§4. The anabicategory of saturated anafunctors

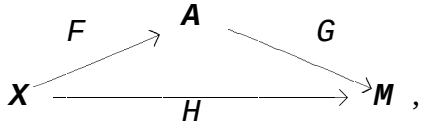
The saturated composition of two saturated anafunctors (or sanafunctors, for short) is obtained by saturating the ordinary composite introduced in §1. Since saturation involves taking quotients of equivalence relations, it is natural to define the composite of sanafunctors up to isomorphism only, by allowing the quotients to be replaced by sets that are bijectively related to them. In this way, we get the meta*ana*bicategory $\mathtt{SANACAT}^{\#}$ of categories and sanafunctors, and the anabicategory $\mathtt{SanaCat}^{\#}$ of small categories and (small) sanafunctors between them. From the point of view of the present paper, $\mathtt{SanaCat}^{\#}$ and $\mathtt{SANACAT}^{\#}$ seem to be the right universes for category theory.

We now assemble the definition of $\mathtt{SanaCat}^{\#}$, the anabicategory of small categories and saturated anafunctors, in explicit terms; the consideration of $\mathtt{SANACAT}^{\#}$ will be left to the reader.

Recall that, for $\boldsymbol{X}$, $\boldsymbol{A}$ small categories, $\mathtt{Sana}(\boldsymbol{X},\boldsymbol{A})$ denotes the category of sanafunctors from $\boldsymbol{X}$ to $\boldsymbol{A}$ and their natural transformations. With $\boldsymbol{X}$, $\boldsymbol{A}$ and $\boldsymbol{M}$ arbitrary small categories, we define the *composition anafunctor*

$$\circ = \circ_{\boldsymbol{X},\boldsymbol{A},\boldsymbol{M}} : \mathtt{Sana}(\boldsymbol{X},\boldsymbol{A}) \times \mathtt{Sana}(\boldsymbol{A},\boldsymbol{M}) \longrightarrow \mathtt{Sana}(\boldsymbol{X},\boldsymbol{M}) \qquad (1)$$

as follows. Recall the criterion for a "bi-anafunctor" stated in §1. after 1.13; according to this, we only have to define, for $F \in \mathtt{Sana}(\boldsymbol{X},\boldsymbol{A})$, $G \in \mathtt{Sana}(\boldsymbol{A},\boldsymbol{M})$ and $H \in \mathtt{Sana}(\boldsymbol{X},\boldsymbol{M})$, that is,



(2)

the specification-sets $|\circ|((F, G), H)$ and the sections

$$(\ )\circ F : \mathtt{Sana}(\boldsymbol{A},\boldsymbol{M}) \longrightarrow \mathtt{Sana}(\boldsymbol{X},\boldsymbol{M}), \qquad (3)$$

$$G\circ(\ ) : \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A}) \longrightarrow \mathrm{Sana}(\boldsymbol{X},\boldsymbol{M}) \qquad (4)$$

appropriately.

For $F\in\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$, $G\in\mathrm{Sana}(\boldsymbol{A},\boldsymbol{M})$ and $H\in\mathrm{Sana}(\boldsymbol{X},\boldsymbol{M})$, $|\circ|\,((F, G), H)$ is defined as the set of all families $\alpha = \langle\alpha_X\rangle_{X\in\mathrm{Ob}(\boldsymbol{X})}$ of maps

$$\alpha_X \in \prod_{(s\in |F|\,(X),\ t\in |G|\,(F_sX))} |H|\,(X, G_tF_sX)$$

$$\alpha_X : (s\in |F|\,(X),\ t\in |G|\,(F_sX)) \longmapsto \alpha_X(s, t)\in |H|\,(X, G_tF_sX)\ . \qquad (5)$$

such that α satisfies the following condition (i).

(i) For all $f:X\to Y$, $s\in |F|\,(X)$, $t\in |G|\,(F_sX)$, $u\in |F|\,(Y)$, $v\in |G|\,(F_uY)$, if $a = \alpha_X(s, t)$, $b = \alpha_Y(u, v)$, we have

$$G_{t,\,v}(F_{s,\,u}(f)) = H_{a,\,b}(f)\ .$$

α as in (5) satisfying (i) is called a *composition specification* (*for* " $H=F\circ G$ ").

Note that, under the notation of (i), $H_a(X) = (G\circ_\alpha F)_a(X) = G_tF_s(X)$ when $\alpha_X(s, t) = a$.

We have that any composition specification α for " $H=F\circ G$ " satisfies the following (ii) and (iii).

(ii) *For any* $X\in\mathrm{Ob}(\boldsymbol{X})$, $M\in\mathrm{Ob}(\boldsymbol{M})$, $a\in |H|\,(X, M)$ *and* $s\in |F|\,X$, *there is* $t\in |G|\,(F_sX)$ *such that* $M = G_tF_sX$ *and* $\alpha_X(s, t) = a$.

Indeed, take any $v\in |G|\,(F_sX)$, and let $b=\alpha_X(s, v)$; since G is saturated, there is a (unique) $t\in |G|\,(F_sX, M)$ such that

$$G_{v,\,t}(1_{F_sX}) = H_{b,\,a}(1_X) : G_vF_sX \longrightarrow M\ .$$

But then for $a' = \alpha_X(s,\,t)$, by (i) we have $H_{b,\,a}(1_X) = H_{b,\,a'}(1_X)$, which, since H is saturated, implies $a' = a$.

(iii) *For all* X, M, s, t, u, v *such that* $G_tF_sX = G_vF_uX = M$, *we have that* $\alpha_X(s,\,t) = \alpha_X(u,\,v)$ *iff* $G_{t,\,v}F_{s,\,u}(1_X) = 1_M$.

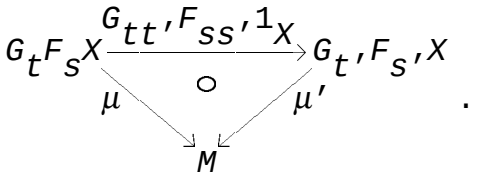
Indeed, by (ii), $G_{t,\,v}F_{s,\,u}(1_X) = H_{ab}(1_X)$ with $a = \alpha_X(s,\,t)$, $b = \alpha_X(u,\,v)$; since H is saturated, $H_{ab}(1_X) = 1_M$ iff $a=b$.

A *partial composition specification for* " $H=G\circ F$ " is a family $\alpha = \langle \alpha_X \rangle_{X\in \mathrm{Ob}(\boldsymbol{X})}$ of functions α_X with domain $\mathrm{dom}(\alpha_X) \subset \bigsqcup_{s\in |F|\,X} |G|\,(F_sX) = \{(s,\,t) : s\in |F|\,(X),\ t\in |G|\,(F_sX)\}$ such that for any $X\in \mathrm{Ob}(\boldsymbol{X})$, there are $s\in |F|\,(X)$, $t\in |G|\,(F_sX)$ with $(s,\,t)\in \mathrm{dom}(\alpha_X)$, and (i) holds for $(s,\,t)\in \mathrm{dom}(\alpha_X)$, $(u,\,v)\in \mathrm{dom}(\alpha_Y)$. Any partial composition specification can be uniquely extended to a complete one for the same relation " $H=G\circ F$ " ; given any $u\in |F|\,(X)$, $v\in |G|\,(F_uX)$, choose $(s,\,t)$ as above, and define $b = \alpha_X(u,\,v)$ so as to satisfy

$$G_{t,\,v}(F_{s,\,u}(1_X)) = H_{a,\,b}(1_X)\ ;$$

a unique such b exists because H is saturated. It is easy to verify that the full specification so defined, also denoted by α , will satisfy (i).

To define the section (3), first of all, we have to show that $|\circ|\,((F,\,G))$ is inhabited (F and G as before). For $H = G\circ_\alpha F$, with suitable α , we take $(G\circ F)^{\#}$; here $G\circ F$ is the composite as anafunctors in the sense of §1, and $(\ )^{\#}$ is the saturation functor. According to the definition of $H = (G\circ F)^{\#}$, a typical element of $|H|\,(X,\,M)$ is an equivalence class $[(s,\,t,\,\mu)]$ with $s\in |F|\,(X)$, $t\in |G|\,(F_sX)$ and $\mu : G_tF_sX \xrightarrow{\cong} M$, where the underlying equivalence relation is defined by

$(s, t, \mu) \sim (s', t', \mu') \iff$ 

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To define the composition specification α, we put $\alpha_X(s, t) = [(s, t, 1_{G_t F_s X})]$ ($s \in |F| X$, $t \in |G| (F_s X)$). With further data as in (i), the definition of $H = (G \circ F)^{\#}$ gives that $H_{[(s, t, 1)], [(u, v, 1)]}(f) = G_{t, v} F_{s, u}(f)$; thus, (i) indeed holds. We have shown that $|\circ| ((F, G))$ is inhabited.

To define the effect of the section (3) on arrows, we take

$$\boldsymbol{X} \xrightarrow{F} \boldsymbol{A} \underset{G'}{\overset{G}{\underset{\downarrow \nu}{\rightrightarrows}}} \boldsymbol{M}$$

and for $\alpha \in |\circ| ((F, G))$, $\alpha' \in |\circ| ((F, G'))$, we define

$$\nu \circ_{\alpha, \alpha', F} : (G \circ_{\alpha} F) \longrightarrow (G' \circ_{\alpha', F}) ; \tag{6}$$

that is, we determine the component

$$(\nu \circ_{\alpha, \alpha', F})_{a, a'} : (G \circ_{\alpha} F)_a X \longrightarrow (G' \circ_{\alpha', F})_{a'} X$$

for any $a \in |G \circ_{\alpha} F| X$, $a' \in |G' \circ_{\alpha', F}| X$. But, by (ii), we can take $s \in |F| X$, $t \in |G| (F_s X)$, $t' \in |G'| (F_s X)$ such that $\alpha_X(s, t) = a$, $\alpha_X(s, t') = a'$; we define

$$(\nu \circ_{\alpha, \alpha', F})_{a, a'} = \nu_{t, t'} : G_t F_s X \longrightarrow G_{t', F_s} X \ . \tag{7}$$

If, instead of s, t, t', we take u, v, v', then by (ii), and the commutativity of

$$\begin{array}{ccc} & G_t F_s X & \xrightarrow{\nu_{t, t'}} G_{t', F_s} X \\ G_{t, \nu}(F_{s, u} 1_X) \Big\downarrow & = 1_{H_a} & 1_{H_{a'}} = \Big\downarrow G_{t', \nu'}(F_{s, u} 1_X) \\ & G_v F_u X & \xrightarrow[\nu_{v, v'}]{} G_{v', F_u} X \end{array}$$

(the naturality of ν), we see that the same value is obtained in (7). We omit the verification that (6) so defined is an nt, and that (3) so defined is an anafunctor.

The definition of (4) is analogous; we get, for

$$\boldsymbol{X} \underset{F'}{\overset{F}{\underset{\downarrow\rho}{\rightrightarrows}}} \boldsymbol{A} \xrightarrow{G} \boldsymbol{M},$$

$$(G\circ_{\alpha,\,\beta}\rho)_{a,\,b} = G_{t,\,u}(\rho_{s,\,s'}) : G_t F_s X \longrightarrow G_u F_{s'} X$$

($\alpha_X(s,\,t) = a$, $\alpha_X(s',\,u) = b$). Finally, condition (ii) in the characterization of "bi"anafunctors is

$$\begin{array}{ccc} G\circ_\alpha F & \xrightarrow{G\circ_{\alpha,\,\beta}\rho} & G\circ_\beta F' \\ {\scriptstyle \nu\circ_{\alpha,\,\alpha'}F}\downarrow & \circ & \downarrow{\scriptstyle \nu\circ_{\beta,\,\beta'}F'} \\ G'\circ_{\alpha'}F & \xrightarrow[G'\circ_{\alpha'\beta'}\rho]{} & G'\circ_{\beta'}F' \end{array},$$

which becomes, in components,

$$\begin{array}{ccc} G_t F_s X & \xrightarrow{G_{tu}\rho_{ss'}} & G_u F'_{s'} X \\ {\scriptstyle \nu_{tt'}}\downarrow & \circ & \downarrow{\scriptstyle \nu_{uu'}} \\ G_{t'} F_s X & \xrightarrow[G_{t'u'}\rho_{ss'}]{} & G_{u'} F'_{s'} X \end{array},$$

which holds by the naturality of ν.

In fact, the composition anafunctor (1) is itself saturated. To see this, let $H = G\circ_\alpha F$ and $h : H \xrightarrow{\cong} K$; by 1.9., h is given by bijections

$$(a \mapsto \bar{a}) : |G\circ_\alpha F|\,(X,\,M) \xrightarrow{\cong} |K|\,(X,\,M)\ .$$

We want to show that there is a unique $\beta \in |\circ|\,((F,\,G),\,K)$ for which $h = 1_{G\circ_{\alpha,\,\beta}1_F}$. For

$X \in \boldsymbol{X}$, $s \in |F|\,X$, $t \in |G|\,(F_s X)$, and $M = G_t F_s X$, let us define $\beta_X(s, t) \overset{\text{def}}{=} \overline{\alpha_X(s, t)}$. We have that β satisfies (i): with $\alpha_X(s, t) = a$, $\alpha_X(u, v) = b$, we have $G_{tv} F_{su}(f) = H_{ab}(f)$ since α is a composition specification; but $H_{ab}(f) = K_{\bar{a}\bar{b}}(f)$ by 1.9.; the desired equality follows.

Let $a \in |G \circ_\alpha F|\,(X, M)$. On the one hand, $h_{a, \bar{a}} = 1_M$ (see 1.9.). On the other hand, choose (s, t) such that $\alpha_X(s, t) = a$; then $\beta_X(s, t) = \bar{a}$, and $(1_G \circ \alpha, \beta 1_F)_{a, \bar{a}} = (1_G \circ \alpha, \beta^F)_{a, \bar{a}} = (1_G)_{t, t} = 1_M$. This shows that $h_{a, \bar{a}} = (1_G \circ \alpha, \beta 1_F)_{a, \bar{a}}$; since we have "enough" pairs of specifications $(a, \bar{a})$ (see 1.7.), it follows that $h = 1_G \circ \alpha, \beta 1_F$. It is also clear that β is uniquely determined.

For any small category $\boldsymbol{X}$, the *identity sanafunctor* $1^{\#}_{\boldsymbol{X}} : \boldsymbol{X} \longrightarrow \boldsymbol{X}$, the saturation of the ordinary identity functor, has $|1^{\#}_{\boldsymbol{X}}|\,(X, X') = \mathrm{Iso}(X, X')$, the set of all isomorphisms from X to X', and, for $f : X \longrightarrow Y$, $i \in |1^{\#}_{\boldsymbol{X}}|\,(X, X')$, $j \in |1^{\#}_{\boldsymbol{X}}|\,(Y, Y')$,

$$(1^{\#}_{\boldsymbol{X}})_{i, j}(f) = j \circ f \circ i^{-1} : X' \longrightarrow Y' .$$

The identity anaobjects in $\mathrm{SanaCat}^{\#}$ are given the ordinary objects $1^{\#}_{\boldsymbol{X}}$.

We may make $\mathrm{SanaCat}^{\#}$ into a saturated anabicategory, by taking the saturation $\ulcorner 1^{\#}_{\boldsymbol{X}} \urcorner^{\#} : \mathbf{1} \xrightarrow{\mathrm{a}} [\boldsymbol{X}, \boldsymbol{X}]$ to be the identity anaobject for $\boldsymbol{X}$; we will not carry out this change.

For $F : \boldsymbol{X} \longrightarrow \boldsymbol{A}$, the component $\lambda_\alpha : 1^{\#}_{\boldsymbol{A}} \circ_\alpha F \xrightarrow{\cong} F$ of the left identity isomorphism $\lambda = \lambda_{\boldsymbol{X}, \boldsymbol{A}} : 1^{\#}_{\boldsymbol{X}} \circ_\alpha (\ ) \xrightarrow{\cong} 1_{[\boldsymbol{X}, \boldsymbol{A}]}$ is given (see 1.9) by the mappings

$$\lambda[X, A] : (a \in |1^{\#}_{\boldsymbol{A}} \circ F|\,(X, A)) \longmapsto (\bar{a} \in |F|\,(X, A)),$$

one for each pair $(X \in \mathrm{Ob}(\boldsymbol{X}), A \in \mathrm{Ob}(\boldsymbol{A}))$, defined thus: if $s \in |F|\,X$, $i \in |1^{\#}_{\boldsymbol{A}}|\,(F_s X)$, $a = \alpha_X(s, i)$, then $\bar{a}$ is determined (through the saturatedness of F) by the conditions

$F_{\bar{a}}X = A$, $F_{s\bar{a}}(1_X) = i$; it is easy to see that this does define $a \mapsto \bar{a}$ uniquely, and the equality (see 1.9) $(1^{\#}_{\boldsymbol{A}} \circ F)_{a,\, b}(f) = F_{\bar{a},\, \bar{b}}(f)$ holds ($f : X \to Y$, $a \in |1^{\#}_{\boldsymbol{A}} \circ F|\,(X)$, $b \in |1^{\#}_{\boldsymbol{A}} \circ F|\,(Y)$), showing that λ_α is properly defined. The naturality of λ has also to be checked. Similarly, we define the right identity isomorphisms.

The associativity isomorphism

$$\alpha = \alpha_{\boldsymbol{X},\boldsymbol{A},\boldsymbol{M},\boldsymbol{P}} : (\circ_{\boldsymbol{X},\boldsymbol{A},\boldsymbol{P}}) \circ (\circ_{\boldsymbol{X},\boldsymbol{A},\boldsymbol{M}} \times [\boldsymbol{M},\boldsymbol{P}]) \xrightarrow{\cong} (\circ_{\boldsymbol{X},\boldsymbol{A},\boldsymbol{P}}) \circ ([\boldsymbol{X},\boldsymbol{A}] \times \circ_{\boldsymbol{A},\boldsymbol{M},\boldsymbol{P}})$$

has components

$$\alpha = \alpha_{\alpha,\, \beta,\, \gamma,\, \delta} : H \circ_\beta (G \circ_\alpha F) \xrightarrow{\cong} (H \circ_\gamma G) \circ_\delta F$$

($F \in \mathtt{Sana}(\boldsymbol{X},\boldsymbol{A})$, $G \in \mathtt{Sana}(\boldsymbol{A},\boldsymbol{M})$, $H \in \mathtt{Sana}(\boldsymbol{M},\boldsymbol{P})$, *etc.*), defined through the renaming transformation $\bar{\alpha}$ given by the maps

$$\bar{\alpha}[X, P] : (a \in |H \circ_\beta (G \circ_\alpha F)|\,(X, P)) \xrightarrow{\cong} (\bar{a} \in |(H \circ_\gamma G) \circ_\delta F|\,(X, P))$$

($X \in \mathtt{Ob}(\boldsymbol{X})$, $P \in \mathtt{Ob}(\boldsymbol{P})$) thus: if $a = \beta_X(\alpha_X(s, t), u)$, then $\bar{a} = \delta_X(s, \gamma_{F_s X}(t, u))$. I leave all verifications, including the associativity and identity coherences, to the reader.

Extending 1.10, we have

$\mathtt{SanaCat}^{\#}$ *and* $\mathtt{AnaCat}$ *are equivalent as anabicategories.*

We have not given the notion of equivalence of anabicategories; the formulation of this notion, and the verification of the last-stated proposition are left to the reader.