§5. The effects of weak versions of the axiom of choice.

In the previous parts of the paper, we left open whether the bicategory $\mathtt{AnaCat}$ is Cartesian closed. In this section, we show that a very weak version of the AC, one that is consistent with the negations of most of the usual special cases of the AC, is sufficient to ensure that the said conclusion holds.

This section is somewhat incomplete; since the first version of this paper was written, further, and partly better, results have been found, in a collaboration of Robert Paré and the author; it is planned that they will be described in [M/P]. On the other hand, it incorporates substantial improvements that were kindly communicated to me by the Referee.

In this section, we sometimes use classical logic throughout; the marking (CL) indicates that the result in question depends on classical logic (the principle of excluded middle).

For sets A, B, $A \cong B$ abbreviates that there is a bijection $A \xrightarrow{\cong} B$. I propose the following axiom of class-set theory.

Small Cardinality Selection Axiom (**SCSA**)**.** There is a class-function assigning, to each set A, a set $\|A\|$ and a bijection $\iota_A : A \xrightarrow{\cong} \|A\|$ such that, for each set B, the class $\{\|A\| : A \cong B\}$ is a set.

Of course, under the Global AC (there is a class-function that assigns to each inhabited set a member of that set), and by using classical logic (whose validity is a consequence of the AC, by the well-known argument of R. Diaconescu (see [Jo], 5.23, p. 141)), we have the SCSA in the strong form when $\langle B \rangle = \{\|A\| : A \cong B\}$ is a singleton; now, $\|A\|$ is the usual cardinality of A; the global version of choice is needed for the function $A \mapsto \iota_A$.

As the Referee has pointed out, and as will be explained below, the SCSA is related to A. Blass' Axiom of Small Violations of Choice (SVC) (see [Bl], section 4., p.41). This axiom is as follows.

SVC. There is a set S such that, for every set A, there exists an ordinal α and a function from a subset of $S\times\alpha$ onto A.

The following global version of the SVC was also pointed out by the Referee; although it appeared implicitly in the first version of this paper, as a fact about the universe of sets constructible from a fixed set.

GSVC. There are a set S and a class-function mapping $S\times\mathbf{Ord}$ onto $\mathbf{V}$, the class of all sets.

In [Bl], Blass shows that the SVC is a "very weak form of the axiom of choice". It holds in $L(T)$, the universe of sets constructible over an arbitrary transitive set T ($T\in L(T)$), and it holds in $HOD(T)$, the universe of hereditarily ordinal definable sets over T, for an arbitrary set T (use T as a single additional parameter, in addition to ordinals, in definitions of members of $HOD(T)$). Moreover, it holds in P. Cohen's symmetric models, derived from generic models, used by Cohen to show the independence of the Axiom of Choice. Any of these facts show that SVC is consistent with the negation of any of a certain class of special cases of the AC; such a special case, call it AC^*, may e.g. be the statement that there is a well-ordering of the set of all reals. If we have a model M of set-theory in which AC^* fails, then there is a (transitive) set T "responsible" for this failure (the transitive closure of $\mathbb{R}$ in the example), and $L^{(M)}(T)$, that is, $L(T)$ taken relative to M, will also exhibit the failure of AC^*; but $L^{(M)}(T)$ satisfies the SVC.

On the other hand, Blass shows the independence of SVC from ZF by a forcing argument using a proper class of forcing conditions.

All the mentioned results of Blass seem to have straightforward variants for the GSVC; in the first version of this paper, a detailed proof was given of the fact that $L(T)$, with T a transitive set, satisfies the GSVC.

The next result, the fact that the GSVC implies the SCSA, is due to the Referee. In the first

version of the paper, I had the weaker result that, assuming that the Axiom of Choice holds in $\mathbf{V}$, we have that any standard model of G-B satisfying the GSVC also satisfies the SCSA.

1.(CL) *The GSVC implies the SCSA.*

Proof. Let, by the GVSC, S be a set and $E:S\times\mathbf{Ord}\twoheadrightarrow\mathbf{V}$ a surjection. For any $a\in\mathbf{V}$, let ρ_a be the least ordinal ρ for which $\{s\in S:E(s,\rho)=a\}\neq\emptyset$, and let $j(a)=\{s:s\in S, E(s,\rho_a)=a\}$; clearly, $j(a)\neq\emptyset$.

Let A be arbitrary set. Define $R_A=\{\rho_a:a\in A\}$, and let $\pi_A:R_A\xrightarrow{\cong}\lambda_A$ be the unique order-preserving bijection of R_A (ordered as the ordinals are) onto an ordinal. Let $\|A\|=\{(j(a),\pi_A(\rho_a)):a\in A\}$, and $i_A:A\longrightarrow\|A\|$ be given by $i_A(a)=(j(a),\pi_A(\rho_a))$. Note that $\|A\|\subset S\times\lambda_A$. Clearly, i_A is a surjection. But if $a,b\in A$ and $i_A(a)=i_A(b)$, then $\rho_a=\rho_b$, and picking any $s\in j(a)=j(b)$, we have that $a=E(s,\rho_a)=E(s,\rho_b)=b$; this shows that i_A is also an injection, and thus a bijection, $i_A:A\xrightarrow{\cong}\|A\|$.

Let B be an arbitrary set. The class C of ordinals that can be mapped into $\mathcal{P}(B)$ in a 1-1 way is a set (Hartog's theorem), since the mapping with domain the set $\mathcal{W}$ of all well-orderings $(W,<_W)$ of subsets W of $\mathcal{P}(B)$, assigning to $(W,<_W)$ the order-type of $(W,<_W)$, is a surjection $\mathcal{W}\twoheadrightarrow C$. Thus, there exist ordinals that cannot be mapped 1-1 into $\mathcal{P}(B)$; let α be any, *e.g.* the least, such.

I claim that for any A such that $A\cong B$, we have that $\|A\|\subset S\times\alpha$. Once this is shown, we obviously have that $\langle B\rangle=\{\|A\|:A\cong B\}$ is a set as required.

If $f:B\twoheadrightarrow\alpha$ is a surjection, then the map $\xi\mapsto f^{-1}(\xi):\alpha\longrightarrow\mathcal{P}(B)$ is 1-1; it follows that B cannot be mapped surjectively onto α. But if $g:B\cong A$, then we have

$$B\xrightarrow[\cong]{g}A\xrightarrow{a\mapsto\rho_a}R\xrightarrow[\cong]{\pi_A}\lambda_A,$$

and if also $\lambda_A\geq\alpha$, then ($\alpha\neq\emptyset$) λ_A, and hence also B, can be mapped surjectively onto α. It follows that if $A\cong B$, then $\lambda_A<\alpha$. Therefore, if $A\cong B$, then $\|A\|\subset S\times\lambda_A\subset S\times\alpha$ as desired.

As a consequence of 2. and remarks made earlier, we have, e.g., that

2.$^{(CL)}$ *For an arbitrary transitive set* T, $\mathsf{L}(T)^{\wedge}$ *is a model of* G-B+SCSA . *In fact, the class-function witnessing the* SCSA *can be chosen to be a definable class* (that is, definable with parameters in the structure $(X, \in \restriction X)$, for $X = \mathsf{L}(T)$).

We turn to the effect of SCSA on anafunctors.

3.$^{(SCSA)}$ (*Assume the* SCSA.) *For any small categories* $\boldsymbol{X}$, $\boldsymbol{A}$, *there is a small full subcategory* $\mathsf{A}(\boldsymbol{X},\boldsymbol{A})$ *of* $\mathsf{Sana}(\boldsymbol{X},\boldsymbol{A})$ *such that the inclusion* $\mathsf{A}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathsf{Sana}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of categories.*

Proof. We use the notation of the statement of the SCSA; we write $\langle B \rangle$ for $\{\|A\| : A \cong B\}$. Let Γ be the *set* of all sets $\mathcal{G}$ for which there are a subset S of $\mathsf{Ob}(\boldsymbol{X}) \times \mathsf{Ob}(\boldsymbol{A})$ and a function $\Phi \in \prod_{(X, A) \in S} \langle \mathsf{Iso}(A, A) \rangle$ ($\mathsf{Iso}(A, A)$ is the set of all isomorphisms $A \xrightarrow{\cong} A$ in $\boldsymbol{A}$) such that $\mathcal{G}$ is of the form

$$\mathcal{G} = \bigsqcup_{(X, A) \in S} \Phi(X, A) \quad \bar{\bar{\mathsf{def}}} \quad \{((X, A), u) : u \in \Phi(X, A)\} ;$$

indeed, it is clear that Γ is a set. The class (in fact, set) of objects of $\mathsf{A}(\boldsymbol{X},\boldsymbol{A})$ is defined to be the set of sanafunctors (saturated anafunctors) $G : \boldsymbol{X} \longrightarrow \boldsymbol{A}$ such that $|G|$, the set of specifications of G, belongs to Γ. Since Γ is a set, and $\boldsymbol{X}$, $\boldsymbol{A}$ are small, it is clear that the class of sanafuntors described is a set; $\mathsf{A}(\boldsymbol{X},\boldsymbol{A})$, a full subcategory of $\mathsf{Sana}(\boldsymbol{X},\boldsymbol{A})$, is small.

To show that the inclusion $\mathsf{A}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathsf{Sana}(\boldsymbol{X},\boldsymbol{A})$ is an equivalence, we exhibit, for any $F \in \mathsf{Sana}(\boldsymbol{X},\boldsymbol{A})$, a sanafunctor $[\![F]\!] \in \mathsf{A}(\boldsymbol{X},\boldsymbol{A})$ and an isomorphism $\iota_F : F \xrightarrow{\cong} [\![F]\!]$. Let F be given. Remember that, since F is saturated, $|F|(X, A) \cong \mathsf{Iso}(A, A)$ and hence, $\| |F|(X, A) \| \in \langle \mathsf{Iso}(A, A) \rangle$ whenever $X \in \boldsymbol{X}$, $A \in \boldsymbol{A}$ and $|F|(X, A)$ is inhabited (see §1., (2')). We put

$$S = \{(X, A) \in \mathrm{Ob}(\boldsymbol{X}) \times \mathrm{Ob}(\boldsymbol{A}) : \ |F|\,(X, A) \ \text{ is inhabited}\},$$

$$|\llbracket F \rrbracket| = \bigsqcup_{(X, A) \in S} \big\| \, |F|\,(X, A) \big\| \ ;$$

this ensures that $|\llbracket F \rrbracket| \in \Gamma$, and thus, once $\llbracket F \rrbracket$ is fully defined, that $\llbracket F \rrbracket \in \mathrm{A}(\boldsymbol{X}, \boldsymbol{A})$. Note that we have

$$|\llbracket F \rrbracket| = \bigsqcup_{X \in \boldsymbol{X},\, A \in \boldsymbol{A}} \big\| \, |F|\,(X, A) \big\| \ ; \tag{1}$$

this holds because, in general for any family $\langle C_i : i \in I \rangle$ of sets,

$$\bigsqcup_{i \in I} C_i = \bigsqcup_{i \in I} \{C_i : \ i \in I, \ C_i \text{ is inhabited}\} \ .$$

Continuing with the definition of $\llbracket F \rrbracket$, we put, for $(X, A) \in S$, $u \in |F|\,(X, A)$,

$$\sigma_{\llbracket F \rrbracket}((X, A), u) = X \ , \quad \tau_{\llbracket F \rrbracket}((X, A), u) = A \ .$$

It follows by (1) that

$$|\llbracket F \rrbracket|\,(X, A) = \{((X, A), u) : \ u \in \big\| \, |F|\,(X, A) \big\|\}$$

for any $X \in \mathrm{Ob}(\boldsymbol{X})$, $A \in \mathrm{Ob}(\boldsymbol{A})$.

We define the natural transformation ι_F via a renaming transformation $\bar{\iota}_F : F \xrightarrow{\equiv} \llbracket F \rrbracket$; the mapping

$$\bar{\iota}_F[X, A] \ : \ |F|\,(X, A) \xrightarrow{\cong} |\llbracket F \rrbracket|(X, A)$$

is defined by

$$\bar{\iota}_F[X, A](v) = ((X, A), \ \iota_{|F|\,(X, A)}(v)) \ ,$$

with $\iota_{|F|\,(X, A)} : |F|\,(X, A) \longrightarrow \big\| \, |F|\,(X, A) \big\|$ being given by the SCSA. The effect of $\llbracket F \rrbracket$ on morphisms is thereby determined: we put

$$[\![F]\!]_{\bar{s},\,\bar{t}}(f) = F_{s,\,t}(f)$$

whenever $f : X \longrightarrow Y$, $s \in |F|\,X$, $t \in |F|\,Y$, $\bar{s} = \bar{\iota}_F[X, A](s)$, $\bar{t} = \bar{\iota}_F[Y, A](t)$. It is clear that the sanafunctor $[\![F]\!]$ and the natural isomorphism $\iota_F : F \xrightarrow{\cong} [\![F]\!]$ are thus well-defined; and as we said above, it follows that $[\![F]\!] \in \mathsf{A}(\boldsymbol{X}, \boldsymbol{A})$.

4.(SCSA) *For any small categories* $\boldsymbol{X}$, $\boldsymbol{A}$, *there is a small full subcategory* $\mathsf{A}(\boldsymbol{X}, \boldsymbol{A})$ *of* $\mathsf{Ana}(\boldsymbol{X}, \boldsymbol{A})$ *such that the inclusion* $\mathsf{A}(\boldsymbol{X}, \boldsymbol{A}) \longrightarrow \mathsf{Ana}(\boldsymbol{X}, \boldsymbol{A})$ *is an equivalence of categories.*

Proof. Combine 3. with 1.10.

The main result of this section is that, under the SCSA, AnaCat is Cartesian closed. In a natural way, in any bicategory, we say that A^X is an *exponential* of (X, A), with *evaluation* $e : X \times A^X \longrightarrow A$, if for any object Y, the functor

$$\begin{array}{rcl} (X \times (\,)) \circ e : \mathsf{Hom}(Y, A^X) & \longrightarrow & \mathsf{Hom}(X \times Y, A) \\ Y \xrightarrow{f} A^X & \longmapsto & (X \times Y \xrightarrow{X \times f} X \times A^X \xrightarrow{e} A) \end{array}$$

is an equivalence of categories; a bicategory is *Cartesian closed* if it is Cartesian and has exponentials of all pairs of objects.

5.(SCSA) AnaCat *is Cartesian closed.*

Proof. By 1.18, Anacat is Cartesian, with the product structure computed as in Cat . 1.14. and 4. imply that exponentials exist; in fact, $\mathrm{A}(\boldsymbol{X},\boldsymbol{A})$ will act as an exponential $\boldsymbol{A}^{\boldsymbol{X}}$ in Anacat, with evaluation morphism $\mathrm{e}:\boldsymbol{X}\times\mathrm{A}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \boldsymbol{A}$ the restriction to $\boldsymbol{X}\times\mathrm{A}(\boldsymbol{X},\boldsymbol{A})$ of the map defined in (8), (8') and (9) in §1.

Since $\mathtt{SanaCat}$ is equivalent (as a bicategory) to $\mathtt{AnaCat}$ (1.12'), we have

6.$^{(\mathrm{SCSA})}$ $\mathtt{SanaCat}$ *is Cartesian closed.*

Terminal object, *product* and *exponentiation* in an anabicategory are defined by modifying the definitions for bicategories. In a anabicategory, we say that $A \xleftarrow{\pi} C \xrightarrow{\pi'} B$ is a *product diagram* if, for any object D, the anafunctor

$$(\pi\circ(\text{-}),\pi'\circ(=)) : \mathtt{Hom}(D, C) \longrightarrow \mathtt{Hom}(D, A)\times\mathtt{Hom}(D, B)$$

is an anaequivalence of categories. To emphasize the possibly obvious, here $\pi\circ(\text{-}):\mathtt{Hom}(D, C) \longrightarrow \mathtt{Hom}(D, A)$ is the section of the composition anafunctor

$$\circ_{D, C, A} : \mathtt{Hom}(D, C)\times\mathtt{Hom}(C, A) \longrightarrow \mathtt{Hom}(D, A)$$

at $\pi\in\mathtt{Hom}(C, A)$; or what is the same, the composite of $\circ_{D, C, A}$ with

$$\mathtt{Hom}(D, C)\times\ulcorner\pi\urcorner : \mathtt{Hom}(D, C)\times\mathbf{1} \longrightarrow \mathtt{Hom}(D, C)\times\mathtt{Hom}(C, A)\ .$$

A^X, with $e:X\times A^X \longrightarrow A$ (*evaluation*), is an *exponential* of (X, A) if for any Y, the anafunctor

$$(X\times(\text{-}))\circ e : \mathtt{Hom}(Y, A^X) \longrightarrow \mathtt{Hom}(X\times Y, A)$$

is an anaequivalence of categories.

T is a *terminal object* if, for any A, $\mathtt{Hom}(A, T) \longrightarrow \mathbf{1}$, with $\mathbf{1}$ the one-object, one-arrow category, is an anaequivalence of categories.

An anabicategory $\mathcal{A}$ is *Cartesian* if it has a terminal object, and binary products of arbitrary pairs of objects, and it is *Cartesian closed* if, in addition, it has exponentials. The definitions of the concepts involved are straightforward versions of the corresponding definitions for

bicategories.

7.(SCSA) $\mathtt{SanaCat}^{\#}$ *is Cartesian closed.*

The proof, whose details I leave to the reader, is based on the fact that the identity mapping is an equivalence of anabicategories between $\mathtt{SanaCat}^{\#}$ and $\mathtt{SanaCat}$, the latter understood as an anabicategory, and on the fact that the Cartesian closed character of an anabicategory is invariant under equivalences. Although the notion of "equivalence of anabicategories" has not been explicitly stated, the reader will not find it difficult to complete the proof.

Let me mention that the main result 5. holds with the SCSA replaced by Blass' SVC. The proof involves a weaker version of the SCSA, which is a consequence of the SVC, and suffices for the conclusion of 5. This weaker version of SCSA will be discussed in the context of indexed category theory over a topos in [M/P]. The interest of the said variant of SCSA is heightened by the fact that it holds in all Grothendieck toposes, whereas direct topos-theoretic translates of SVC do not.

To end the paper, we will consider consequences of the Axiom of Regularity. Actually, only a consequence having the form of a weak version of the Axiom of Choice of the said axiom is used; this consequence I call the

Axiom of Hierarchy (**AH**)**.** There is a family $\langle V_\alpha : \alpha \in \mathbf{Ord} \rangle$ of sets indexed by the ordinals whose union $\bigcup_{\alpha \in \mathbf{Ord}} V_\alpha$ is the class of all sets.

As is well-known, the AH is made true by the Axiom of Regularity through the von-Neumann hierarchy $\langle \mathbf{V}_\alpha : \alpha \in \mathbf{Ord} \rangle$ of pure (regular) sets. In what follows, we will assume the AH, and also the validity of classical logic.

The AH gives a metafunction assigning to any class A a subset $\bar{A} \subset A$ such that if A is non-empty, then so is $\bar{A}$; if A is non-empty, $\bar{A} = A \cap V_\alpha$ for α the least α for which $A \cap V_\alpha \neq \emptyset$, $\bar{A} = \emptyset$ otherwise . Combined with the GAC, we have a meta-choice-function,

assigning to every non-empty class an element of it. Therefore, using the proof of 1.11, we can conclude the following strengthening of 1.11:

8.(GAC,AH) *The inclusion* $\mathsf{FUN}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathsf{ANA}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories.*

(This was observed by the Referee, correcting a careless statement I originally made.)

We have

9.(AH,CL) *The inclusion* $\mathsf{ANA}_{\mathsf{WS}}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathsf{ANA}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories. In particular, if* $\boldsymbol{X}$ *is a small category, the inclusion* $\mathsf{Ana}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathsf{ANA}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories.*

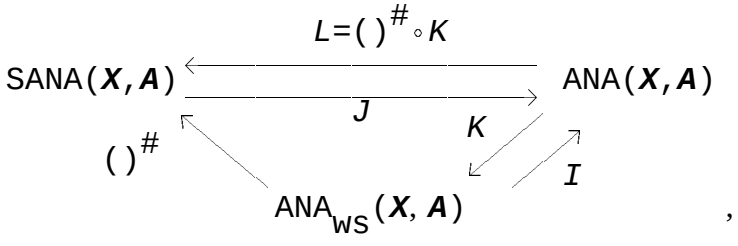
Indeed, given any $F \in \mathsf{ANA}(\boldsymbol{X},\boldsymbol{A})$, define $\bar{F} \in \mathsf{Ana}(\boldsymbol{X},\boldsymbol{A})$ by $|\bar{F}|\, X = (\,|F|\, X)^{-}$ (with the latter $(\,)^{-}$ understood as in $A \mapsto \bar{A}$ above), and otherwise restricting the data of F appropriately; 1.1.(iii) for $\bar{F}$ is ensured by the construction, and all other laws of "anafunctor" are automatically true for $\bar{F}$. We have the natural isomorphism $h : \bar{F} \xrightarrow{\cong} F$ whose components $h_{s,\,s}$ for $s \in |\bar{F}|\,(X, A)$ ($\subset |F|\,(X, A)$) are identities.

We can now generalize, under the AH, 1.10 to obtain

10.(AH,CL) *The inclusion* $\mathsf{SANA}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathsf{ANA}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories, provided* $\boldsymbol{A}$ *is locally small.*

To see this, note that in the proof of 1.10, the construction of $F^{\#}$ is legitimate provided $\boldsymbol{A}$ is locally small, and $F : \boldsymbol{X} \xrightarrow{a} \boldsymbol{A}$ is weakly small; the class $S_{X,\,A}$ introduced there is a set in this case, and we may consider the set of equivalence classes of the equivalence relation $\sim$ on $S_{X,\,A}$ as needed there. (Note, however, that $F^{\#}$ constructed is not necessarily weakly small.)

We obtain the functor $()^{\#}:\mathrm{ANA}_{ws}(\boldsymbol{X},\boldsymbol{A})\to\mathrm{SANA}(\boldsymbol{X},\boldsymbol{A})$ such that for every $F\in\mathrm{ANA}_{ws}(\boldsymbol{X},\boldsymbol{A})$, $F\cong F^{\#}$ with a canonical isomorphism; that is, in



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where I and J are inclusions, and K is the quasi-inverse of I given by 9., we have $J\circ()^{\#}\cong I$. But then

$$J\circ L\cong J\circ()^{\#}\circ K\cong I\circ K\cong \mathrm{Id}\,, \tag{2}$$

and

$$J\circ L\circ J = J\circ()^{\#}\circ K\circ J\cong I\circ K\circ J\cong J = J\circ \mathrm{Id}\,,$$

which, since J is full and faithful, implies that

$$L\circ J\cong \mathrm{Id}\ . \tag{3}$$

(2) and (3) show what we want.

As an immediate consequence of 9. and 2.6, we have

11.(AH,CL) *Suppose that the category* $\boldsymbol{A}$ *has small limits, and* $\boldsymbol{X}$ *is a small category. Then* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *and* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *have specified small limits.*