$); here, the notation $\langle\vec{a}\rangle$ is used in the sense given to it in the line after 5.(7'). The subscript w is to serve as a warning that this is a "non-standard" meaning for truth ($\models$); the coherent functor $M:S\to\mathrm{Set}$ is not supposed to respect the full logical structure of S, hence it does not necessarily "recognize" the full meaning of φ; M is a "weak model for $\mathbf{L}$-formulas". We have that for $U:\mathbf{K}\to\mathrm{Set}$, and $M\xleftarrow{m}U\xrightarrow{n}N$, $(U,m,n):M\xrightarrow[*]{}N$ iff for all $\mathcal{X}$ and φ as above, and for any $\langle c_x\rangle_{x\in\mathcal{X}}\in U[\mathcal{X}]$,

$$M\models_{\mathrm{w}}\varphi[\langle mc_x\rangle_{x\in\mathcal{X}}]\implies N\models_{\mathrm{w}}\varphi[\langle nc_x\rangle_{x\in\mathcal{X}}] .$$

Note that when $U=\emptyset$, $(\emptyset,\emptyset,\emptyset):M\xrightarrow[*]{}N$ means that $M(F[\emptyset:\varphi])=1\Rightarrow N(G[\emptyset:\varphi])=1$.

Let

$$\begin{array}{ccc} S & \xrightarrow{\bar{I}_0} & S+_{\#}T \\ {\scriptstyle F}\big\uparrow & & \big\uparrow{\scriptstyle \bar{I}_1} \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array} \qquad \bar{\alpha}=(\bar{A},\ \bar{\alpha}_0,\ \bar{\alpha}_1) : \bar{I}_0 F \xleftrightarrow[\#]{} \bar{I}_1 G \tag{4}$$

be the entity that is "initial" among all

$$\begin{array}{ccc} S & \xrightarrow{H} & Q \\ {\scriptstyle F}\big\uparrow & & \big\uparrow{\scriptstyle I} \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array} \qquad \alpha=(A,\ \alpha_0,\ \alpha_1) : HF \xleftrightarrow[\#]{} IG\ , \tag{4'}$$

in the following natural sense, amounting to a modification of the definition of $S+_{\boldsymbol{L}}T$. The category $\mathrm{C}^{\#}_{Q}$, for Q a coherent category, has objects (4'), and arrows

$$(\varphi : H \to H',\ \psi : I \to I',\ \gamma : A \to A') : (H,\ I,\ \alpha) \longrightarrow (H',\ I',\ \alpha')$$

($\alpha=(A,\ \alpha_0,\ \alpha_1)$, $\alpha'=(A',\ \alpha'_0,\ \alpha'_1)$) such that (2') holds; it is important that here φ , ψ and γ are not restricted to be isomorphisms. For any coherent category R , and $\Gamma\in\mathrm{C}^{\#}_{Q}$, we have

$$\Gamma^{*} : \mathrm{Coh}(Q,\ R) \longrightarrow \mathrm{C}^{\#}_{R}$$

where $\mathrm{Coh}(Q,\ R)$ is the category of coherent functors $Q \to R$, a full subcategory of R^{Q} . The universal property of $S+_{\#}T$ is that, for Γ given by (4), for any coherent R , Γ^{*} is a surjective equivalence of categories.

The construction of $S+_{\#}T$ is similar to that of $S+_{\boldsymbol{L}}T$. $S+_{\#}T$ is the Lindenbaum-Tarski category of a *coherent* theory $Q^{\#}_{0}$; the language of $Q^{\#}_{0}$ is the same as that for Q_0 given above for $S+_{\boldsymbol{L}}T$. We include (coherent) axioms to ensure

$$(\bar{\alpha}_0)^{*}_{[\mathcal{X}]}(F[\mathcal{X}:\theta]) =_{\bar{A}[\mathcal{X}]} (\bar{\alpha}_1)^{*}_{[\mathcal{X}]}(G[\mathcal{X}:\theta])$$

for each $\mathcal{X}$, θ as above. Note that the (ordinary, Set-valued) models of $S+_{\#}T$ are essentially the same as triples (M, N, u), with $M \in \mathrm{Mod}(S)$, $N \in \mathrm{Mod}(T)$ and $u : M \underset{\#}{\longleftrightarrow} N$.

(4) may be referred to as the $\#$-*pushout of* $(F:\boldsymbol{L}\to S,\ G:\boldsymbol{L}\to T)$.

Notice that there is a coherent comparison functor $J : S+_{\#}T \longrightarrow S+_{\boldsymbol{L}}T$ for which $J\bar{I}_0 = I_0$, $J\bar{I}_1 = I_1$ and $J\bar{\alpha} = \alpha$. The reason is the universal property of $S+_{\#}T$, and the fact that, for Heyting functors $\Phi : S \to R$, $\Psi : T \to R$, $\alpha : \Phi \underset{\boldsymbol{L}}{\longleftrightarrow} \Psi$ implies $\alpha : \Phi \underset{\#}{\longleftrightarrow} \Psi$.

Any diagram

$$\begin{array}{ccc} S & \xrightarrow{H} & Q \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle I} \\ \boldsymbol{L} & \xrightarrow[G]{} & T \end{array} \qquad\qquad HF \xleftarrow{\alpha_0} A \xrightarrow{\alpha_1} IG$$

involving (at least) coherent categories and coherent functors, is said to have the *interpolation property* if the following holds: whenever $\mathcal{X}$ is a finite context for $\boldsymbol{L}$, $\sigma \in \mathrm{S}_S(F[\mathcal{X}])$, $\tau \in \mathrm{S}_T(G[\mathcal{X}])$ and $(\alpha_0)^*_{[\mathcal{X}]}(H\sigma) \le_{A[\mathcal{X}]} (\alpha_1)^*_{[\mathcal{X}]}(I\tau)$, then there is an $\boldsymbol{L}$-formula θ (of FOLDS) such that $\sigma \le_{F[\mathcal{X}]} F[\mathcal{X}:\theta]$ and $G[\mathcal{X}:\theta] \le_{G[\mathcal{X}]} \tau$.

Using the (Kripke) completeness theorem for intuitionistic logic (for any small Heyting category S, there is a conservative Heyting functor $S \to \mathrm{Set}^{\boldsymbol{C}}$), it is easy to see that (1) is a weakened form of saying that the $\boldsymbol{L}$-pushout diagrams have the interpolation property. Thus, (1) will follow from

(5) Both the $\#$-pushout and the $\boldsymbol{L}$-pushout of a pair $(F:\boldsymbol{L}\to S,\ G:\boldsymbol{L}\to T)$, with S and T small Heyting categories, have the interpolation property. Moreover, the comparison map $J : S+_{\#}T \longrightarrow S+_{\boldsymbol{L}}T$ is conservative; thus, the assertion for the $\boldsymbol{L}$-pushout is a consequence of that for the $\#$-$\boldsymbol{L}$-pushout.

For the proof of (5), we will employ the method described in [M4] (and adapted there from [G]).

Let $M\in\mathrm{Mod}(S)$. $M\restriction L \overset{=}{\mathrm{def}} M\circ F$, and $M\restriction\boldsymbol{K} \overset{=}{\mathrm{def}} M\circ F\circ j$, for the inclusion $j:\boldsymbol{K}\to\boldsymbol{L}$. For $W\in\mathrm{Set}^{\boldsymbol{K}}$, an arrow $m:W\to M$ means an arrow $m:W\to M\restriction\boldsymbol{K}$.

We write L_S for the underlying graph of the category S, and regard it as a vocabulary for intuitionistic first-order logic. (Now, S is a general small Heyting category; in particular, what follows will also be applied to T.) For a finite sequence $\vec{x}=\langle x_i\rangle_{i<n}$ of distinct variables , by $[\vec{x}]$ we mean a chosen product $X_0\times X_1\times\ldots\times X_{n-1}$, where $x_i:X_i$. For any (first-order) formula φ over L_S, with free variables in $\vec{x}$, we have $[\vec{x}:\varphi]$, a subobject of $[\vec{x}]$, the "internal interpretation of φ in the context $\vec{x}$ in S"; see [MR1].

We will use the coherent theory $\mathrm{T}_S^{\mathrm{coh}}=(\mathrm{L}_S,\Sigma_S^{\mathrm{coh}})$, the internal theory of S as a coherent category introduced in [MR1]. $\mathrm{Mod}(S)$ is identical to $\mathrm{Mod}(\mathrm{T}_S^{\mathrm{coh}})$, the category of models of the theory $\mathrm{T}_S^{\mathrm{coh}}$ with ordinary homomorphisms as arrows. For a coherent formula φ with free variables in $\vec{x}$, $M([\vec{x}:\varphi])$, a subset of $M([\vec{x}])$, is identical to the ordinary interpretation of φ, $\{\vec{a}:M\models\varphi[\vec{a}/\vec{x}]\}$, modulo the canonical isomorphism $j:\prod_{i<n}X_i\to M([\vec{x}])$ ($\vec{x}=\langle x_i\rangle_{i<n}$, $x_i:X_i$) ; that is, $M([\vec{x}:\varphi])$ = $j(\{\vec{a}:M\models\varphi[\vec{a}/\vec{x}]\})$. For coherent formulas φ and ψ over L_S, with free variables included in $\vec{x}$,

$$\mathrm{T}_S^{\mathrm{coh}}\models\varphi\underset{\vec{x}}{\Longrightarrow}\psi\ \text{(that is, for all } M\in\mathrm{Mod}(S),\ M\models\forall\vec{x}(\varphi\to\psi)\text{) iff}$$

$$[\vec{x}:\varphi]\leq_{[\vec{x}]}[\vec{x}:\psi]\ ;$$

in other words, a coherent entailment is an ordinary semantic consequence of $\mathrm{T}_S^{\mathrm{coh}}$ iff it is internally true in S; this is but a form of the (Gödel) completeness theorem for coherent logic.

Now, we refer to $F:\boldsymbol{L}\to S$ as well. Let $x\mapsto\underline{x}$ a 1-1 mapping of variables of FOLDS over $\boldsymbol{L}$ into variables over L_S so that $\underline{x}:F(\mathrm{K}_x)$. Let, for any finite context $\mathcal{X}$ of $\boldsymbol{L}$-variables, $\mathrm{E}(\underline{\mathcal{X}})$ denote the formula

$$\bigwedge\{(Fp)(\underline{x}) =_{FK_p} \underline{x_{x,p}} : x\in\mathcal{X},\ p\in K_x|\boldsymbol{K}\}\ .$$

This formula describes that the $\underline{x}$ for $x\in\mathcal{X}$ fit together via the maps Fp , $p\in K_x|\boldsymbol{K}$, as dictated by the structure of the context $\mathcal{X}$.

Recall $F[\mathcal{X}]$ defined as a certain pullback; we have a monomorphism $m: F[\mathcal{X}] \rightarrowtail [\underline{\mathcal{X}}]$ for which $\pi_{\underline{x}}\circ m = \pi_x$ ($x\in\mathcal{X}$); here, we refer to the evident projections. In fact, m represents the subobject $[\underline{\mathcal{X}}:\mathrm{E}(\underline{\mathcal{X}})]$ of $[\underline{\mathcal{X}}]$. If $\Phi = [n: |\Phi| \rightarrowtail F[\mathcal{X}]]$ is any subobject of $F[\mathcal{X}]$, then $\underline{\Phi} \overset{=}{\mathrm{def}} [mn: |\Phi| \rightarrowtail [\underline{\mathcal{X}}]]$ is a subobject of $[\underline{\mathcal{X}}]$. We have a formula $\underline{\Phi}(\underline{\mathcal{X}})$ with free variables in $\underline{\mathcal{X}}$ such that $[\underline{\mathcal{X}}:\underline{\Phi}(\underline{\mathcal{X}})] = \underline{\Phi}$;

$$\underline{\Phi}(\underline{\mathcal{X}}) \overset{=}{\mathrm{def}} \exists z\in |\Phi| \bigwedge_{x\in\mathcal{X}} (\pi_x\circ n)(z)=\underline{x}$$

($\pi_x : F[\mathcal{X}] \longrightarrow FK_x$). When φ is an $\boldsymbol{L}$-formula in FOLDS, with $\mathrm{Var}(\varphi)\subset\mathcal{X}$, and we take $\Phi=F[\mathcal{X}:\varphi]\in \mathrm{S}(F[\mathcal{X}])$, we get $\underline{\varphi}(\underline{\mathcal{X}}) \overset{=}{\mathrm{def}} \underline{F[\mathcal{X}:\varphi]}(\underline{\mathcal{X}})$.

Note that if $M\in\mathrm{Mod}(S)$, then for $\langle a_x\rangle_{x\in\mathcal{X}}\in F[\mathcal{X}]$,

$$M\models_{\mathrm{w}}\varphi[\langle a_x\rangle_{x\in\mathcal{X}}] \iff M\models\underline{\varphi}(\underline{\mathcal{X}})[a_x/\underline{x}]_{x\in\mathcal{X}}\ . \tag{5'}$$

If $\mathrm{Var}(\varphi)\subset\mathcal{X}\subset\mathcal{X}'$, then

$$[\underline{\mathcal{X}}':\mathrm{E}(\underline{\mathcal{X}}')\wedge\underline{\varphi}(\underline{\mathcal{X}})] = [\underline{\mathcal{X}}':\underline{\varphi}(\underline{\mathcal{X}}')] \tag{6}$$

as is easily seen.

Let $\mathcal{X}\subset\mathcal{Y}$ be finite contexts over $\boldsymbol{L}$; assume $\mathrm{Var}(\varphi)\subset\mathcal{Y}$. Let us write $\forall(\mathcal{Y}-\mathcal{X})\varphi$ for the formula $\forall z_1\forall z_2\ldots\forall z_n\varphi$, where $\langle z_i\rangle_{i=1}^{n}$ is a repetition-free enumeration of the set $\mathcal{Y}$-$\mathcal{X}$ such that for all $j\le n$, $\mathcal{X}\cup\{x_i : i\le j\}$ is a context (an enumeration in a non-decreasing order of the level of K_z will ensure this; the formula $\forall z_1\forall z_2\ldots\forall z_n\varphi$ is well-formed as a consequence. $\forall(\mathcal{Y}-\mathcal{X})\varphi$ is not quite uniquely determined, but it is, up to logical equivalence). We have the equality:

$$[\underline{\mathcal{X}}:\underline{(\forall(\mathcal{Y}-\mathcal{X})\varphi)}(\underline{\mathcal{X}})] = [\underline{\mathcal{X}}:\mathrm{E}(\underline{\mathcal{X}})\wedge\forall(\underline{\mathcal{Y}}-\underline{\mathcal{X}})(\mathrm{E}(\underline{\mathcal{Y}})\rightarrow\underline{\varphi}(\underline{\mathcal{Y}}))]\ ; \tag{7}$$

here, $\forall(\underline{\mathcal{Y}-\mathcal{X}})$ stands for $\forall \underline{z_1} \forall \underline{z_2} \ldots \forall \underline{z_n}$ for $\mathcal{Y}-\mathcal{X} = \{z_1, \ldots z_n\}$ as above. This is easy to show by induction on the cardinality of $\mathcal{Y}$-$\mathcal{X}$.

Other easily seen equalities we will use are

$$[\mathcal{Y}\colon \underline{\bigwedge_{i<m} \varphi_i}(\underline{\mathcal{Y}})] = [\mathcal{Y}\colon \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{Y}})] \ , \tag{8}$$

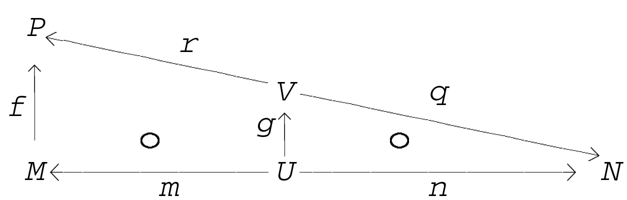
$$[\mathcal{Y}\colon \underline{\bigvee_{j<n} \psi_j}(\underline{\mathcal{Y}})] = [\mathcal{Y}\colon \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{Y}})] \ , \tag{9}$$

$$[\underline{\mathcal{Y}}\colon (\underline{\varphi \rightarrow \psi})(\underline{\mathcal{Y}})] = [\underline{\mathcal{Y}}\colon \mathrm{E}(\underline{\mathcal{Y}}) \wedge (\underline{\varphi}(\underline{\mathcal{Y}}) \rightarrow \underline{\psi}(\underline{\mathcal{Y}}))] \ , \tag{10}$$

under the natural conditions on the parameters involved.

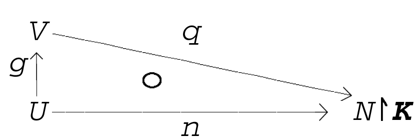
The following is the analog of Lemma 3 of [M4].

(11) Suppose $M \in \mathrm{Mod}(S)$, $N \in \mathrm{Mod}(T)$ and $(U, m, n)\colon M \underset{*}{\longrightarrow} N$. Then we have $P \in \mathrm{Mod}(S)$, $(f\colon M \rightarrow P) \in \mathrm{Mod}(S)$, $g\colon U \rightarrow V$ and $(V, r, q)\colon P \underset{*}{\longleftrightarrow} N$ such that q is very surjective, and



(12)

Proof. We first construct



in $\mathrm{Set}^{\boldsymbol{K}}$ such that $q\colon V \twoheadrightarrow N{\restriction}\boldsymbol{K}$ is very surjective and g is a monomorphism. We put

$U_0 \xrightarrow{n_0} N{\restriction}\boldsymbol{K} = U \xrightarrow{n} N{\restriction}\boldsymbol{K}$, and by recursion on $i<k$, the height of $\boldsymbol{K}$, assuming $U_i \xrightarrow{n_i} N{\restriction}\boldsymbol{K}$ defined, we define $U_{i+1} \xrightarrow{n_{i+1}} N{\restriction}\boldsymbol{K}$ as follows. For all $K\in\boldsymbol{K}$ except when the level of K equals i, we put $U_{i+1}K = U_iK$. When $K\in\boldsymbol{K}_i$, we put, for all $\vec{a}\in U_i[K]=U_{i+1}[K]$, $U_{i+1}K(\vec{a}) = U_iK(\vec{a})\bigsqcup(N{\restriction}\boldsymbol{K})K(n_i(\vec{a}))$; here, we use the notation of 1.(3). (This means that $U_{i+1}K = \bigsqcup_{\vec{a}\in U_i[K]}(U_iK(\vec{a})\bigsqcup(N{\restriction}\boldsymbol{K})K(n_i(\vec{a})))$.) We have the map $g_{i,\ i+1}: U_i \to U_{i+1}$ whose component at each $K\notin\boldsymbol{K}_i$ is the identity, and whose component at $K\in\boldsymbol{K}_i$ on the fiber over $\vec{a}\in U_i[K]$ is the coproduct coprojection $U_iK(\vec{a}) \longrightarrow U_{i+1}K(\vec{a})$. The component of n_{i+1} at each $K\notin\boldsymbol{K}_i$ is that of n_i. For $K\in\boldsymbol{K}_i$, $(n_{i+1})_K: U_{i+1}K \to (N{\restriction}\boldsymbol{K})K$ maps the image of $b\in U_iK(\vec{a})$ in $U_{i+1}K(\vec{a})$ under the first coprojection to $(n_i)_K(b)$, and the image of $b\in(N{\restriction}\boldsymbol{K})K(n_i(\vec{a}))$ in $U_{i+1}K(\vec{a})$ under the second coprojection to b itself. We have that $n_{i+1}\circ g_{i,\ i+1}=n_i$. Having defined all $U_i \xrightarrow{n_i} N{\restriction}\boldsymbol{K}$, we let $V = \operatorname{colim}_{i<k} U_{i+1}$ ($= U_{k-1}$ when $k<\omega$), with the $g_{i,\ i+1}$ as connecting maps, and $q = \operatorname{colim} n_i$. g is the coprojection $U_0 \to V$. It is fairly clear that V, g and q so constructed are appropriate.

We may assume that g is an inclusion (that is, each of its components g_K is an inclusion of sets).

Consider the (infinite) contexts $\mathcal{Y}_U \subset \mathcal{Y}_V$ associated with U and V as in §4. For $x\in\mathcal{Y}_V - \mathcal{Y}_U$, let $\underline{x}$ denote a variable for ordinary multisorted logic over L_S, of the sort $F(\mathrm{K}_x)$; the mapping $x \mapsto \underline{x}$ is 1-1. For any $A\in S$, $a\in M(A)$, let $\underline{(A,\ a)}$, abbreviated as $\underline{a}$, be a variable of sort A; assume that the $\underline{a}$ are different from the $\underline{x}$. With

$$C \underset{\mathrm{def}}{=} \{\underline{x} : x\in\mathcal{Y}_V - \mathcal{Y}_U\} \dot{\cup} \{\underline{a} : A\in S,\ a\in M(A)\},$$

by a C-formula we mean one over L_S whose free variables all belong to C.

For $x\in\mathcal{Y}_U$, $m(\mathrm{a}(x))$ is an element of M, thus $\underline{m(\mathrm{a}(x))}$ belongs to the second term in C. When $x\in\mathcal{Y}_U$, let $\underline{x}$ stand for $\underline{m(\mathrm{a}(x))}$. (Recall the correspondence between the elements of $\mathcal{Y}_V$ and those of V; for any fixed $K\in\boldsymbol{K}$, $d \mapsto y^V_{K,\ d}$ is a bijection

$V(K) \xrightarrow{\cong} \{x\in\mathcal{Y}_V : K_x = K\}$, with inverse $x \mapsto \mathrm{a}(x)$). Now, $\underline{x}\in C$ is defined for all $x\in\mathcal{Y}_V$, and we have $\underline{x}:FK_x$.

We now write down a set Σ of formulas over the language L_S with free variables in the set C. Σ is the union of the following five sets of formulas in classical first order logic:

$$\Sigma_S^{\mathrm{coh}} \tag{13.1}$$

$$\{o(\underline{a}) =_B \underline{b} \ : \ (o{:}A\to B)\in S,\ a\in MA,\ b\in MB,\ (Mo)(a)=b\} \tag{13.2}$$

$$\{(Fp)(\underline{x}) =_{FK_p} \underline{x_{x,p}} \ : \ x\in\mathcal{Y}_V,\ p\in K_x|\boldsymbol{K}\} \tag{13.3}$$

$$\{\underline{\varphi}(\underline{\mathcal{X}}) \ : \ \mathrm{Var}(\varphi)\subset\mathcal{X}\subset\mathcal{Y}_V\ ,\ N\models_W\varphi[\langle qx\rangle_{x\in\mathcal{X}}]\} \tag{13.4}$$

$$\{\neg(\underline{\psi}(\underline{\mathcal{X}})) \ : \ \mathrm{Var}(\psi)\subset\mathcal{X}\subset\mathcal{Y}_V\ ,\ N\not\models_W\psi[\langle qx\rangle_{x\in\mathcal{X}}]\} \tag{13.5}$$

(note that $N\not\models_W\psi[\langle nx\rangle_{x\in\mathcal{X}}]$ is not the same as $N\models_W(\neg\psi)[\langle nx\rangle_{x\in\mathcal{X}}]$!).

Let us understand the free variables in Σ as individual constants. Assume that Σ is consistent (satisfiable); let $(P, \hat{c})_{c\in C}$ be a model of Σ. Then, by (13.1), $P\in\mathrm{Mod}(S)$. By (13.2), $f = \langle f_A\rangle_{A\in S}$ for which $f_A(a) = \hat{\underline{a}}$ $(A\in S,\ a\in MA)$ is a natural transformation $f{:}M\to P$. By (13.3), $r = \langle r_K\rangle_{K\in\boldsymbol{K}}$ for which $r_K(d) = (\underline{y^V_{K,d}})^{\wedge}$ whenever $K\in\boldsymbol{K}$, $d\in VK$ is a natural transformation $r{:}V\to P{\restriction}\boldsymbol{K}$. Since for $c\in U$, $\underline{y^V_{K,c}} = \underline{m(\mathrm{a}(y^V_{K,c}))} = \underline{m(c)}$, we have the left-hand commutativity in (12). Finally, by (13.4) and (13.5), $(V, r, q){:}P \underset{*}{\leftrightarrow} N$ (see (5')). We have verified that the consistency of Σ establishes (11).

Let us prove that Σ is satisfiable. Assume that a finite subset Φ of Σ is not satisfiable. Φ involves a finite number of C-variables. There is a finite context $\mathcal{Y}\subset\mathcal{U}_V$ and a finite set $\mathcal{A}$ of elements $a = (A\in S,\ a\in MA)$ of M such that all formulas in Φ have free variables from $\underline{\mathcal{Y}}\cup\underline{\mathcal{A}}$; $\underline{\mathcal{Y}} = \{\underline{y} : y\in\mathcal{Y}\}$, $\underline{\mathcal{A}} = \{\underline{a} : a\in\mathcal{A}\}$. Let Θ denote the set $\Phi\cap(13.2)$; for all formulas $\theta\in\Theta$, $\mathrm{Var}(\theta)\subset\underline{\mathcal{A}}$. By increasing Φ, we may assume that it is a subset of

$$\Sigma_S^{\mathrm{coh}} \cup \Theta \cup \mathrm{E}'(\underline{\mathcal{Y}}) \cup \{\underline{\varphi_i}(\underline{\mathcal{U}_i}) : i<m\} \cup \{\neg(\underline{\psi_j}(\underline{\mathcal{V}_j})) : j<n\} \tag{14}$$

where $\mathrm{E}'(\underline{\mathcal{Y}})$ is the set whose union is $\mathrm{E}(\underline{\mathcal{Y}})$, each $\underline{\varphi_i}(\underline{\mathcal{U}_i})$ belongs to (13.4), each $\neg(\underline{\psi_j}(\underline{\mathcal{V}_j}))$ belongs to (13.5), and $\mathcal{U}_i \subset \mathcal{Y}$, $\mathcal{V}_j \subset \mathcal{Y}$. Let $\theta = \bigwedge \Theta$.

The inconsistency of (14) is the same as saying that

$$\mathrm{T}_S^{\mathrm{coh}} \vDash \theta \wedge \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{U}_i}) \xRightarrow[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}]{} \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{V}_j}) .$$

By our remarks above (completeness), this is the same as

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \theta \wedge \mathrm{E}(\underline{\mathcal{X}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{U}_i})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{V}_j})] .$$

By (6), this may be rewritten as

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \theta \wedge \mathrm{E}(\underline{\mathcal{Y}}) \wedge \bigwedge_{i<m} \underline{\varphi_i}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \bigvee_{j<n} \underline{\psi_j}(\underline{\mathcal{Y}})] .$$

With $\varphi = \bigwedge_{i<m} \varphi_i$, $\psi = \bigvee_{j<n} \psi_j$, we see that $\underline{\varphi}(\underline{\mathcal{Y}}) \in$ (13.4), $\neg(\underline{\psi}(\underline{\mathcal{Y}})) \in$ (13.5).

Also using (8), (9), we have

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \theta \wedge \mathrm{E}(\underline{\mathcal{Y}}) \wedge \underline{\varphi}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \underline{\psi}(\underline{\mathcal{Y}})]$$

In other words,

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \theta \wedge \mathrm{E}(\underline{\mathcal{Y}})] \wedge [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \underline{\varphi}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \underline{\psi}(\underline{\mathcal{Y}})] ,$$

and as a consequence, using the Heyting implication in $\mathrm{S}([\underline{\mathcal{A}} \cup \underline{\mathcal{Y}}])$,

$$[\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \theta \wedge \mathrm{E}(\underline{\mathcal{Y}})] \leq [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \underline{\varphi}(\underline{\mathcal{Y}})] \longrightarrow [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \underline{\psi}(\underline{\mathcal{Y}})] = [\underline{\mathcal{A}} \cup \underline{\mathcal{Y}} \colon \underline{\varphi}(\underline{\mathcal{Y}}) \rightarrow \underline{\psi}(\underline{\mathcal{Y}})] .$$

By (10), it follows that

$$[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ \theta\wedge\mathrm{E}(\underline{\mathcal{Y}})] \le [\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ (\underline{\varphi\to\psi})(\underline{\mathcal{Y}})]$$

and

$$[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ \theta] \le [\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\ \mathrm{E}(\underline{\mathcal{Y}}) \longrightarrow (\underline{\varphi\to\psi})(\underline{\mathcal{Y}})] \ . \tag{15}$$

Let $\mathcal{X}=\mathcal{Y}\cap\mathcal{Y}_{\mathrm{U}}$. We have that $\underline{\mathcal{X}} = \underline{\mathcal{A}}\cap\mathcal{Y}\subset\underline{\mathcal{A}}$, and $\underline{\mathcal{A}}\cup\underline{\mathcal{Y}} = \underline{\mathcal{A}}\,\dot{\cup}\,(\underline{\mathcal{Y}}-\underline{\mathcal{X}})$. Let $\pi:[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}]\to[\underline{\mathcal{A}}]$ be the projection, let $\tau = \mathrm{E}(\underline{\mathcal{Y}}) \longrightarrow (\underline{\varphi\to\psi})(\underline{\mathcal{Y}})$. As was mentioned above, $\mathrm{Var}(\theta)\subset\underline{\mathcal{A}}$. Using $\pi^{*}\dashv\forall_{\pi}$,

$$\pi^{*}[\underline{\mathcal{A}}:\theta] \le [\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau] \iff [\underline{\mathcal{A}}:\theta] \le \forall_{\pi}[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau] \ .$$

By (15), it follows that $[\underline{\mathcal{A}}:\theta] \le \forall_{\pi}[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau]$. Now, $\forall_{\pi}[\underline{\mathcal{A}}\cup\underline{\mathcal{Y}}:\tau] = [\underline{\mathcal{A}}:\forall(\underline{\mathcal{Y}}-\underline{\mathcal{X}})\tau]$. We conclude

$$[\underline{\mathcal{A}}:\ \theta\wedge\mathrm{E}(\underline{\mathcal{X}})] \le [\underline{\mathcal{A}}:\ \mathrm{E}(\underline{\mathcal{X}})\wedge\forall(\underline{\mathcal{Y}}-\underline{\mathcal{X}}).\mathrm{E}(\underline{\mathcal{Y}}) \longrightarrow (\underline{\varphi\to\psi})(\underline{\mathcal{Y}}).]$$

and by (7) ,

$$[\underline{\mathcal{A}}:\ \theta\wedge\mathrm{E}(\underline{\mathcal{X}})] \le [\underline{\mathcal{A}}:\ (\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\to\psi)}(\underline{\mathcal{X}})] \ . \tag{16}$$

By the definition of $\mathrm{E}(\underline{\mathcal{X}})$, $\mathrm{M}\models\mathrm{E}(\underline{\mathcal{X}})(m(\mathrm{a}(x))/\underline{x})_{x\in\mathcal{X}}$. But, for $x\in\mathcal{X}$, $\underline{x}=\underline{a}$ for $a=m(\mathrm{a}(x))$; thus, $\mathrm{M}\models\mathrm{E}(\underline{\mathcal{X}})[a/\underline{a}]_{a\in\mathcal{A}}$. By $\Theta\subset(13.4)$, $M\models\theta[a/\underline{a}]_{a\in\mathcal{A}}$. By (16), we conclude that $M\models(\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\to\psi)}(\underline{\mathcal{X}})[a/\underline{a}]_{a\in\mathcal{A}}$, that is,

$$M\models(\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\to\psi)}(\underline{\mathcal{X}})(m(\mathrm{a}(x))/\underline{x})_{x\in\mathcal{X}} \ .$$

By $(U, m, n): M \xrightarrow[*]{} N$, we conclude

$$N\models(\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\to\psi)}(\underline{\mathcal{X}})[q(\mathrm{a}(x))/\underline{x}]_{x\in\mathcal{X}}$$

(q extends n). By the choice of φ and ψ,

$$N\models\varphi[q(\mathrm{a}(y))/\underline{y}]_{y\in\mathcal{X}} \ ,$$

$$N \not\models \psi[q(\mathrm{a}(y))/\underline{y}]_{y\in\mathcal{X}} \ .$$

Also,

$$N \models \mathrm{E}(\underline{\mathcal{Y}})[[q(\mathrm{a}(y))/\underline{y}]_{y\in\mathcal{X}} \ .$$

However,

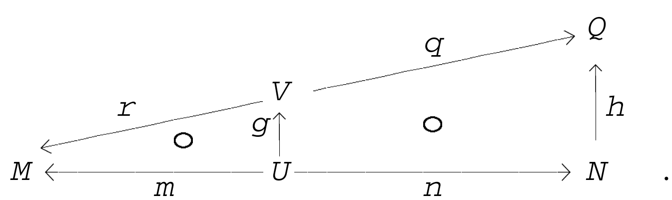
$$[\underline{\mathcal{Y}}\colon\ (\underline{\forall(\mathcal{Y}-\mathcal{X})(\varphi\to\psi)}(\underline{\mathcal{X}}) \wedge \varphi \wedge \mathrm{E}(\underline{\mathcal{Y}})] \le [\underline{\mathcal{Y}}\colon\ \psi] \ .$$

The last five displays contain a direct contradiction.

This completes the proof of (11).

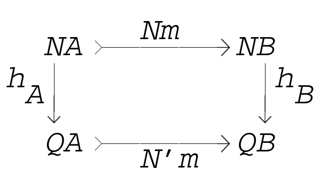
The following is essentially simpler than (11); it is the analog in our context of Lemma 4 of [M4].

(17) Suppose $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$ and $(U, m, n)\colon M \overset{}{\underset{*}{\leftrightarrow}} N$. Then we have $Q\in\mathrm{Mod}(T)$, $(h\colon N\to Q)\in\mathrm{Mod}(T)$, $g\colon U\to V$ and $(V, r, q)\colon M\underset{*}{\leftrightarrow} Q$ such that r is very surjective, h is pure, and



(12)

($h\colon N\to Q$ being *pure* means that the naturality squares



corresponding to monomorphisms $m\in T$ are pullbacks.)

Combining (11) and (17) in an "alternating chain" argument (see the proof of Lemma 2 in [M4]), we obtain

(18) Suppose $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$ and $(U, m, n): M \underset{*}{\longrightarrow} N$. Then there are $M'\in\mathrm{Mod}(S)$, $N'\in\mathrm{Mod}(T)$, $g: U\to U'$, $f: M\to M'$, $h: N\to N'$ and $(U', m', n'): M' \underset{\#}{\longleftrightarrow} N'$ (in particular, m' and n' are very surjective) such that h is pure, and

$$\begin{array}{ccccc} M' & \xleftarrow{\;m'\;} & U' & \xrightarrow{\;n'\;} & N' \\ {\scriptstyle f}\uparrow & \circ & \uparrow{\scriptstyle g} & \circ & \uparrow{\scriptstyle h} \\ M & \xleftarrow[\;m\;]{} & U & \xrightarrow[\;n\;]{} & N \end{array} \quad . \tag{18'}$$

(Observe the asymmetry; $(U, m, n): M \underset{*}{\longrightarrow} N$, and not the other way around; h, but not f, is required to be pure.)

Let us prove the assertion, contained in (5), that (4) has the interpolation property. Let σ and τ be as in the interpolation property, assume the hypotheses, and also that the conclusion fails. That is,

(19) $(\bar{\alpha}_0)^*_{[\mathcal{X}]}(\bar{I}_0\sigma) \leq_{\bar{A}[\mathcal{X}]} (\bar{\alpha}_1)^*_{[\mathcal{X}]}(\bar{I}_1\tau)$;

however,

(20) there is no $\mathbf{L}$-formula θ (of FOLDS) such that $\sigma \leq_{F[\mathcal{X}]} F[\mathcal{X}:\theta]$ and $G[\mathcal{X}:\theta] \leq_{G[\mathcal{X}]} \tau$.

I claim that (20) implies that

(21) there are $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$ and $(F_{\mathcal{X}}, \vec{a}, \vec{b}): M \underset{*}{\longrightarrow} N$ such that $M\models_{\mathrm{w}}\sigma[\vec{a}]$ and $N\not\models_{\mathrm{w}}\tau[\vec{b}]$.

Let $x \mapsto \underline{x}$ be a 1-1 map of variables $x \in \mathcal{X}$ into variables over L_T, $\underline{x}: G\mathrm{K}_X$. Let θ range over $\mathcal{L}$-formulas with $\mathrm{Var}(\theta) \subset \mathcal{X}$. $\mathrm{E}'[\underline{\mathcal{X}}]$, $\underline{\theta}(\underline{\mathcal{X}})$ and $\underline{\tau}(\underline{\mathcal{X}})$ were defined before. Consider the set

$$\Sigma_T^{\mathrm{coh}} \cup \mathrm{E}'[\underline{\mathcal{X}}] \cup \{\underline{\theta}(\underline{\mathcal{X}}) : \sigma \leq_{F[\mathcal{X}]} F[\mathcal{X}:\theta]\} \cup \{\neg(\underline{\tau}(\underline{\mathcal{X}}))\} \ . \tag{22}$$

If this were inconsistent, we would easily conclude that there is θ with $G[\mathcal{X}:\theta] \leq_{G[\mathcal{X}]} \tau$, contrary to (20). Let $(N; \bar{x}/\underline{x})_{x \in \mathcal{X}}$ be a model for (19). Next, let $x \mapsto \underset{\sim}{x}$ be a 1-1 map of variables $x \in \mathcal{X}$ into variables over L_S, $\underset{\sim}{x}: F\mathrm{K}_X$, and consider

$$\Sigma_S^{\mathrm{coh}} \cup \mathrm{E}'[\underset{\sim}{\mathcal{X}}] \cup \{\neg(\underline{\theta}(\underset{\sim}{\mathcal{X}})) : (N; \bar{x}/\underline{x})_{x \in \mathcal{X}} \not\models_{\mathrm{w}} \underline{\theta}(\underline{\mathcal{X}})\} \cup \{\underline{\sigma}(\underset{\sim}{\mathcal{X}})\} \ . \tag{23}$$

This is easily seen to be consistent by the fact that $(N; \bar{x}/\underline{x})_{x \in \mathcal{X}}$ is a model of (22). Now, if $(M; \tilde{x}/\underset{\sim}{x})_{x \in \mathcal{X}}$ is a model of (23), then with $\vec{a} = \langle \tilde{x} \rangle_{x \in \mathcal{X}}$, $\vec{b} = \langle \bar{x} \rangle_{x \in \mathcal{X}}$ we have (21).

Now, apply (18) to $(F_{\mathcal{X}}, \vec{a}, \vec{b}) : M \xrightarrow[*]{} N$ as $(U, m, n) : M \xrightarrow[*]{} N$; we obtain that

(24) there are $M' \in \mathrm{Mod}(S)$, $N' \in \mathrm{Mod}(T)$ and $(V, m, n) : M \underset{\#}{\leftrightarrow} N$ such that $M \models_{\mathrm{w}} \sigma[\vec{a}]$ and $N \not\models_{\mathrm{w}} \tau[\vec{b}]$.

(Indeed, h being pure ensures that $N \not\models_{\mathrm{w}} \tau[\vec{b}]$.) On the other hand, by (24) and the universal property of (4), there is $P: S +_{\#} T \to \mathrm{Set}$ such that $P\bar{I}_0 = M$, $P\bar{I}_1 = N$ and $P(\bar{A}, \bar{\alpha}_0, \bar{\alpha}_1) = (V, m, n)$. Applying these to (19), we get $m^*_{[\mathcal{X}]}(M\sigma) \leq_{V[\mathcal{X}]} n^*_{[\mathcal{X}]}(N\tau)$, which contradicts the conjunction of $M \models_{\mathrm{w}} \sigma[\vec{a}]$ and $N \not\models_{\mathrm{w}} \tau[\vec{b}]$.

It remains to prove the other assertion of (4), namely that the comparison J is conservative.

For any small coherent category R, we have the evaluation functor $e: R \to \mathrm{Set}^{\mathrm{Mod}(R)}$, a conservative coherent functor, and if R is Heyting, e is Heyting (Kripke-Joyal theorem; see [M4]).

We show, in analogy to Proposition 7 of [M4], that

(25) For $R=S+_{\#}T$, the composites $T\xrightarrow{\bar{I}_1}R\xrightarrow{e}\mathrm{Set}^{\mathrm{Mod}(R)}$, $S\xrightarrow{\bar{I}_0}R\xrightarrow{e}\mathrm{Set}^{\mathrm{Mod}(R)}$ are Heyting.

The argument is similar to that in *loc.cit.* We deal with the first composite; the second is symmetric. Upon an analysis similar to that in *loc.cit.*, we see that what we need is this:

given $k:A\to B$ in T, $X\in S(A)$, $M\in\mathrm{Mod}(S)$, $N\in\mathrm{Mod}(T)$, $u=(U,m,n):M\underset{\#}{\leftrightarrow}N$, $y\in NB$ such that $y\notin N(\forall_k X)$,

there are $M'\in\mathrm{Mod}(S)$, $N'\in\mathrm{Mod}(T)$, $u'=(U',m',n):M'\underset{\#}{\leftrightarrow}N'$ and $(f:M\to M', h:N\to N', g:U\to U'):(M,N,u)\to(M',N',u')$, an arrow in $C^{\#}_{\mathrm{Set}}$, and $x\in N'(A)-N'(X)$ such that $h_B(y)=(N'k)(x)$.

As in *loc.cit.*, we have $N^*\in\mathrm{Mod}(T)$, $h^*:N\to N^*$, $x^*\in N^*(A)-N^*(X)$ such that $h^*_B(y)=(N^*k)(x^*)$. We build a commutative diagram

$$\begin{array}{ccccc}
M' & \xleftarrow{m'} & U' & \xrightarrow{n'} & N' \\
{\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h'} \\
M & \xleftarrow[m]{} & U & \xrightarrow[(h^*\restriction\boldsymbol{K})\circ n]{} & N^* \\
{\scriptstyle 1_M}\uparrow & & {\scriptstyle 1_U}\uparrow & & \uparrow{\scriptstyle h^*} \\
M & \xleftarrow[m]{} & U & \xrightarrow[n]{} & N
\end{array}.$$

The lower half is already constructed. The important remark is that $(U,m,n):M\underset{*}{\leftrightarrow}N$ implies that $(U,m,(h^*\restriction\boldsymbol{K})\circ n):M\underset{*}{\longrightarrow}N$. Then, by (18), we have the rest such that, in addition, $(U',m',n):M'\underset{\#}{\leftrightarrow}N'$ and h' is pure. Taking the vertical composites, in particular $h=h'\circ h^*$, and $x=(h')_A(x^*)$, noting the purity of h^*, we have what we want.

Having (25), the proof of the conservativeness of $\mathcal{J}$ is as in *loc.cit.*

This completes the proof of (5) and (1).

The results proved may be applied to characterizations of formulas invariant under equivalence of categories, of diagrams of categories, and of bicategories, in category theory done in intuitionistic set-theory. However, the condition of being invariant under equivalence cannot, in most cases, be stated by using the traditional concept of equivalence. Note that in the proofs of 6.(5), 6.(23), 7.(5), one direction (passing from an $\mathbf{L}$-equivalence to a categorical equivalence) uses the Axiom of Choice, not available in intuitionistic set-theory. [M2] introduces "ana"-versions of certain concepts, among others functors of categories and functors of bicategories, that can be used in this context. The condition of invariance under categorical equivalence has to be strengthened, in general, to invariance under categorical anaequivalence, to have the characterizations analogous to the ones we proved for classical logic.

Let us note that statement (5), being in essence of a syntactical (arithmetical) nature, can be proved constructively, in intuitionistic set theory, by a general transfer result of H. Friedman [Fr]; thus, (5) is available when doing category theory intuitionistically. However, to be able to apply (5), the assumption of invariance under equivalence has to be available in the "provable" sense.

In the case of equivalence of categories, essentially because now there is no need to pass to a notion of "anacategory", we do have the direct analog of 6.(3) for intuitionistic logic. In particular:

(25') Let $\varphi(\mathcal{X})$ be a first-order formula on a finite diagram $\mathcal{X}$ of objects and arrows in the language of categories. Suppose that it is provable in intuitionistic set-theory that the property of $\varphi(\mathcal{X})$ being true is preserved and reflected along equivalence functors. Then there is a formula $\theta(\mathcal{X})$ in FOLDS over $\mathbf{L}_{\mathtt{cat}}$ such that $\forall\mathcal{X}(\varphi\longleftrightarrow\theta^{*})$ is provable in intuitionistic predicate calculus from the axioms of category (here, θ^* is the usual translate of θ into ordinary multisorted logic, given in §1).

In the rest of this Appendix, we discuss (simple) Craig interpolation and Beth definability for FOLDS.

For specificity, we consider FOLDS in the sense of classical FOLDS with (restricted) equality; theories *etc.* below are to be understood accordingly.

First, let us put ourselves in the context of Appendix A. Suppose $\boldsymbol{L}_1$ is a vocabulary. A *subvocabulary* of $\boldsymbol{L}_1$ is a subset $\boldsymbol{L}$ of $\boldsymbol{L}_1$ which itself is a vocabulary. Note that the set-theoretical intersection and union of any number of vocabularies are always again vocabularies.

In terms of the terminology of §1, instead of the above notions, we would use the following. Let $\boldsymbol{L}$, $\boldsymbol{L}_1$ DSV's, $i:\boldsymbol{L}\to\boldsymbol{L}_1$ a functor. I call i an *inclusion of* DSV's if it is (a) 1-1 on objects, (b) for any object R of the category $\boldsymbol{L}$, $R\in\mathrm{Rel}(\boldsymbol{L})$ iff $iR\in\mathrm{Rel}(\boldsymbol{L}_1)$, and (c) for every $A\in\boldsymbol{L}$, i induces a bijection $A|\boldsymbol{L}\to iA|\boldsymbol{L}_1$. Obviously, i preserves levels. A *sub-DSV* $\boldsymbol{L}$ of $\boldsymbol{L}_1$ is given by an inclusion $i:\boldsymbol{L}\to\boldsymbol{L}_1$ of DSV's for which i acts as the identity (i is a "real" inclusion). If we have inclusions $i_1:\boldsymbol{L}\to\boldsymbol{L}_1$, $i_2:\boldsymbol{L}\to\boldsymbol{L}_2$, we may consider the pushout $\boldsymbol{L}_1+_{\boldsymbol{L}}\boldsymbol{L}_2$; as a category, it is a pushout in the ordinary sense; the relations of $\boldsymbol{L}_1+_{\boldsymbol{L}}\boldsymbol{L}_2$ are defined to be the images of those of $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$; clearly, the coprojections $\boldsymbol{L}_1\to\boldsymbol{L}_1+_{\boldsymbol{L}}\boldsymbol{L}_2$, $\boldsymbol{L}_2\to\boldsymbol{L}_1+_{\boldsymbol{L}}\boldsymbol{L}_2$ are inclusions too.

Let us use the terminology of Appendix A. Suppose that T_1 is a theory in FOLDS over $\boldsymbol{L}_1$, and $\boldsymbol{L}\subset\boldsymbol{L}_1$. Then $T_1\restriction\boldsymbol{L}$ denotes the theory $(\boldsymbol{L},\mathrm{Cn}_{\boldsymbol{L}}(T_1))$, where $\mathrm{Cn}_{\boldsymbol{L}}(T_1)$ is the set of $\boldsymbol{L}$-consequences (in classical FOLDS) of T_1. (A small point to make here is that an $\boldsymbol{L}$-formula is not necessarily an $\boldsymbol{L}_1$-formula, despite the fact that $\boldsymbol{L}\subset\boldsymbol{L}_1$. The reason is that a kind K in $\boldsymbol{L}$ may be maximal in $\boldsymbol{L}$, but not maximal in $\boldsymbol{L}_1$, in which case equality on K is allowed in FOLDS over $\boldsymbol{L}$, but not in FOLDS over $\boldsymbol{L}_1$. The definition of $\mathrm{Cn}_{\boldsymbol{L}}(T_1)$ is that it is the set of all $\boldsymbol{L}$-sentences *which are also* $\boldsymbol{L}_1$*-sentences*, and which are consequences of T_1.) If T_i is a theory over $\boldsymbol{L}_i$ (i=1,2), then $T_1\cup T_2$ is the theory over $\boldsymbol{L}_1\cup\boldsymbol{L}_2$ for which $\Sigma_{T_1\cup T_2}=\Sigma_{T_1}\cup\Sigma_{T_2}$. When two theories S_1 and S_2 are over the same language $\boldsymbol{L}$, then $S_1\cup S_2$ is also over $\boldsymbol{L}$.

In the §1 terminology, when T_i is a theory over $\mathbf{L}_i$ (i=1,2), we can define the "pushout" theory $T_1 +_{\mathbf{L}} T_2$ in the obvious way.

We will revert to the Appendix-A terminology.

Craig Interpolation for classical FOLDS. Suppose $\mathbf{L}_1$, $\mathbf{L}_2$ are vocabularies (for FOLDS), $\mathbf{L} = \mathbf{L}_1 \cap \mathbf{L}_2$, T_i is a theory over $\mathbf{L}_i$ $(i=1, 2)$. Then $T_1 \cup T_2$ is consistent if and only if $(T_1 \restriction \mathbf{L}) \cup (T_2 \restriction \mathbf{L})$ is consistent.

Of course, only the "if" part requires proof.

Let us illustrate the meaning of the above statement of the Craig interpolation theorem for FOLDS.

Suppose σ_i is a sentence over $\mathbf{L}_i$ $(i=1, 2)$, and $\sigma_1 \models \sigma_2$. Consider T_1 over $\mathbf{L}_1$ whose single axiom is σ_1, and T_2 over $\mathbf{L}_2$ whose single axiom is $\neg\sigma_2$. Then, $T_1 \cup T_2$ is inconsistent; hence, so is $(T_1 \restriction \mathbf{L}) \cup (T_2 \restriction \mathbf{L})$. This means that there are sentences θ_1, θ_2 over $\mathbf{L}$ such that $\sigma_1 \models \theta_1$, $\neg\sigma_2 \models \theta_2$ and $\{\theta_1, \theta_2\}$ is inconsistent; but then $\sigma_1 \models \theta_1$ and $\theta_1 \models \sigma_2$; we have the usual form of interpolation.

There is a generalization of the above statement of interpolation, obtained by allowing individual constants in the theories. A *vocabulary* $\mathbf{L}$ *with individual constants* is a set of the form $\mathbf{L} = \mathbf{L}_0 \cup \mathcal{C}$, where $\mathbf{L}_0$ is a vocabulary, and $\mathcal{C}$ is a (not necessarily finite) context of variables (individual constants) such that for $c \in \mathcal{C}$, $K_c \in \mathbf{L}_0$. Intersection and union of vocabularies with individual constants is again such. An $\mathbf{L}$-*sentence* is an $\mathbf{L}_0$-formula with all free variables in $\mathcal{C}$. A *structure* M for $\mathbf{L}$ is one, say M_0, for $\mathbf{L}_0$, together with an interpretation of the $\mathcal{C}$-symbols: some $\langle a_c \rangle_{c \in \mathcal{C}} \in M_0[\mathcal{C}]$. For an $\mathbf{L}$-sentence φ, $M \models \varphi \underset{\text{def}}{\Longleftrightarrow} M_0 \models \varphi[\langle a_c \rangle_{c \in \mathcal{C}}]$. A *theory over* $\mathbf{L}$ is given by any set of $\mathbf{L}$-sentences; a model of the theory is an $\mathbf{L}$-structure satisfying all the axioms. Now, all the terms in the above statement of the Craig interpolation theorem have natural meanings when $\mathbf{L}_1$, $\mathbf{L}_2$ are vocabularies with individual constants; the theorem remains correct in the generalized form.

In the well-known manner, the Beth definability theorem can be deduced from Craig interpolation, by using individual constants. We obtain

Beth definability theorem for FOLDS. Suppose T is a theory in FOLDS, $\mathbf{L}\subset\mathbf{L}_T$, $\mathcal{X}$ is a finite context for $\mathbf{L}$, and φ is an $\mathbf{L}_T$-formula with $\mathrm{Var}(\varphi)\subset\mathcal{X}$. Suppose that for any two models M_1, M_2 of T, if $M_1\restriction\mathbf{L}=M_2\restriction\mathbf{L}$, then $M_1[\mathcal{X}:\varphi]=M_2[\mathcal{X}:\varphi]$. Then there is an $\mathbf{L}$-formula θ with $\mathrm{Var}(\theta)\subset\mathcal{X}$ such that $M[\mathcal{X}:\varphi]=M[\mathcal{X}:\theta]$ for all models M of T.

For the proof, make two copies $\mathbf{L}_1$, $\mathbf{L}_2$ of the vocabulary $\mathbf{L}_T$, by renaming all kinds and relations $A\in\mathbf{L}_T-\mathbf{L}$ in two distinct ways as A_1 and A_2, and by putting $\mathbf{L}_i=\mathbf{L}\cup\{A_i:A\in\mathbf{L}_T-\mathbf{L}\}$; $\mathbf{L}_1\cap\mathbf{L}_2=\mathbf{L}$. For any $\mathbf{L}\cup\{\mathcal{X}\}$-sentence ψ, we have the $\mathbf{L}_i\cup\{\mathcal{X}\}$-sentence ψ_i, with the same free variables (in $\mathcal{X}$), obtained by the appropriate renaming. Applied to all members of Σ_T, this gives Σ_i, a set of $\mathbf{L}_i$-sentences. Consider the theories $T_1=(\mathbf{L}_1\cup\mathcal{X},\Sigma_1\cup\{\varphi_1\})$, $T_2=(\mathbf{L}_2\cup\mathcal{X},\Sigma_2\cup\{\neg\varphi_2\})$ over vocabularies $\mathbf{L}_1\cup\mathcal{X}$, $\mathbf{L}_2\cup\mathcal{X}$ with individual constants. Craig interpolation applied for T_1 and T_2 gives the desired conclusion.

We make some preparations for the proof of the Craig interpolation theorem.

Recall our definition of saturation in §5. We make some modifications on it.

Let us fix the DSV $\mathbf{L}$; $\mathbf{K}$ is its category of kinds. First of all, in contrast to §5, we now want to deal with logic with equality; formulas now may have equality. The definitions up to " $\mathcal{Y}$-$\mathbf{L}$-saturated " remain the same, except for the change in what counts as a formula. Consider a context $\mathcal{Y}$, and a $\mathcal{Y}$-set Φ of formulas; all formulas in Φ have variables in the context $\mathcal{Y}\dot{\cup}\{x\}$. Let us say that Φ is *low* if K_x is *low*, that is, it is not a maximal element of $\mathbf{K}$. This is the same as to say that no equality predicate is allowed on K_x.

The $\mathbf{L}$-structure M is said to be *strictly* $\mathcal{Y}$-$\mathbf{L}$-*saturated* if for every $\vec{a}\in M[\mathcal{Y}]$ and every $\mathcal{Y}$-set Φ, if Φ is finitely satisfiable in $(M,\vec{a})$, then (1) Φ is satisfiable in $(M,\vec{a})$, and (2) if Φ is a low set, then Φ is satisfiable by an element a for which $a\neq a_y$ for all $y\in\mathcal{Y}$; here, $\vec{a}=\langle a_y\rangle_{y\in\mathcal{Y}}$. We say that M is *strictly* κ-$\mathbf{L}$-*saturated* if it is strictly $\mathcal{Y}$-saturated for all $\mathcal{Y}$ of cardinality $<\kappa$.

There are two issues: existence and uniqueness; let's deal with existence first. To that end, we give a simple general construction.

Let M, N be $\boldsymbol{L}$-structures. We write $M\prec_{\boldsymbol{L}}N$ if M is a subfunctor of N (note that both M and N are functors $\boldsymbol{L}\to\mathrm{Set}$), and for any $\mathcal{X}$, $\vec{a}\in M[\mathcal{X}]$ ($\subset N[\mathcal{X}]$), $M\models\varphi[\vec{a}]$ iff $N\models\varphi[\vec{a}]$.

(26) Let M be any $\boldsymbol{L}$-structure, K a low kind, $\vec{a}\in M[K]$, and $MK(\vec{a})\neq\emptyset$. We can construct another structure N such that $M\prec_{\boldsymbol{L}}N$ and $MK(\vec{a})\subsetneq NK(\vec{a})$.

For simplicity, we assume that M is separated (the MK are pairwise disjoint). Let $b\in MK(\vec{a})$.

Let $U=M\restriction\boldsymbol{K}$. Construct $V:\boldsymbol{K}\to\mathrm{Set}$ as follows. Say of $x\in|U|$ that it is *above* b if there is $f:K'\to K$ (possibly the identity) such that $(Uf)x=b$. Note that

(27) if $g:K_1\to K_2$, $x_1\in UK_1$ and $x_2=(Ug)(x_1)$, then if x_2 is above b, so is x_1.

Introduce a new element $\bar{x}$ for every x above b, distinct from each other and from the elements of U. Put $VK'=UK'\dot{\cup}\{\bar{x}:\ x\in UK'\text{ above }b\}$. The effect of V on arrows is defined so that U is a subfunctor of V, and by the following determinations. For $g:K_1\to K_2$, $x_1\in UK_1$ above b, let $x_2=(Ug)(x_1)$; $\bar{x}_1\in VK_1\overset{Vg}{\underset{\mathrm{def}}{\longmapsto}}\bar{x}_2$ if x_2 is above b, $\bar{x}_1\in VK_1\overset{Vg}{\underset{\mathrm{def}}{\longmapsto}}x_2$ otherwise. It is easy to see, using (27), that V is a functor, we have the inclusion $i:U\to V$, and we have the retraction $r:V\to U$ for which $\bar{x}\overset{r}{\longmapsto}x$; $ri=1_U$. I claim that r is very surjective. If $\vec{y}=\langle y_p\rangle_{p\in K|\boldsymbol{K}}\in V[K]$, $\vec{y}\overset{r}{\longmapsto}\vec{x}$, $x\in UK(\vec{x})$, then if x is not above b, then no y_p is above b and $x\in VK(\vec{y})$, and of course $x\overset{r}{\longmapsto}x$; but if x is above b, then $\bar{x}\in VK(\vec{y})$, and of course $\bar{x}\overset{r}{\longmapsto}x$.

Returning to M, using the very surjective $r:V\to U$, define $N=r^{*}M$ (see §5). When we regard M and N as structures for $\boldsymbol{L}^{\mathrm{eq}}$, with standard equality for the equality predicates, then still $N=r^{*}M$. This amounts to the following: if K' is a maximal kind, $\vec{y}\in V[K']$, $y_1, y_2\in VK'(\vec{y})$, $\vec{y}\overset{r}{\longmapsto}\vec{x}$, $y_1\overset{r}{\longmapsto}x_1$, $y_2\overset{r}{\longmapsto}x_2$, then $x_1=x_2$ implies $y_1=y_2$. If

$x_1=x_2$, the only way $y_1\neq y_2$ could be the case is that x_1 is above b, $y_1=x_1$ and $y_2=\bar{x}_1$ (or the other way around). However, if so, then since $K'\neq K$ (K is low), we have $p:K'\to K$ proper such that $(Up)x_1=b$, hence, $(Up)\bar{x}_1=\bar{b}$, and, since $b\neq\bar{b}$, $y_1=x_1$, $y_2=\bar{x}_1$ cannot both be in $VK'(\vec{y})$ for the same $\vec{y}$, contradiction.

We have, by 5.(1), that θ_r is elementary (with respect to logic over $\mathbf{L}^{\mathrm{eq}}$ *without* equality; i.e., with respect to logic over $\mathbf{L}$ *with* equality). Combining this with $ri=1_U$, we immediately obtain that $\theta_i:M\to N$ is elementary, that is, $M\prec_{\mathbf{L}}N$ as desired. This proves (26).

The usual proof of the existence of saturated models (see [CK]), using unions of elementary chains, is now easily supplemented by uses of (26) to provide

(28) For any infinite cardinal $\kappa\geq\#\mathbf{L}$ ($\mathbf{L}$ any vocabulary with individual constants), any consistent theory T over $\mathbf{L}$ has a strictly κ^+, $\mathbf{L}$–saturated model of cardinality $\leq 2^{\kappa}$.

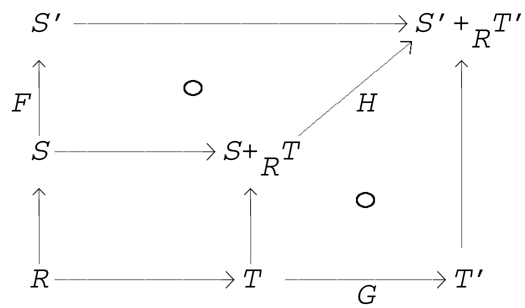
(29) If M, N are strictly κ, $\mathbf{L}$–saturated $\mathbf{L}$-structures, $M\equiv_{\mathbf{L}}N$, both of cardinality $\leq\kappa$, then they are isomorphic.

Proof. Inspecting the proof of 5.(4), we see that we can make both maps m and n bijective. This suffices.

Proof of Craig. Suppose $(T_1\restriction\mathbf{L})\cup(T_2\restriction\mathbf{L})$ is consistent. Let M be a model of it; M is an $\mathbf{L}$-structure. Let Σ be the set of all sentences in FOLDS over $\mathbf{L}$ that are true in M; $T=(\mathbf{L},\Sigma)$. Both $T_1\cup T$ and $T_2\cup T$ are consistent; if not, we would have (say) $\tau\in\Sigma$ such that $T_1\models\neg\tau$; but then, by definition, $\neg\tau\in\Sigma_{T_1\restriction\mathbf{L}}$, hence $M\models\neg\tau$; contradiction to $\tau\in\Sigma$.

Choose $\lambda\geq\#\mathbf{L}_1$, $\geq\#\mathbf{L}_2$ such that $\kappa=\lambda^+=2^{\lambda}$. By (28), let $M_i\models T_i\cup T$ $(i=1,2)$ strictly κ, $\mathbf{L}_i$-saturated, of cardinality $\leq\kappa$. Then $M_i\restriction\mathbf{L}$ is also strictly κ, $\mathbf{L}_i$–saturated, of cardinality $\leq\kappa$. By (29), there is an isomorphism $f:M_1\restriction\mathbf{L}\xrightarrow{\cong}M_2\restriction\mathbf{L}$. There is M_2' and an isomorphism $g: M_2' \xrightarrow{\cong} M_2$ such that $M_2' \restriction \boldsymbol{L} = M_1 \restriction \boldsymbol{L}$ (and $g \restriction \boldsymbol{L} = f$). But then the $\boldsymbol{L}_1 \cup \boldsymbol{L}_2$-structure N for which $N \restriction \boldsymbol{L}_1 = M_1$, $N \restriction \boldsymbol{L}_2 = M_2'$, is a model of $T_1 \cup T_2$.

Finally, let us note that Craig interpolation and Beth definability hold for intuitionistic FOLDS. Looking at the above formulation for classical FOLDS, we are led to the following formulation:



F , G conservative $\Longrightarrow$ H conservative

This is to be understood in a suitable doctrine. Above we proved, in essence, this in the doctrine of $\wedge\vee\neg\exists$-fibrations (see §3) restricted to fibrations obtained from simple base-categories as described in §4 , with arrows restricted to inclusions as defined above. The claim is that the same holds when we switch to $\wedge\vee\rightarrow\exists\forall$-fibrations. The proof is along the lines we presented in the first part of this Appendix.

References

[BD] J. C. Baez and J. Dolan, Higher dimensional algebra and topological quantum field theory. Preprint (University of California, Riverside).

[Ba] J. Barwise, Back and forth through infinitary logic. In: *Studies in Model Theory,* ed. M. Morley, MAA Studies in Mathematics, vol. **8**, The Mathematical Association of America, 1973; pp. 5-34.

[Be] J. Benabou, Introduction to bicategories. In: Lecture Notes in Math., no. **47**, Springer Verlag, 1967; pp. 1-77.

[Ben] P. Benaceraff, What numbers could not be. *Philosophical Review,* **74** (1965), 47-73.

[B] G. Blanc, Equivalence naturelle et formules logiques en theorie des categories. *Archiv math. Logik* **19** (1978), 131-137.

[C] J. Cartmell, Generalised algebraic theories and contextual categories. *Annals of Pure and Applied Logic* **32** (1986), 209-243.

[CK] C. C. Chang and H. J. Keisler, *Model Theory.* Third edition, North-Holland, 1990.

[Co] M. Coste, Localisation, spectra and sheaf representation. In: *Applications of sheaves,* Lecture Notes in Math., no. **753**, Springer Verlag, 1979; pp. 212-238.

[F] P. Freyd, Properties invariant within equivalence types of categories. In: *Algebra, Topology and Category Theories,* ed. A. Heller and M. Tierney, Academic Press, New York, 1976; pp. 55-61.

[Fr] H. Friedman, Classically and intuitionistically provable recursive functions. In: *Higher Set Theory,* Lecture Notes in Math., no. **669**, Springer Verlag, 1978; pp. 21-28.

[FS] P. J. Freyd and A. Scedrov, *Categories, Allegories*. North-Holland, 1990.

[G] D. M. Gabbay, Semantic proof of Craig's interpolation theorem for intuitionistic logic and extensions, Part I. In: *Logic Colloquium '69* (R. O. Gandy and C. M. E. Yates, eds.), North-Holland; pp. 391-402.

[GPS] R. Gordon, A. J. Power and R. Street, Coherence for tricategories. Preprint (Temple University); to appear in *Memoirs Amer. Math. Soc.*

[J] T. Jech, Set Theory. Academic Press, New York, 1978.

[Kel] G. M. Kelly, Complete functors in homology I. Chain maps and endomorphisms. *Proc. Cambridge Phil. Soc.* **60** (1964), 721-735.

[K] S. C. Kleene, *Introduction to Metamathematics*. Ninth reprint, North-Holland, 1988.

[L] F. W. Lawvere, More on graphic toposes. *Cah. de Top. et Geom. Diff.* **32** (1991), 5-10.

[L2] F. W. Lawvere, Equality in hyperdoctrines and the comprehension schema an adjoint functor. In: *Proceedings of the New York Symposium on Applications of Categorical Logic* (ed. A. Heller), Amer. Math. Soc., Providence, 1970.

[M L] S. Mac Lane, *Categories for the Working Mathematician.* Springer-Verlag, 1971.

[M1] M. Makkai, Generalized sketches as a framework for completeness theorems. To appear in *J. Pure and Applied Algebra.*

[M2] M. Makkai, Avoiding the axiom of choice in general category theory. To appear in *J. Pure and Applied Algebra.*

[M3] M. Makkai, The fibrational formulation of intuitionistic predicate logic I: Completeness according to Godel, Kripke and Lauchli. *Notre Dame Journal of Formal Logic* **34** (1993), 334-377 (Part 1) and 471-498 (Part 2).

[M4] M. Makkai, On Gabbay's proof of the Craig interpolation theorem for intuitionistic predicate logic. To appear *in Notre Dame Journal of Formal Logic.*

[M5] M. Makkai, Ultraproducts and categorical logic. In: *Methods in Mathematical Logic,* ed. C. A. DiPrisco, Lecture Notes in Math., no. **1130**, Springer-Verlag, 1985; pp. 222-309.

[MR1] M. Makkai and G. E. Reyes, *First Order Categorical Logic*. Lecture Notes in Mathematics, no. **611**, Springer-Verlag, 1977.

[MR2] M. Makkai and G. E. Reyes, Completeness theorems for intuitionistic and modal logic in a categorical setting. *Annals of Pure and Applied Logic* 72 (1995), 25-101.

[MP] M. Makkai and Robert Pare, *Accessible Categories.* Contemporary Mathematics, v. **104**. American Mathematical Society, 1989.

[M-L] P. Martin-Löf, An Intutionistic Theory of Types. In: *Proc. Bristol Logic Colloquium,* North-Holland, 1973.

[P] A. Preller, A language for category theory in which natural equivalence implies elementary equivalence of models. *Zeitschrift f. math. Logik und Grundlagen d. Math.* **31** (1985), 227-234.

[S] R. Street, Fibrations in bicategories. *Cah. de Top. et Geom. Diff.* **21** (1980), 111-160.