§3. Anabicategories

In a two-dimensional category, for a given pair of objects (0-cells), the totality of arrows (1-cells) from one to the other form a category; in short, arrows are objects in categories, and thus, *one should attempt to determine them only up to isomorphism*. This means that, for a given triple of 0-cells A, B, C composition of arrows $A \to B$ with arrows $B \to C$, instead of being a functor as it is in a bicategory, should be an anafunctor, preferably a saturated one.

The definition of *anabicategory* is obtained from that of bicategory by natural modifications, amounting to replacing functors by anafunctors in all places. There is a good abstract way of saying this: an anabicategory is ANACAT-*enriched bicategory*. I will not attempt to make this completely explicit, although I believe this is the right way of looking at the concept. From the description that follows, the reader may get a reasonably good idea of the notion of an $\mathcal{V}$-*enriched bicategory*, for any Cartesian (in fact, for any *monoidal*) bicategory $\mathcal{V}$. The point is that *the more general notion* ($\mathcal{V}$-enriched bicategory) *is easier to grasp than the more concrete one* (ANACAT-enriched bicategory), because there are fewer elements of the abstract situation available for manipulation than there are in the concrete one. This was already fondly pointed out by me in [Ma1] in the context of the notion of 2-category.

An *anabicategory* $\mathcal{A}$ has data 1.(i) to (vi), satisfying conditions (vii) and (viii):

1. **(i)** *Objects* (0-cells).

(ii) For any pair of objects A, B a category $\mathcal{A}(A, B)$, or simply $[A, B]$, of *arrows* (1-cells) $A \longrightarrow B$ (the objects of $[A, B]$), and *2-cells* as α in $A \overset{f}{\underset{g}{\Downarrow \alpha}} B$ (the arrows of the category $[A, B]$).

(iii) For any object A, an *identity* anaobject of $[A, A]$ (an anafunctor $1_A : \mathbf{1} \longrightarrow [A, A]$).

(iv) For any objects A, B, C, an anafunctor

$$\circ_{A,\,B,\,C} : [A,\,B]\times[B,\,C] \longrightarrow [A,\,C]$$

(*composition*).

(v) $\lambda_{A,\,B} : 1_B \circ (\ ) \xrightarrow{\cong} 1_{[A,\,B]}$ and $\rho_{A,\,B} : (\ ) \circ 1_A \xrightarrow{\cong} 1_{[A,\,B]}$ (*left* and *right identity isomorphisms*); here, $1_B \circ (\ )$ is an abbreviated notation for the composite (in ANACAT) of

$$[A,\,B] \xrightarrow{\cong} [A,\,B]\times\mathbf{1} \xrightarrow{[A,\,B]\times 1_B} [A,\,B]\times[B,\,B] \xrightarrow{\circ_{A,\,B,\,B}} [A,\,B]\ ;$$

the natural transformations $\lambda = \lambda_{A,\,B}$, $\rho = \rho_{A,\,B}$ have components

$$\lambda_{a;\,j} : 1_{B,\,j} \circ_a f \xrightarrow{\cong} f\ ,\quad \rho_{b;\,i} : f \circ_b 1_{A,\,i} \xrightarrow{\cong} f$$

$(f : A \to B\ ,\ i \in |1_A|\ ,\ j \in |1_B|\ ,\ a \in |\circ_{A,\,B,\,B}|\,(f,\,1_{B,\,j})\ ,\ b \in |\circ_{A,\,A,\,B}|\,(1_{A,\,i},\,f)\ )$.

(vi) For any objects A, B, C, D, the *associativity isomorphism*, a 2-cell $\alpha_{A,\,B,\,C,\,D}$ between composite arrows as shown in

$$\begin{array}{ccc} [A,\,B]\times[B,\,C]\times[C,\,D] & \xrightarrow{\circ_{A,\,B,\,C}\times[C,\,D]} & [A,\,C]\times[C,\,D] \\ {\scriptstyle [A,\,B]\times\circ_{B,\,C,\,D}}\downarrow & \swarrow\ \alpha_{A,\,B,\,C,\,D}\ \cong & \downarrow{\scriptstyle \circ_{A,\,C,\,D}} \\ [A,\,B]\times[B,\,D] & \xrightarrow[\circ_{A,\,B,\,D}]{} & [A,\,D] \end{array}\quad .$$

The natural transformation $\alpha = \alpha_{A,\,B,\,C,\,D}$ has components

$$\alpha_{a,\,b,\,c,\,d} \ :\ h \circ_b (g \circ_a f) \xrightarrow{\cong} (h \circ_c g) \circ_d f$$

$(\ f : A \to B\ ,\ g : B \to C\ ,\ h : C \to D\ ,\ a \in |\circ_{A,\,B,\,C}|\,(f,\,g)\ ,\ b \in |\circ_{A,\,C,\,D}|\,(g \circ_a f,\,h)\ ,$
$c \in |\circ_{B,\,C,\,D}|\,(g,\,h)\ ,\ d \in |\circ_{A,\,B,\,D}|\,(f,\,h \circ_c g)\ )$.

Remark. To simplify notation, here and below we ignore the canonical equivalences and

isomorphisms that are part of the Cartesian bicategorical structure of ANACAT . Thus, when we want to state the notion of " $\mathcal{V}$-enriched bicategory", there is another object in the diagram, and the left-hand side looks like

$$
\begin{array}{ccc}
([A, B]\otimes[B, C])\otimes[C, D] & & \\
\simeq \downarrow \hat{p} & & \\
[A, B]\otimes([B, C]\otimes[C, D]) & & \\
\Big\downarrow [A, B]\otimes\circ_{B, C, D} & & \\
[A, B]\otimes[B, D] & , &
\end{array}
$$

with $\hat{p}$ the appropriate canonical equivalence in $\mathcal{A}$. In ANACAT , $\hat{p}$ is in fact an isomorphism.

(vii) (*Associativity coherence*). Let A, B, C, D and E be objects. With a logic that should be easy to guess, we make the following abbreviations:

$$
\begin{array}{ll}
0 & = [A, B]\times[B, C]\times[C, D]\times[D, E] \\
1 & = [A, C]\times[C, D]\times[D, E] \\
2 & = [A, B]\times[B, D]\times[D, E] \\
3 & = [A, B]\times[B, C]\times[C, E] \\
12 & = [A, D]\times[D, E] \\
13 & = [A, C]\times[C, E] \\
23 & = [A, B]\times[B, E] \\
123 & = [A, E]
\end{array}
$$

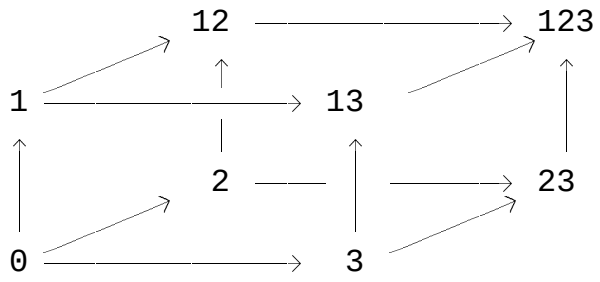
$$
\begin{array}{lll}
\langle 0, 1\rangle & = \circ_{A, B, C}\times[C, D]\times[D, E] & : 0 \longrightarrow 1 \\
\langle 0, 2\rangle & = [A, B]\times\circ_{B, C, D}\times[D, E] & : 0 \longrightarrow 2 \\
\langle 0, 3\rangle & = [A, B]\times[B, C]\times\circ_{C, D, E} & : 0 \longrightarrow 3 \\
\langle 1, 12\rangle & = \circ_{A, C, D}\times[D, E] & : 1 \longrightarrow 12 \\
\langle 1, 13\rangle & = [A, C]\circ_{C, D, E} & : 1 \longrightarrow 13 \\
\langle 2, 12\rangle & = \circ_{A, B, D}\times[D, E] & : 2 \longrightarrow 12 \\
\langle 2, 23\rangle & = [A, B]\times\circ_{B, D, E} & : 2 \longrightarrow 23 \\
\langle 3, 13\rangle & = \circ_{A, B, C}\times[C, E] & : 3 \longrightarrow 13 \\
\langle 3, 23\rangle & = [A, B]\times\circ_{B, C, E} & : 3 \longrightarrow 23
\end{array}
$$

$\langle 12, 123\rangle = \circ_{A,\,D,\,E} : 12 \longrightarrow 123$

$\langle 13, 123\rangle = \circ_{A,\,C,\,E} : 13 \longrightarrow 123$

$\langle 23, 123\rangle = \circ_{A,\,B,\,E} : 23 \longrightarrow 123$

We have the cube



with the edges the anafunctors indicated. Each face of the cube has a 2-cell in it as follows:

the front face:

$(13) = \mathrm{id} : \langle 1,13\rangle\langle 0,1\rangle \longrightarrow \langle 3,13\rangle\langle 0,3\rangle$

(both composites are canonically isomorphic to $\circ_{A,\,B,\,C} \times \circ_{C,\,D,\,E}$; taking the products in ANACAT as in CAT, the isomorphism is the identity);

the back face:

$(13)2 = \alpha_{A,\,B,\,D,\,E} : \langle 12, 123\rangle\langle 2, 12\rangle \longrightarrow \langle 23, 123\rangle\langle 2, 23\rangle;$

the bottom face:

$(23) = [A, B] \times \alpha_{B,\,C,\,D,\,E} : \langle 2, 23\rangle\langle 0, 2\rangle \longrightarrow \langle 3, 23\rangle\langle 0, 3\rangle;$

the top face:

$(23)1 = \alpha_{A,\,C,\,D,\,E} : \langle 12, 123\rangle\langle 1, 12\rangle \longrightarrow \langle 13, 123\rangle\langle 1, 13\rangle;$

the left face:

$(12) = \alpha_{A,\,B,\,C,\,D} \times [D, E] : \langle 1, 12\rangle\langle 0, 1\rangle \longrightarrow \langle 2, 12\rangle\langle 0, 2\rangle;$

the right face:

$(12)3 = \alpha_{A,\,B,\,C,\,E} : \langle 13, 123\rangle\langle 3, 13\rangle \longrightarrow \langle 23, 123\rangle\langle 3, 23\rangle.$

With triple composites meant as associated to the left, there are six composite anafunctors from 0 to 123:

$\langle 123\rangle = \langle 12,123\rangle\langle 1,12\rangle\langle 0,1\rangle,$

$\langle 132\rangle = \langle 13,123\rangle\langle 1,13\rangle\langle 0,1\rangle,$

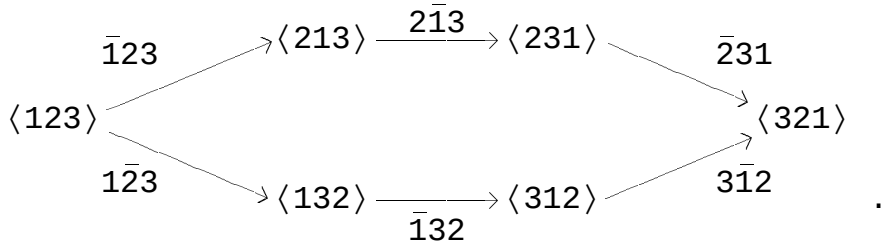
$\langle 213\rangle = \langle 12,123\rangle\langle 2,12\rangle\langle 0,2\rangle,$

$$\langle 231\rangle = \langle 23,123\rangle\langle 2,23\rangle\langle 0,2\rangle\ ,$$
$$\langle 312\rangle = \langle 13,123\rangle\langle 3,13\rangle\langle 0,3\rangle\ ,$$
$$\langle 321\rangle = \langle 23,123\rangle\langle 3,23\rangle\langle 0,3\rangle\ ;$$

and six composite natural transformations between the latter, which (after ignoring canonical isomorphisms in ANACAT) are as follows:

$$\bar{1}23 = (12)\circ\langle 12,123\rangle \quad : \langle 123\rangle \longrightarrow \langle 213\rangle\ ,$$
$$1\bar{2}3 = \langle 0,1\rangle\circ(23)1 \quad : \langle 123\rangle \longrightarrow \langle 132\rangle\ ,$$
$$2\bar{1}3 = \langle 0,2\rangle\circ(13)2 \quad : \langle 213\rangle \longrightarrow \langle 231\rangle\ ,$$
$$\bar{2}31 = \langle 23,123\rangle\circ(23) \quad : \langle 231\rangle \longrightarrow \langle 321\rangle\ ,$$
$$\bar{1}32 = \langle 13,123\rangle\circ(13) \quad : \langle 132\rangle \longrightarrow \langle 312\rangle\ ,$$
$$3\bar{1}2 = \langle 0,3\rangle\circ(12)3 \quad : \langle 312\rangle \longrightarrow \langle 321\rangle\ .$$

We obtain the following diagram of natural transformations:

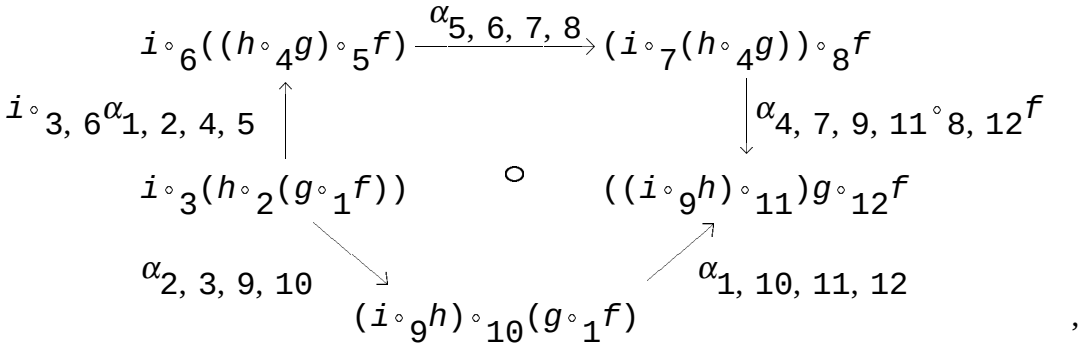


.

The associativity-coherence condition is that the last diagram be commutative, for any choice of A , B , C , D and E . This is essentially a pentagon since $\bar{1}32$ is (can be taken to be) the identity; the above way of presenting it is more symmetric.

When we present the last diagram componentwise, we take a 4-tuple $(f{:}A\to B,\ g{:}B\to C,\ h{:}C\to D,\ i{:}D\to E)$ of arrows, and compute their (five) composites with all possible bracketings. Since there are twelve distinct compositions to be performed (corresponding to the twelve edges of the cube), we will use twelve arbitrary specifications for ten composition anafunctors (ten because any one of them is determined by selecting three out of the five given objects); two of them are used twice (see the formulas above for the composites $0 \longrightarrow 123$). The condition becomes the coherence pentagon ((1.1)(A.C.) in [Be1],

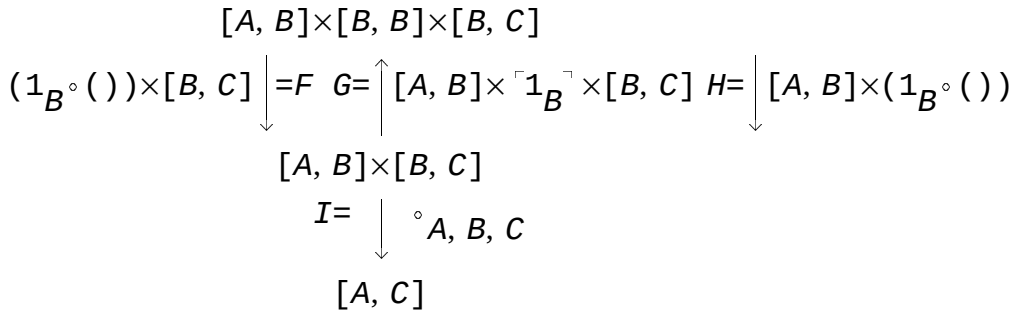
p. 6), with the twelve arbitrary specifications appropriately added; explicitly



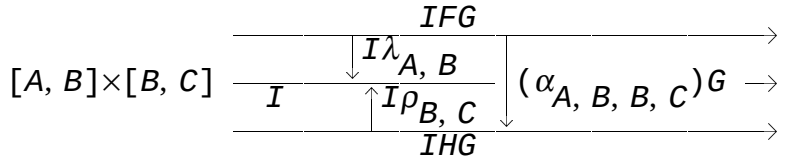
,

with the numbers 1 to 12 standing for appropriate, but otherwise arbitrary, specifications.

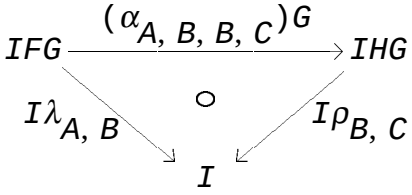
(viii) (*Identity coherence*) Consider



and

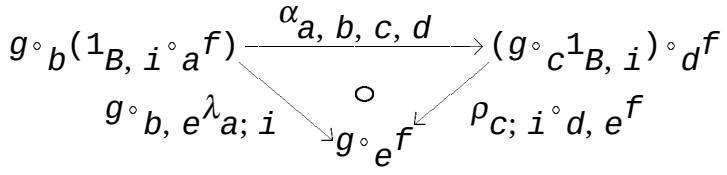


The condition is the commutativity



.

In components, this becomes



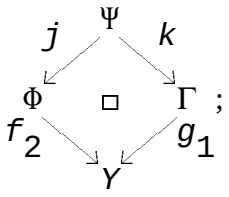
($f:A\to B$, $g:B\to C$, *etc.*).

A one-object bicategory is a monoidal category; its objects are the 1-cells of the bicategory, arrows the 2-cells, tensor-product is composition of 1-cells, the (two-sided) unit I is the identity 1-cell 1_* . A one-object anabicategory is an *anamonoidal category*. An anamonoidal category is a category $\boldsymbol{A}$, with, among others, a unit anaobject $I=\mathrm{I}_{\boldsymbol{A}}$ of $\boldsymbol{A}$ (that is, an anafunctor $I:\mathbf{1}\to\boldsymbol{A}$) and a tensor-product anafunctor $\otimes:\boldsymbol{A}\times\boldsymbol{A}\longrightarrow\boldsymbol{A}$; for objects A , B , we have tensors $A\otimes_a B$, for any specification $a\in|\otimes|(A,B)$. (Note the changed order in $A\otimes_a B$ of the arguments with respect to the notation of composites.) To say that $\otimes$ is saturated is to say that whenever $\mu:A\otimes_a B\xrightarrow{\cong}C$, there is a unique $b\in|\otimes|(A,B)$ such that $C=A\otimes_b B$ and $\mu=1_A\otimes_{a,\,b}1_B$; if so, and if I is saturated too, we say that $\boldsymbol{A}$ is a *saturated* anamonoidal category.

2. Example. Let R be a commutative ring (with 1); Mod_R the category of R-modules. Mod_R is a saturated anamonoidal category, with the following canonical structure. $|\otimes|(A,B)$ is the set of all entities $a=(C,\eta:(A,B)\longrightarrow C)$ where η is a universal (R-)bilinear map from (A,B) : for any bilinear $\varphi:(A,B)\longrightarrow D$, there is unique $f:C\longrightarrow D$ such that $\varphi=f\circ\eta$; we put $A\otimes_a B=C$. Given also $b=(F,\theta:(D,E)\longrightarrow F)\in|\otimes|(D,E)$, and $f:A\to D$, $g:B\to E$, $f\otimes_{a,\,b}g:A\otimes_a B\longrightarrow D\otimes_b E$ is the unique arrow $h:C\to F$ for which $\theta=f\circ\eta$. The unit anaobject is the saturation $\ulcorner R\urcorner^{\#}:\mathbf{1}\to\mathrm{Mod}_R$ of $\ulcorner R\urcorner:\mathbf{1}\to\mathrm{Mod}_R$ picking out R as a module over itself. I leave the definition of the identity and associativity isomorphisms to the reader. This is the usual definition of Mod_R as a monoidal category, except for the absence of a non-canonical choice of the tensors, and a somewhat artificial-looking way of dealing with the unit object. (We could leave the unit object as it usually is, given by $\ulcorner R\urcorner:\mathbf{1}\to\mathrm{Mod}_R$ (as an anafunctor), but then Mod_R is not saturated; we will see the advantages of saturation for anamonoidal categories when we discuss morphisms

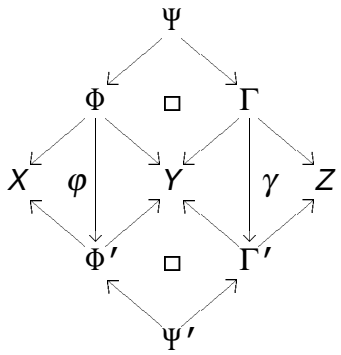
between them.) The usual verification of the monoidal properties works with no essential change to show that we indeed have an anamonoidal category. It is in fact saturated; the part of this fact that concerns the tensor-operation is essentially equivalent to saying that the universal bilinear map is unique up to a unique isomorphism.

3. Example. Given a category $\boldsymbol{X}$ with (not necessarily specified) pullbacks, we have the anabicategory $\mathtt{Span}(\boldsymbol{X})$ of *spans* in $\boldsymbol{X}$, defined as follows. The objects are the objects of $\boldsymbol{X}$, an arrow $f:X\longrightarrow Y$ in $\mathtt{Span}(\boldsymbol{X})$ is a pair $f=(X\xleftarrow{f_1}\Phi\xrightarrow{f_2}Y)$ of arrows in $\boldsymbol{X}$; with also $f'=(X\xleftarrow{f'_1}\Phi'\xrightarrow{f'_2}Y):X\to Y$, a 2-cell $\varphi:f\to f'$ is an arrow $\varphi:\Phi\to\Phi'$ with $f'_1\varphi=f_1$, $f'_2\varphi=f_2$; composition of 2-cells is composition in $\boldsymbol{X}$. With f as above, and $g=(Y\xleftarrow{g_1}\Gamma\xrightarrow{g_2}Z):Y\to Z$, we define $|\circ_{X,\,Y,\,Z}|(f,g)$ as the set of pullback diagrams

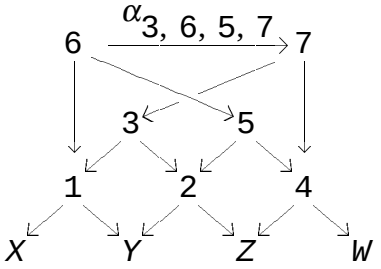


(1)

denoting (1) by a, we define $g\circ_a f$ to be $(X\xleftarrow{f_1\varphi}\Psi\xrightarrow{g_2k}Z):X\to Z$. Given, in addition to data as above, also $g'=(Y\xleftarrow{g'_1}\Gamma'\xrightarrow{g'_2}Z):Y\to Z$, $a'\in|\circ_{X,\,Y,\,Z}|(f',g')$, $\gamma:g\to g'$, $\varphi\circ_{a,\,a'}\gamma:g\circ_a f\longrightarrow g'\circ_{a'}f'$ is defined by the arrow $\psi:\Psi\longrightarrow\Psi'$ we can put in the diagram



by the universal property of the lower pullback. The identity $1_X : X \to X$ is given by the span $(X \xleftarrow{1_X} X \xrightarrow{1_X} X)$ (the identity anaobject of $\mathtt{Span}(\boldsymbol{X})(X, X)$ is now an ordinary object). The identity and associativity isomorphisms for the composition are given by canonical isomorphisms of pullbacks; the picture



(2)

contains the construction of the associativity isomorphism

$$\alpha_{3,6,5,7} : 4 \circ_6 (2 \circ_3 1) \xrightarrow{\cong} (4 \circ_5 2) \circ_7 1 \ ;$$

in (2), each of the items 3, 5, 6 and 7 is obtained by taking a pullback. The rest of the definition and the verification that all this gives an anabicategory are left to the reader. In fact, the definition is the usual one ((2.6) in [Be1]) for the bicategory of spans, with the non-canonical choices removed. The composition functors are saturated in this example.

A morphism of anabicategories will be called an *anafunctor* (*of anabicategories*); this is the natural counterpart of "homomorphism of bicategories" (see [Be1]), or functor of bicategories as we called it above. With $\mathcal{X}$ and $\mathcal{A}$ anabicategories, an anafunctor $F : \mathcal{X} \longrightarrow \mathcal{A}$ is given by data 4.(i)-(iv), satisfying 4.(v)-(vi), all given below. Again, we actually have a more general notion, that of an anafunctor of $\mathcal{V}$-enriched bicategories.

4. **(i)** A class $|F|$, with maps $\sigma : |F| \longrightarrow |\boldsymbol{X}|$ ("source"), $\tau : |F| \longrightarrow |\boldsymbol{A}|$ ("target"). $|F|$ is the class of *specifications*; $x \in |F|$ "specifies the value $\tau(x)$ at the argument $\sigma(x)$ ". For $X \in \boldsymbol{X}$, we write $|F| X$ for the class $\{x \in |F| : \sigma(x) = X\}$, and $F_X(X)$ for $\tau(x)$; the notation $F_X(X)$ presumes that $x \in |F| X$. As before, we write $|F| (X, A)$ for $\{x \in |F| X : F_X(X) = A\}$.

(I note that it would be natural to make $|F|\ (X, A)$ into a category; indeed, *saturation* for F will mean an *additional structure*, with $|F|\ (X, A)$ made the object-class of a groupoid; I will not pursue this in this paper, but see [Ma2]).

(ii) For any $x \in |F|\ X$, $y \in |F|\ Y$, an anafunctor $F_{x,\,y} : \mathcal{X}(X, Y) \longrightarrow \mathcal{A}(F_x X, F_y Y)$.

(iii) (*Composition isomorphism*) For x and y as before, and $z \in |F|\ Z$, an isomorphism natural transformation $F_{x,\,y,\,z}$ as in

$$\begin{array}{ccc} \mathcal{X}(X, Y) \times \mathcal{X}(Y, Z) & \xrightarrow{\quad \circ X, Y, Z \quad} & \mathcal{X}(X, Z) \\ \downarrow{\scriptstyle F_{x,\,y} \times F_{y,\,z}} & \overset{\cong}{\nearrow}{\scriptstyle F_{x,\,y,\,z}} & \downarrow{\scriptstyle F_{x,\,z}} \\ \mathcal{A}(F_x X, F_y Y) \times \mathcal{A}(F_y Y, F_z Z) & \xrightarrow[\quad \circ F_x X, F_y Y, F_z Z \quad]{} & \mathcal{A}(F_x X, F_z Z) \end{array} \quad .$$

The components of $F_{x,\,y,\,z}$ are 2-cells of the form

$$F_{x,\,y,\,z};\ s,\ t,\ u;\ a,\ b : (F_{y,\,z};\ t\, g) \circ_b (F_{x,\,y};\ s\, f) \xrightarrow{\ \cong\ } F_{x,\,z};\ u\, (g \circ_a f) \ . \qquad (3)$$

(iv) (*Identity isomorphism*) For any $X \in \mathcal{X}$ and $x \in |F|\ X$, an isomorphism F_x as in

$$\begin{array}{ccc} & & \mathcal{X}(X, X) \\ & {\scriptstyle 1_X} \nearrow & \\ \mathbf{1} & {\scriptstyle F_x} \overset{\cong}{\nearrow} & \downarrow{\scriptstyle F_{x,\,x}} \\ & {\scriptstyle 1_{F_x X}} \searrow & \\ & & \mathcal{A}(F_x X, F_x X) \end{array} \quad ;$$

the components of F_x are of the form $F_x;\ u,\ i,\ j : 1_{F_x X},\ j \xrightarrow{\ \cong\ } F_{x,\,x};\ u\, (1_X,\ i)$.

(v) (*Composition coherence*) We arrive at the coherence condition for the transition isomorphisms $F_{X,\,y,\,z}$ without explicit use of further specifications by drawing diagrams of anafunctors and natural transformations in the style of 1.(vii). Assume

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

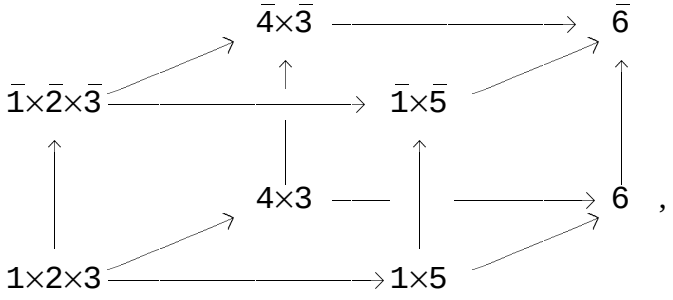
and

$$x \in |F|\,X\,,\ \ y \in |F|\,Y\,,\ \ z \in |F|\,Z\,,\ \ w \in |F|\,W\,.$$

Let us abbreviate:

$$\bar{X} = F_X X\,,\ \ \bar{Y} = F_y Y\,,\ \ \bar{Z} = F_Z Z\,,\ \ \bar{W} = F_W W\,,$$
$$1 = \mathcal{X}(X,\,Y),\ \ 2 = \mathcal{X}(Y,\,Z),\ \ 3 = \mathcal{X}(Z,\,W),\ \ 4 = \mathcal{X}(X,\,Z),\ \ 5 = \mathcal{X}(Y,\,W),\ \ 6 = \mathcal{X}(X,\,W),$$
$$\bar{1} = \mathcal{A}(\bar{X},\,\bar{Y}),\ \ \bar{2} = \mathcal{A}(\bar{Y},\,\bar{Z}),\ \ \bar{3} = \mathcal{A}(\bar{Z},\,\bar{W}),\ \ \bar{4} = \mathcal{A}(\bar{X},\,\bar{Z}),\ \ \bar{5} = \mathcal{A}(\bar{Y},\,\bar{W}),\ \ \bar{6} = \mathcal{A}(\bar{X},\,\bar{W})\,.$$

and consider



with edges

$$\begin{aligned}
\langle 1\rangle &= F_{X,\,y}\times F_{y,\,z}\times F_{z,\,w} && :\ 1\times 2\times 3 \longrightarrow \bar{1}\times\bar{2}\times\bar{3}\rangle\,,\\
\langle 2\rangle &= \mathcal{X}(X,\,Y)\times\circ_{Y,\,Z,\,W} && :\ 1\times 2\times 3 \longrightarrow 1\times 5\,,\\
\langle 3\rangle &= \mathcal{A}(\bar{X},\,\bar{Y})\times\circ_{\bar{Y},\,\bar{Z},\,\bar{W}} && :\ \bar{1}\times\bar{2}\times\bar{3} \longrightarrow \bar{1}\times\bar{5}\,,\\
\langle 4\rangle &= F_{X,\,y}\times F_{y,\,w} && :\ 1\times 5 \longrightarrow \bar{1}\times\bar{5}\,,\\
\langle 5\rangle &= \circ_{X,\,Y,\,Z}\times\mathcal{X}(Z,\,W) && :\ 1\times 2\times 3 \longrightarrow 4\times 3\,,\\
\langle 6\rangle &= \circ_{\bar{X},\,\bar{Y},\,\bar{Z}}\times\mathcal{A}(\bar{Z},\,\bar{W}) && :\ \bar{1}\times\bar{2}\times\bar{3} \longrightarrow \bar{4}\times\bar{3}\,,
\end{aligned}$$

$$
\begin{aligned}
\langle 7\rangle &= F_{X,\,Z}{}^{\times}F_{Z,\,W} && : 4\times 3 \longrightarrow \bar{4}\times\bar{3}\,,\\
\langle 8\rangle &= {}^{\circ}{}_{X,\,Z,\,W} && : 4\times 3 \longrightarrow 6\,,\\
\langle 9\rangle &= {}^{\circ}{}_{X,\,Y,\,W} && : 1\times 5 \longrightarrow 6\,,\\
\langle 10\rangle &= {}^{\circ}{}_{\bar{X},\,\bar{Y},\,\bar{W}} && : \bar{1}\times\bar{5} \longrightarrow \bar{6}\,,\\
\langle 11\rangle &= F_{X,\,W} && : 6 \longrightarrow \bar{6}\,,\\
\langle 12\rangle &= {}^{\circ}{}_{\bar{X},\,\bar{Z},\,\bar{W}} && : \bar{4}\times\bar{3} \longrightarrow \bar{6}\,.
\end{aligned}
$$

The front and left faces are themselves composites as follows:

$$
\begin{array}{ccc}
\bar{1}\times\bar{2}\times\bar{3} & \xrightarrow{\quad\langle 3\rangle\quad} & \bar{1}\times\bar{5}\\
\uparrow{\scriptstyle\langle 1.2\rangle} & & {\scriptstyle\langle 4.2\rangle}\uparrow\\
1\times\bar{2}\times\bar{3} & \xrightarrow{\quad\langle 13\rangle\quad} & 1\times\bar{5}\\
\uparrow{\scriptstyle\langle 1.1\rangle} & & {\scriptstyle\langle 4.1\rangle}\uparrow\\
1\times 2\times 3 & \xrightarrow[\quad\langle 2\rangle\quad]{} & 1\times 5
\end{array}
\qquad\qquad
\begin{array}{ccc}
\bar{1}\times\bar{2}\times\bar{3} & \xrightarrow{\quad\langle 6\rangle\quad} & \bar{4}\times\bar{3}\\
\uparrow{\scriptstyle\langle 1.4\rangle} & & {\scriptstyle\langle 7.2\rangle}\uparrow\\
1\times 2\times\bar{3} & \xrightarrow{\quad\langle 14\rangle\quad} & 4\times\bar{3}\\
\uparrow{\scriptstyle\langle 1.3\rangle} & & {\scriptstyle\langle 7.1\rangle}\uparrow\\
1\times 2\times 3 & \xrightarrow[\quad\langle 5\rangle\quad]{} & 4\times 3
\end{array}\,.
$$

where the arrows are the ones obtained by the above pattern. The faces of the cube each have an isomorphism 2-cell in them as follows:

the lower front face:

$\mathcal{X}(X,\,Y)\times F_{y,\,z,\,w}$: $\langle 13\rangle\langle 1.1\rangle \longrightarrow \langle 4.1\rangle\langle 2\rangle$;

the upper front face:

id : $F_{X,\,y}{}^{\times}{}_{\circ}{}_{\bar{Y},\,\bar{Z},\,\bar{W}} \longrightarrow F_{X,\,y}{}^{\times}{}_{\circ}{}_{\bar{Y},\,\bar{Z},\,\bar{W}}$;

the (complete) front face:

$[1] = (F_{X,\,y}{}^{\times}{}_{\circ}{}_{\bar{Y},\,\bar{Z},\,\bar{W}})(\mathcal{X}(X,\,Y)\times F_{y,\,z,\,w})$: $\langle 3\rangle\langle 1\rangle \longrightarrow \langle 4\rangle\langle 2\rangle$;

the back face:

$[2] = F_{X,\,z,\,w}$: $\langle 15\rangle\langle 10\rangle \longrightarrow \langle 14\rangle\langle 11\rangle$;

the bottom face:

$[3] = \alpha_{X,\,Y,\,Z,\,W}$: $\langle 11\rangle\langle 8\rangle \longrightarrow \langle 12\rangle\langle 2\rangle$;

the top face:

$[4] = \alpha_{\bar{X},\,\bar{Y},\,\bar{Z},\,\bar{W}}$: $\langle 15\rangle\langle 9\rangle \longrightarrow \langle 13\rangle\langle 6\rangle$;

the lower left face:

id : ${}^{\circ}{}_{X,\,Y,\,Z}{}^{\times}F_{Z,\,W} \longrightarrow {}^{\circ}{}_{X,\,Y,\,Z}{}^{\times}F_{Z,\,W}$;

the upper left face:

$$F_{X,\,y,\,z}\times\mathcal{A}(\bar{Z},\,\bar{W}) \quad : \quad \langle 6\rangle\langle 1.4\rangle \longrightarrow \langle 7.2\rangle\langle 14\rangle\,;$$

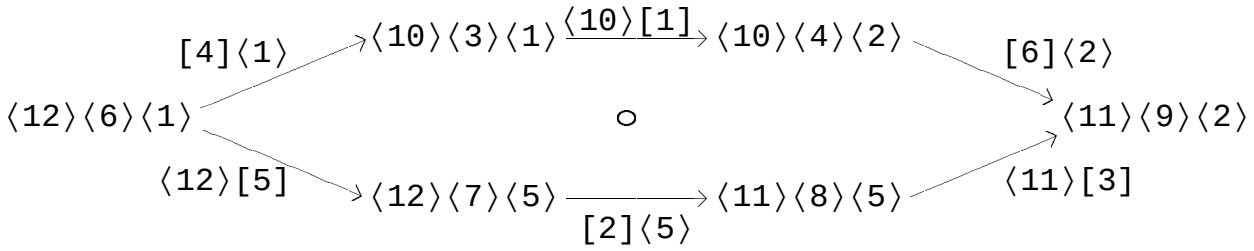
the (complete) left face:

$$[5] = (F_{X,\,y,\,z}\times\mathcal{A}(\bar{Z},\,\bar{W}))(\circ_{X,\,Y,\,Z}\times F_{Z,\,w}) \quad : \langle 6\rangle\langle 1\rangle \longrightarrow \langle 7\rangle\langle 5\rangle\,;$$

the right face:

$$[6] = F_{X,\,y,\,w} \qquad : \langle 10\rangle\langle 4\rangle \longrightarrow \langle 11\rangle\langle 9\rangle\,.$$

We have six composite arrows from $1\times 2\times 3$ to $\bar{6}$ in the cube, and six 2-cells between them; the commutativity of the resulting hexagon is the requirement:



The componentwise form of the condition can be obtained by chasing an arbitrary element $(f,\,g,\,h)\in 1\times 2\times 3$ around. It can also be obtained directly from the corresponding condition for bicategories (see (4.1)(M.1) in [Be1]), by putting in further indices in a meaningful way. Starting with $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, $x\in |F|\,X$, $y\in |F|\,Y$, $z\in |F|\,Z$, $w\in |F|\,W$, and $\bar{1}$, ..., $\bar{7}$ and 1, ..., 10 from appropriate specification-sets, we use the abbreviations

$$\bar{F}f = F_{X,\,y;\,\bar{1}}(f)\,,\ \bar{F}g = F_{y,\,z;\,\bar{2}}(g)\,,\ \bar{F}h = F_{z,\,w;\,\bar{4}}(h)\,,$$
$$\bar{F}(g\circ_2 f) = F_{X,\,z,\,\bar{3}}(g\circ_2 f)\,,\ \bar{F}(h\circ_6 g) = F_{y,\,w;\,\bar{5}}(h\circ_6 g)\,,$$
$$\bar{F}(h\circ_9(g\circ_2 f)) = F_{X,\,w;\,\bar{6}}(h\circ_9(g\circ_2 f))\,,$$
$$\bar{F}((h\circ_6 g)\circ_{10} f)) = F_{X,\,w;\,\bar{7}}((h\circ_6 g)\circ_{10} f))\,,$$
$$\bar{F}_{1,\,2} = F_{X,\,y,\,z;\,\bar{1},\,\bar{2},\,\bar{3};\,1,\,2}\,,\ \bar{F}_{5,\,6} = F_{y,\,z,\,w;\,\bar{2},\,\bar{4},\,\bar{5};\,5,\,6}\,,$$
$$\bar{F}_{4,\,9} = F_{X,\,z,\,w;\,\bar{3},\,\bar{4},\,\bar{6};\,4,\,9}\,,\ \bar{F}_{8,\,10} = F_{X,\,y,\,w;\,\bar{1},\,\bar{5},\,\bar{7};\,8,\,10}\,,$$
$$\bar{F}\alpha_{2,\,9,\,6,\,10} = F_{X,\,w;\,\bar{6},\,\bar{7}}(\alpha_{2,\,9,\,6,\,10})\,.$$

The following diagram is to be commutative:

$$\begin{array}{ccc}
\bar{F}h \circ_3 (\bar{F}g \circ_1 \bar{F}f) & \xrightarrow{\alpha_{1,\,3,\,5,\,7}} & (\bar{F}h \circ_5 \bar{F}g) \circ_7 \bar{F}f \\
{\scriptstyle \bar{F}h \circ_{3,\,4} \bar{F}_{1,\,2}} \downarrow & & \downarrow {\scriptstyle \bar{F}_{5,\,6} \circ_{7,\,8} \bar{F}f} \\
\bar{F}h \circ_4 \bar{F}(g \circ_2 f) & \circlearrowright & \bar{F}(h \circ_6 g) \circ_8 \bar{F}f \\
{\scriptstyle \bar{F}_{4,\,9}} \downarrow & & \downarrow {\scriptstyle \bar{F}_{8,\,10}} \\
\bar{F}(h \circ_9 (g \circ_2 f) & \xrightarrow[\bar{F}\alpha_{2,\,9,\,6,\,10}]{} & \bar{F}((h \circ_6 g) \circ_{10} f)
\end{array}$$

(vi) (*Identity coherence*) We give the componentwise form of the condition only. For $f : X \longrightarrow Y$ in $\mathcal{X}$, $x \in |F|\,X$, $y \in |F|\,Y$, and appropriate specifications $\bar{1}$, $\bar{2}$, $\bar{3}$, 1, 2, 3, i, j,

$$\begin{array}{ccc}
F_{y,\,y;\,\bar{1}}(1_{Y,\,i}) \circ_1 F_{x,\,y;\,\bar{3}}\ (f) & \xleftarrow{F_{y;\,\bar{1}} \circ_{1,\,2} F_{x,\,y;\,\bar{3}}(f)} & 1_{F_y(Y),\,j} \circ_2 F_{x,\,y;\,\bar{3}}(f) \\
\downarrow {\scriptstyle F_{x,\,y,\,y;\,\bar{3},\,\bar{1},\,\bar{2};\,1,\,3}} & \circlearrowright & \downarrow {\scriptstyle \lambda_{2,\,j}} \\
F_{x,\,y;\,\bar{2}}(1_{Y,\,i} \circ_3 f) & \xrightarrow[F_{x,\,y;\,\bar{2},\,\bar{3}}(\lambda_{3,\,i})]{} & F_{x,\,y;\,\bar{3}}(f)
\end{array} ,$$

and a similar commutativity involving the ρ's.

From now on, we assume that the anabicategory $\mathcal{A}$ has saturated identity functors $1_A : \mathbf{1} \longrightarrow [A, A]$ (1.(iii)), and saturated composition functors $\circ_{A,\,B,\,C} : [A, B] \times [B, C] \longrightarrow [A, C]$ (1.(iv)); in this case, let us call $\mathcal{A}$ *saturated*. For $\mathcal{A}$ saturated, there is a useful reformulation of the notion of anafunctor $\mathcal{X} \longrightarrow \mathcal{A}$, trading in the associativity isomorphisms for certain operations on specifications.

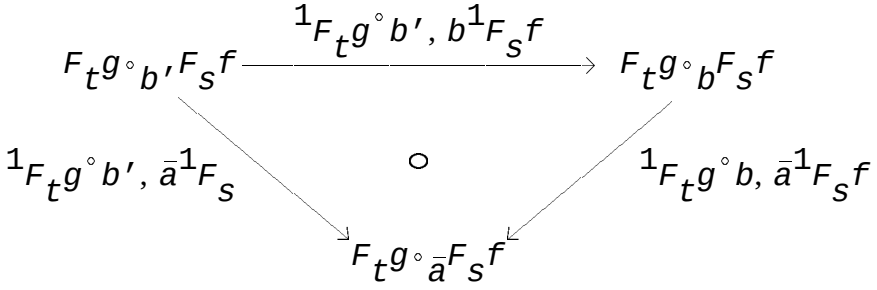
Let $F : \mathcal{X} \longrightarrow \mathcal{A}$ be an anafunctor of anabicategories. For simplicity, assume that the sets $|F|\,X$ are singletons. With the notation of 4.(iii), in the context of (3) there, with the subscripts x, y, z dropped, the saturation of the composition functor

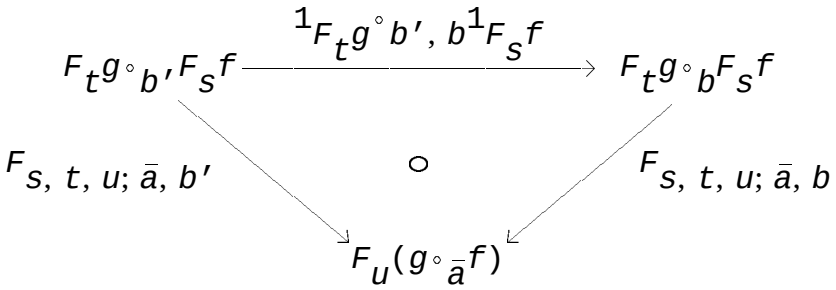
$$\circ = \circ_{X,\,Y,\,Z} : \mathcal{A}(FX, FY) \times \mathcal{A}(FY, FZ) \longrightarrow \mathcal{A}(FX, FZ) , \qquad (4)$$

gives a uniquely determined $\bar{a} \in |\circ|$ such that $(F_t g) \circ_{\bar{a}} (F_s f) = F_u (g \circ_a f)$ and

$$ {}^1F_t g \circ b, \bar{a} {}^1F_s f = F\langle s, t, u; a, b \rangle . \tag{5} $$

Here, $\bar{a}$ depends on (f, g and) s, t, u, a and b; however, the dependence on b is illusory. Consider the following commutative diagrams:





(the first is the functoriality of (4), the second is the naturality of $F_{X, Y, Z}$ of 4.(iii)). Together they show that if $\bar{a}$ is suitable for b, then it is suitable for b', with the other data being the same. Thus, we have $\bar{a}$ as a function of s, t, a and u:

4. (iii*) (*Action on specifications*)

$$ \bar{a} = F\langle s, t, u; a \rangle \qquad (s \in |F|\, f,\ t \in |F|\, g,\ a \in |\circ|\, ((f, g))\,, $$
$$ u \in |F|\, (g \circ_a f);\ \bar{a} \in |\circ|\, ((F_s f, F_t g))\ ) $$

such that

$$ F_u (g \circ_a f) = F_t (g) \circ_{\bar{a}} F_s (f) $$

and

$$ F_{u, u'} (\gamma \circ_a, a' \varphi) = (F_{t, t'} (\gamma)) \circ_{\bar{a}, \bar{a}'} (F_{s, s'} (\varphi)) $$

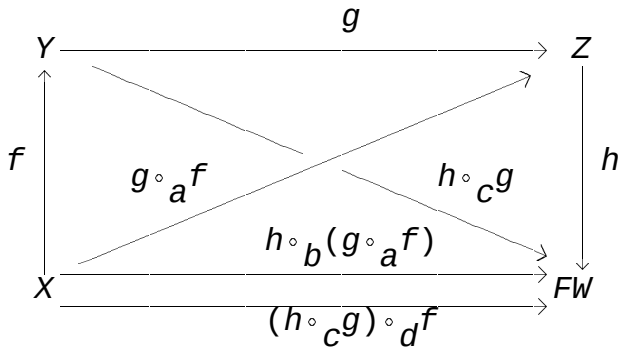
$(\varphi: f \to f'\ ,\ \gamma: g \to g'\ ,\ \bar{a}' = F(s', t', u'; a')$; the last equality corresponds to the naturality of $F_{X, Y, Z}$ of 4.(iii)).

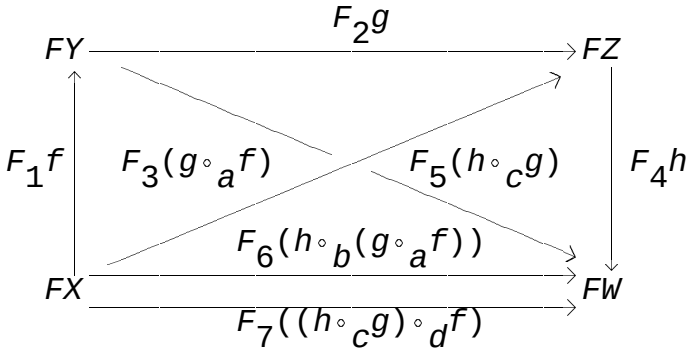
Next, we realize that in this context composition coherence (4.(v)) has as a consequence the identity

4. (v*) (*Action on associativity isomorphisms*)

$$F_{6, 7}(\alpha_{a, b, c, d}) = \alpha_{\bar{a}, \bar{b}, \bar{c}, \bar{d}}\ ;$$

here reference is made to the data





and $\bar{a} = F(1, 2, 3; a)$, $\bar{b} = F(3, 4, 6; b)$, $\bar{c} = F(2, 4, 5; c)$, $\bar{d} = F(1, 5, 7; d)$.

The identity isomorphisms (4.(iv)) can similarly be traded in for specifications. Let $A \in \mathcal{A}$. The

saturation of 1_A means that whenever $\mu : 1_{A,\,i} \xrightarrow{\cong} \ell$, there is a unique $j \in |1_A|$ such that $\ell = 1_{A,\,j}$ and $\mu = 1_{A,\,i,\,j}$. The identity isomorphism of 4.(iv) has components

$$F_{X,\,u;\,i,\,j} : 1_{FX,\,j} \xrightarrow{\cong} F_u(1_{X,\,i}) \ .$$

Given the data in the last display, by the saturation of 1_A, there is unique $\bar{i}$ such that $1_{FX,\,\bar{i}} = F_u(1_{X,\,i})$ and

$$1_{FX,\,j,\,\bar{i}} = F_{X,\,u;\,i,\,j} \ . \qquad (5')$$

Again, $\bar{i}$ here does not depend on j; $\bar{i} = F(u;\,i)$. Thus, the data 4.(iv) can be replaced by

4. (iv*) $\bar{i} = F(u;\,i)$ $\quad (\ i \in |\ulcorner 1_X \urcorner|\ ,\ u \in |F|\,(1_{X,\,i})\ )$

such that

$$F_u(1_{X,\,i}) = 1_{FX,\,\bar{i}}$$

and

$$F_{u,\,u'}(1_{X,\,i,\,i'}) = 1_{FX,\,\bar{i},\,\bar{i}'}$$

($\bar{i} = F(u;\,i)$, $\bar{i}' = F(u';\,i')$).

Then 4.(vi) becomes

4. (vi*) $F_{t,\,u}(\lambda_{a,\,i}) = \lambda_{\bar{a},\,\bar{i}}$ $\quad (\ f : X \to Y\ ,\ \lambda_{a,\,i} : 1_{Y,\,i} \circ_a f \xrightarrow{\cong} f\ ,\ s \in |F|\,f\ ,$ $t \in |F_{Y,\,Y}|\,(1_{Y,\,i})$, $u \in |F|\,(1_{Y,\,i} \circ_a f)$; $\bar{a} = F(s,\,t,\,u;\,a)$, the latter from 4.(iii*), and $\bar{i} = F(t;\,i)$), and similarly for ρ.

Conversely, even without assuming that $\mathcal{A}$ is saturated, in the definition of an anafunctor $F : \mathcal{X} \longrightarrow \mathcal{A}$ with all $|F|\,X$ singletons, we can dispense with the composition isomorphisms 4.(iii) and the identity isomorphisms 4.(iv), and instead, use the data 4.(iii*), 4.(iv*), satisfying

4.(v*) and 4.(vi*). Indeed, if F is given by 4.(i),(ii),(iii*),(iv*),(v*) and (vi*), there is a unique anafunctor $F:\mathcal{X}\longrightarrow\mathcal{A}$ in the original sense that is related to the F we start with as just described; we use (5) and (5') as the definition of composition isomorphisms and identity isomorphisms, respectively. The essential point is that 4.(v) and 4.(vi) and will hold.

An *anamonoidal functor* is, by definition, an anafunctor between anabicategories with one object, with the one specification set for the object-function being a singleton. Explicitly, with $\boldsymbol{X}$ and $\boldsymbol{A}$ anamonoidal categories, $\boldsymbol{A}$ saturated, an anamonoidal functor $F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ is given by an anafunctor $F:\boldsymbol{X}\longrightarrow\boldsymbol{A}$ of the ordinary categories $\boldsymbol{X}$ and $\boldsymbol{A}$, a mapping assigning to each $s\in|F|X$, $t\in|F|Y$, $a\in|\otimes|(X,Y)$, $u\in|F|(A\otimes_a B)$ a specification $\bar{a}=F(s,t,u;a)\in|\otimes|(F_sX,F_tY)$ such that

$$F_u(X\otimes_a Y)=(F_sX)\otimes_{\bar{a}}(F_tY)\,,$$

$$F_{u,u'}(f\otimes_{a,a'}g)=(F_{s,s'}f)\otimes_{\bar{a},\bar{a}'}(F_{t,t'}g)\quad(f:X\to X',\ g:Y\to Y',\ \textit{etc.}),$$

and a mapping assigning to each $i\in|\mathrm{I}_{\boldsymbol{X}}|$, $u\in|F|(\mathrm{I}_{\boldsymbol{X},i})$, a specification $\bar{i}=F(u;i)$ such that

$$F(\mathrm{I}_{\boldsymbol{X},i})=\mathrm{I}_{\boldsymbol{A},\bar{i}}$$

$$F(\mathrm{I}_{\boldsymbol{X},i,j})=\mathrm{I}_{\boldsymbol{A},\bar{i},\bar{j}}$$

and such that the conditions 4.(v*), 4.(vi*) hold.

In case F is a functor (each $|F|X$ is a singleton), $\bar{a}$ becomes a function of a alone, thus F is given with a function also denoted $F:|\otimes|(X,Y)\longrightarrow|\otimes|(FX,FY)$ and another function $F:|\mathrm{I}_{\boldsymbol{X}}|\longrightarrow|\mathrm{I}_{\boldsymbol{A}}|$ such that

$$F(X\otimes_a Y)=(FX)\otimes_{Fa}(FY)\,,$$

$$F(f\otimes_{a,a'}g)=(Ff)\otimes_{Fa,Fa'}(Fg)\quad(f:X\to X',\ g:Y\to Y',\ \textit{etc.}),$$

$F(\alpha_{a,\,b,\,c,\,d}) = \alpha_{Fa,\,Fb,\,Fc,\,Fd}$,

$F(\lambda_{a,\,i}) = \lambda_{Fa,\,Fi}$, $F(\rho_{a,\,i}) = \rho_{Fa,\,Fi}$ ($\lambda_{a,\,i}: \mathrm{I}_{\boldsymbol{X}},i \otimes a \xrightarrow{\cong} A$, *etc.*).

We may call such an F a *monoidal functor* between the anamonoidal categories; a monoidal functor is precisely a structure preserving map of anamonoidal categories. It is pointed out next that monoidal functors of anamonoidal categories correspond *exactly* to the (not necessarily strict) monoidal functors of monoidal categories in the usual sense [E/K].

First, given any monoidal category $\boldsymbol{X}$ in the usual sense, we can construct its *saturation* $\boldsymbol{X}^{\#}$, a saturated anamonoidal category, as follows. The underlying category of $\boldsymbol{X}^{\#}$ is the same as that of $\boldsymbol{X}$; the anaoperation $\otimes^{\#}: \boldsymbol{X} \times \boldsymbol{X} \longrightarrow \boldsymbol{X}$ is the saturation of the functor $\otimes: \boldsymbol{X} \times \boldsymbol{X} \longrightarrow \boldsymbol{X}$. This means that $|\otimes^{\#}|\,(X, Y)$ is the class of all entities $a = (X, Y, \mu: X \otimes Y \xrightarrow{\cong} Z)$, with $X \otimes_a Y = Z$, and for $f: X \to X'$, $g: Y \to Y'$, $f \otimes_{a,\,a'} g = \mu' \circ (f \otimes g) \circ \mu^{-1}$. Similarly, the unit anaobject of $\boldsymbol{X}^{\#}$ is the saturation of $\ulcorner I \urcorner : \mathbf{1} \to \boldsymbol{X}$.

Now, given the monoidal categories $\boldsymbol{X}$ and $\boldsymbol{A}$, we can form their saturations $\boldsymbol{X}^{\#}$, $\boldsymbol{A}^{\#}$, and we have a *bijection*

$$\mathrm{Hom}(\boldsymbol{X}, \boldsymbol{A}) \xrightarrow{\cong} \mathrm{Hom}(\boldsymbol{X}^{\#}, \boldsymbol{A}^{\#}) \,, \qquad (5'')$$

where the first Hom is the collection of all monoidal functors $\boldsymbol{X} \longrightarrow \boldsymbol{A}$ in the usual sense, the second all monoidal functors $\boldsymbol{X}^{\#} \longrightarrow \boldsymbol{A}^{\#}$ as defined above. This bijection is given as follows: to $F: \boldsymbol{X} \longrightarrow \boldsymbol{A}$ corresponds $F^{\#}: \boldsymbol{X}^{\#} \longrightarrow \boldsymbol{A}^{\#}$, where for a as above,

$$F^{\#}(a) = (FX, FY, F(\mu) \circ i_{X,\,Y}: FX \otimes FY \xrightarrow{\cong} FZ) \,,$$

where $i_{X,\,Y}: FX \otimes FY \xrightarrow{\cong} F(X \otimes Y)$ is the canonical isomorphism given with F ; the action of $F^{\#}$ on identity specifications is defined similarly. The verification of the bijection (5'') is omitted (see also [Ma3]).

2. Example (continued). Let $\varphi : R \longrightarrow S$ be a homomorphism of commutative rings. With $\mathtt{Mod}_R$, $\mathtt{Mod}_S$ understood as "the" usual monoidal categories, it is common knowledge that we have a monoidal functor

$$F = (\ ) \otimes_R S : \mathtt{Mod}_R \longrightarrow \mathtt{Mod}_S ; \tag{6}$$

this is because

(*) $(X \otimes_R S) \otimes_S (Y \otimes_R S)$ is canonically isomorphic to $(X \otimes_R Y) \otimes_R S$ ($X, Y \in \mathtt{Mod}_R$).

However, in (6) there are two places where non-canonical choices have been made: one is the definition of the monoidal categories, the other is the definition of the functor itself. Making the categories anamonoidal categories in the way explained before, and making the functor an anamonoidal functor, we avoid all non-canonical choices.

Explicitly, F as an anafunctor $F : \mathtt{Mod}_R \longrightarrow \mathtt{Mod}_S$ is given by $|F|\, X \stackrel{\text{def}}{=}$ the class of all universal R-bilinear maps of the form $(X, S) \longrightarrow Y$; for s the last-displayed map, $F_S(X)$ is Y as an S-module (in the usual sense); the action of F on morphisms is the straightforward one. The action 4.(iii*) on specifications is obtained by making the proof of (*) explicit. Given universal R-bilinear maps

$$(X, S) \stackrel{1}{\longrightarrow} \bar{X},$$
$$(Y, S) \stackrel{2}{\longrightarrow} \bar{Y},$$
$$(X, Y) \stackrel{a}{\longrightarrow} Z,$$
$$(Z, S) \stackrel{3}{\longrightarrow} \bar{Z},$$

we construct a canonically defined S-bilinear map

$$(\bar{X}, \bar{Y}) \stackrel{\bar{a}}{\longrightarrow} \bar{Z}, \tag{7}$$

briefly described as follows. The composite

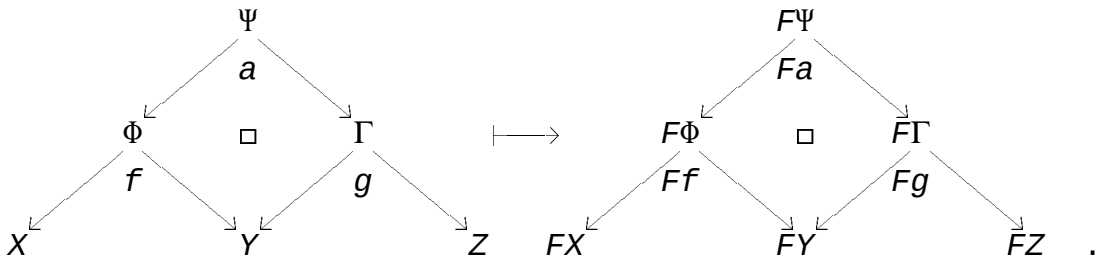
$$(X, S, Y, S) \longrightarrow (Z, S) \xrightarrow{3} \bar{Z}$$
$$(x, u, y, v) \longmapsto (a(x, y), u \cdot v) \longmapsto 3(a(x, y), u \cdot v)$$

is R-linear in each variable. By the universal property of 1, this gives rise to the componentwise R-linear map

$$(\bar{X}, Y, S) \longrightarrow \bar{Z},$$

and that of 2 then gives (7) canonically; we can then show that (7) has the required universal property. This is the description of the action 4.(iii*) of F on specifications. The rest of the data and conditions are left to the reader to provide.

3. Example (continued). Let $F : \mathbf{X} \longrightarrow \mathbf{A}$ be a pullback-preserving functor between categories with pullbacks. F gives rise to an anafunctor $\hat{F} : \mathrm{Span}(\mathbf{X}) \longrightarrow \mathrm{Span}(\mathbf{A})$ of anabicategories. The action of $\hat{F}$ on objects is that of F; the action on arrows of $\hat{F}$ is induced by that of F; now, all specification-sets $|\hat{F}| X$ are singletons, $|\hat{F}| X = \{X\}$, and the effect of F on arrows, $F_{X, Y} : \mathrm{Span}(\mathbf{X})(X, Y) \longrightarrow \mathrm{Span}(\mathbf{A})(FX, FY)$, is an ordinary functor, for each $X, Y \in \mathbf{X}$. The identity isomorphisms are identities. Instead of the composition isomorphisms, we give the action 4.(iii*) of $\hat{F}$ on specifications; remember there are no s, t, u now. Indeed, this is nothing but the action of F again; $a \in |\circ| (f, g)$ is a pullback square, $\bar{a}$ is its F-image:



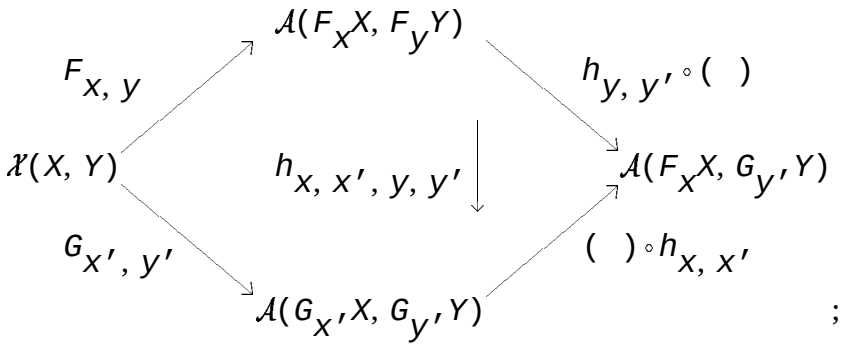
It is immediate that we have 4.(v*): the diagram (2) is mapped into a similar diagram by F.

A *transformation* $h : F \longrightarrow G$, of anafunctors $\mathcal{X} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{A}$ of anabicategories, is given by data

5.(i), (ii) and conditions 5.(iii), (iv).

5. **(i)** For $X \in \mathcal{X}$, $x \in |F|X$, $x' \in |G|X$, an anaobject $h_{x,x'}$ of the hom-category $\mathcal{A}(F_X X, G_{X'} X)$, $h_{x,x'} : \mathbf{1} \xrightarrow{a} \mathcal{A}(F_X X, G_{X'} X)$. $h_{x,x'}$ has components $h_{x,x',m} : F_X X \longrightarrow G_{X'} X$ ($m \in |h_{x,x'}|$).

(ii) For the data in (i) and $Y \in \mathcal{X}$, $y \in |F|Y$, $y' \in |G|Y$, a natural transformation $h_{x,x',y,y'}$ as in



;

Here, $h_{y,y'} \circ (\ )$ abbreviates the composite of

$$\mathcal{A}(F_X X, F_Y Y) \cong \mathbf{1} \times \mathcal{A}(F_X X, F_Y Y) \xrightarrow{h_{y,y'} \times \cdots} \mathcal{A}(F_Y Y, G_{Y'} Y) \times \mathcal{A}(F_X X, F_Y Y) \xrightarrow{\circ} \mathcal{A}(F_X X, G_{Y'} Y)$$

$h_{x,x',y,y'}$ has components

$$h_{x,x',y,y';s,s';a,b;m,n} : h_{y,y';n} \circ_a F_{x,y;s(f)} \longrightarrow G_{x',y';s'(f)} \circ_b h_{x,x';m} .$$

The conditions that follow will be given in their componentwise forms only.

(iii) With reference to

$$X \xrightarrow{f} Y \xrightarrow{g} Z , \quad X \xrightarrow{g \circ_1 f} Y ,$$

$$\bar{F}f = F_{x,\,y;\,s}(f)\,,\ \bar{F}g = F_{y,\,z;\,t}(g)\,,\ \bar{F}(g\circ_1 f) = F_{x,\,z;\,u}(g\circ_1 f)\,,$$
$$\bar{G}f = G_{x',\,y';\,s'}(f)\,,\ \bar{G}g = G_{y',\,z';\,t'}(g)\,,\ \bar{G}(g\circ_1 f) = G_{x',\,z';\,u'}(g\circ_1 f)\,,$$
$$\bar{F}_{1,\,4} = F_{x,\,y,\,z;\,s,\,t,\,u;\,1,\,4}\,,\ \bar{G}_{1,\,6} = G_{x',\,y',\,z';\,s',\,t',\,u';\,1,\,6}\,,$$
$$\bar{h}_X = h_{x,\,x',\,m}\,,\ \bar{h}_Y = h_{y,\,y',\,n}\,,\ \bar{h}_Z = h_{z,\,z',\,p}\,,$$
$$h_{2,\,3} = h_{x,\,x',\,z,\,z';\,u,\,u';\,2,\,3;\,m,\,p}\,,\ h_{8,\,12} = h_{y,\,y',\,z,\,z';\,t,\,t';\,8,\,12;\,n,\,p}\,,$$
$$h_{14,\,10} = h_{x,\,x',\,y,\,y';\,s,\,s';\,14,\,10;\,m,\,n}\,,$$
$$\alpha 1 = \alpha_{4,\,5,\,8,\,9}\,,\ \alpha 2 = \alpha_{10,\,11,\,6,\,7}\,,\ \alpha 3 = \alpha_{14,\,15,\,12,\,13}\,,$$

$$\begin{array}{ccc}
\bar{h}_{Z\circ 2}\bar{F}(g\circ_1 f) & \xrightarrow{\quad h_{2,\,3}\quad} & \bar{G}(g\circ_1 f)\circ_3 \bar{h}_X \\
{\scriptstyle \bar{h}_{Z\circ 5,\,2}\bar{F}_{1,\,4}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \bar{G}_{6,\,1\circ 7,\,3}\bar{h}_X} \\
\bar{h}_{Z\circ 5}(\bar{F}g\circ_4 \bar{F}f) & & (\bar{G}g\circ_6 \bar{G}f)\circ_7 \bar{h}_X \\
{\scriptstyle \alpha 1}\downarrow{\scriptstyle \cong} & \circ & {\scriptstyle \cong}\uparrow{\scriptstyle \alpha 2} \\
(\bar{h}_{Z\circ 8}\bar{F}g)\circ_9 \bar{F}f & & \bar{G}g\circ_{11}(\bar{G}f\circ_{10}\bar{h}_X) \\
{\scriptstyle h_{8,\,12\circ 9,\,13}\bar{F}f}\downarrow & & \uparrow{\scriptstyle \bar{G}g\circ_{15,\,11}h_{14,\,10}} \\
(\bar{G}g\circ_{12}\bar{h}_Y)\circ_{13}\bar{F}f & \xleftarrow[\alpha 3]{\cong} & \bar{G}g\circ_{15}(\bar{h}_Y\circ_{14}\bar{F}f)
\end{array}\,.$$

(iv) With all appropriate values of the parameters,

$$\begin{array}{ccccc}
h_{x,\,x';\,m\circ 1}1_{F_X X,\,j} & \xrightarrow[\cong]{\rho_{1,\,j}} & h_{x,\,x'} & \xleftarrow[\cong]{\lambda_{2,\,k}} & 1_{G_{X'},X,\,k\circ 2}h_{x,\,x';\,m} \\
{\scriptstyle h_{x,\,x'\circ 3,\,1}F_{X;\,s}}\uparrow & \circ & & \circ & \uparrow{\scriptstyle G_{X';\,t\circ 4,\,2}h_{x,\,x'}} \\
h_{x,\,x';\,m\circ 3}F_{x,\,x;\,s}1_{X,\,i} & & \xrightarrow[h_{x,\,x',\,x,\,x';\,s,\,t;\,3,\,4;\,m,\,m}]{} & & G_{x',\,x';\,t}1_{X,\,i\circ 4}h_{x,\,x';\,m}.
\end{array}$$

The transformation h is *natural* [the more usual terminology would be: *strong*] if each natural transformation $h_{x,\,x',\,y,\,y'}$ is an isomorphism.

Consider saturated anamonoidal categories and anamonoidal functors between them: $\boldsymbol{X} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{A}$. A transformation $h : F \longrightarrow G$ (in the sense of one between anafunctors of

anabicategories) involves an anaobject $h_{*,\,*}$ of $\boldsymbol{A}$. Let us take this to be $\mathrm{I}_{\boldsymbol{A}}$, the unit anaobject of $\boldsymbol{A}$; let us call such transformations h *monoidal*. Then

$$\bar{h} \;\bar{\bar{\mathrm{def}}}\; h_{s,\,s'};\, a,\, b;\, m,\, m : F_s(X) \otimes_a \mathrm{I}_{\boldsymbol{A},\,m} \longrightarrow \mathrm{I}_{\boldsymbol{A},\,m} \otimes_b G_{s'}(X)$$

gives rise, via the identity isomorphisms, to $h_{s,\,s'} : F_s(X) \longrightarrow G_{s'}(X)$:

$$\begin{array}{ccc} F_s(X) \otimes_a \mathrm{I}_{\boldsymbol{A},\,m} & \xrightarrow{\quad\bar{h}\quad} & \mathrm{I}_{\boldsymbol{A},\,m} \otimes_b G_{s'}(X) \\ {\scriptstyle \rho_{a;\,m}}\downarrow \cong & \circ & \cong \downarrow {\scriptstyle \lambda_{b;\,m}} \\ F_s(X) & \xrightarrow[\quad h_{s,\,s'}\quad]{} & G_{s'}(X) \end{array} \qquad ;$$

indeed, we easily see that the resulting lower horizontal arrow is independent of the parameters a, b and m. Inspection shows that conditions 5.(iii) and 5.(iv) reduce to the following:

$$\begin{array}{ccc} F_u(X \otimes_a Y) & \xrightarrow{\quad h_{u,\,u'}\quad} & G_{u'}(X \otimes_a Y) \\ \| & \| & \| \\ F_s X \otimes_{\bar{a}} F_t Y & \xrightarrow[\quad h_{s,\,s'} \otimes_{\bar{a},\,\hat{a}} h_{t,\,t'}\quad]{} & G_{s'} X \otimes_{\hat{a}} G_{t'} Y \end{array} \quad \text{, i.e., } h_{u,\,u'} = h_{s,\,s'} \otimes_{\bar{a},\,\hat{a}} h_{t,\,t'} \;,$$

with $\bar{a} = F(s,\, t,\, u;\, a)$ and $\hat{a} = G(s',\, t',\, u';\, a)$; and

$$h_{u,\,\mathrm{v}} = \mathrm{I}_{\boldsymbol{A},F(u;\,i),G(v;\,i)} \;\left(: \mathrm{I}_{\boldsymbol{A},F(u;\,i)} \longrightarrow \mathrm{I}_{\boldsymbol{A},G(v;\,i)} \right)$$

(here $i \in |\mathrm{I}_{\boldsymbol{X}}|$, $u \in |F|\,(\mathrm{I}_{\boldsymbol{X},\,i})$, $v \in |G|\,(\mathrm{I}_{\boldsymbol{X},\,i})$; recall that $F(u;\, i), G(\mathrm{v};\, i) \in |\mathrm{I}_{\boldsymbol{A}}|$.)

When F and G are (monoidal) functors, these data and conditions further reduce to

$$h_A : FA \longrightarrow GA \;,$$

$$h_{A \otimes_a B} = h_A \otimes_{Fa,\,Ga} h_B \;,$$

$$h_{\mathrm{I}_{\boldsymbol{X},\,i}} = \mathrm{I}_{\boldsymbol{A},\,Fi,\,Gi}\,.$$

Now, the mapping (5") becomes an *isomorphism of categories*

$$\mathrm{Hom}(\boldsymbol{X},\boldsymbol{A}) \xrightarrow{\cong} \mathrm{Hom}(\boldsymbol{X}^{\#},\boldsymbol{A}^{\#})$$

between the category $\mathrm{Hom}(\boldsymbol{X},\boldsymbol{A})$ of monoidal functors $\boldsymbol{X}\to\boldsymbol{A}$ and monoidal natural transformations in the classical sense (see [E/K]), and the category $\mathrm{Hom}(\boldsymbol{X}^{\#},\boldsymbol{A}^{\#})$ of the monoidal functors between the saturations $\boldsymbol{X}^{\#}$, $\boldsymbol{A}^{\#}$ and their monoidal transformations.