§5. Internal topology

5.1. Internal cHa's and their representation.

Let E be any topos. A cHa-object in E (or: an internal cHa in E, or even: a cHa in E) is a tuple $(H, \leq, 1, \wedge, \bigvee)$ satisfying the axioms for cHa's in the logic in E; in other words, H is an object of E,

$\leq$: $H\times H \to \Omega$ (a binary relation on H)

1: $1 \to H$ (a 'global element' of H or: a zero-ary operation on H)

$\wedge$: $H\times H \to H$

$\bigvee$: $P(H) \to H$;

two of the axioms written out explicitly:

$$\underset{\substack{x\in H\\ \Sigma\in P(H)}}{\overset{E}{\models}} [x \geq \bigvee(\Sigma) \leftrightarrow \forall_{y\in H}(y\in\Sigma \to x\geq y)]$$

[we put the variables x, Σ under the validity sign $\models$ equivalently to writing $\forall_{x\in H,\Sigma\in P(H)}$ in front of the formula; this axiom says that $\bigvee$ is the sup operation for the relation $\leq$, another axiom would say $\leq$ is a partial order, etc.]

$$\underset{x,\Sigma}{\models} (\bigvee\Sigma)\wedge x = \bigvee(\Sigma(\wedge)x)$$

where $\Sigma(\wedge)x$ is the term $\{y\wedge x\colon y\in\Sigma\}$, i.e. $\Sigma(\wedge)x \mathrel{\cdot{=}\cdot} \{z\in H\colon \exists_{y\in H}(y\in\Sigma \,\&\, z = (y\wedge x))\}$. The other axioms are similarly natural translations of axioms for ordinary cHa's.

A morphism between cHa's in E, $h\colon (H_1,\leq,\ldots) \to (H_2,\leq,\ldots)$, is a morphism (in E) between the 'underlying sets' $h\colon H_1 \to H_2$ satisfying

$$\overset{E}{\models} \text{"}h \text{ preserves } \leq,\ 1,\ \wedge \text{ and } \bigvee\text{"}.$$

Written out in detail, preserving $\bigvee$ means

$$\models^{\mathcal{E}}_{\Sigma\in \mathcal{P}(H_1)} \bigvee_{H_2} h[\Sigma] = h(\bigvee_{H_1} \Sigma)$$

where $h[\Sigma] \cdot{=}\cdot \{h(y):\ y\in\Sigma\} \cdot{=}\cdot \{z\in H_2:\ \exists_{y\in H_1}(y\in\Sigma \wedge h(y) = z)\}$. Composition of morphisms of cHa's in $\mathcal{E}$ is defined automatically as composition of morphisms in $\mathcal{E}$ (of course, one has to check that the composition of two cHa-morphism is one such again, etc.). In this way we have defined the category $\underset{\sim}{\text{cHa}}(\mathcal{E})$ of cHa's in $\mathcal{E}$.

Now, we take $\mathcal{E}$ to be Sh(L) for a cHa L. Given the cHa L, an L-cHa, or a cHa-extension of L, is a cHa H together with a cHa morphism L $\to$ H. The category of cHa-extensions of L, L-$\underset{\sim}{\text{cHa}}$, has all such arrows L $\to$ H as objects, and a morphism from (L $\to$ H) to (L $\to$ K) is a cHa-morphism H $\to$ K such that the diagram

$$\begin{tikzcd} & H \arrow[d] \\ L \arrow[ur] \arrow[dr] & \\ & K \end{tikzcd}$$

commutes. (If you are familiar with rings, this situation is similar to the category of R-algebras, with R a fixed ring.)

The main aim of this subsection is to prove the theorem saying that the categories $\underset{\sim}{\text{cHa}}$(Sh(L)) and L-cHa are equivalent, in fact by a very natural functor. This theorem is due to Fourman and Scott (see FS, 8.13 Theorem, p. 388) and, independently, to André Joyal (personal communication).

We now define the functor Γ: $\underset{\sim}{\text{cHa}}$(Sh(L)) $\to$ L-$\underset{\sim}{\text{cHa}}$. For any category $\mathcal{C}$ with a terminal object 1, the functor represented by 1, $P = \mathrm{Hom}_{\mathcal{C}}(1,-)$ (or: the global section functor) is the functor Γ: $\mathcal{C} \to$ SET with $\Gamma(A) = \mathrm{Hom}_{\mathcal{C}}(1,A)$, and for $A \xrightarrow{f} B$, $\Gamma(f)$ is the map $\mathrm{Hom}_{\mathcal{C}}(1,A) \to \mathrm{Hom}_{\mathcal{C}}(1,B)$ that associates with any $1 \to A$ the composite $1 \to A \xrightarrow{f} B$. For a Grothendieck topos $\mathcal{C} = \mathcal{E}$, Γ is the right adjoint of the unique continuous SET $\to \Gamma$ ($^\vee$ in the case of $\mathcal{E}$ = Sh(L)), so, in particular, it preserves all projective limits.

The functor, also denoted Γ, from $\mathrm{cHa}(\mathrm{Sh}(L)) \to L\text{-}\mathrm{cHa}$ is derived from $\Gamma: \mathcal{E} \to \mathrm{SET}$ for $\mathcal{E} = \mathrm{Sh}(L)$. - Dealing with $\mathcal{E} = \mathrm{Sh}(L)$, we will talk about L-sets, but will feel free to assume that they are actually 'sheaves', i.e. complete and separated. Some further preparatory remarks:

1. If X is a 'sheaf' (i.e. it is a complete and separated L-set) then a 'global section' $\underbrace{1 \to X}$ is represented by a unique function

an arrow in L-sets,
1: terminal object of L-sets

$\{*\} \to |X|$ such that its value (at *) is a global element of X, i.e. some $x \in |X|$ such that $Ex = 1\ (= 1_L)$; in brief, global sections are the same as global elements.

2. The syntax of the logical language allows the use of individual constants, i.e. zero-ary operations. Given $\mathcal{E}$, the arrows $1 \xrightarrow{c} X$ in $\mathcal{E}$ can be employed in formulas as individual constants; for such a c, 'x = c' or '$x = \underline{c}$' is a well formed formula [and we don't have to write c(y) for c with y a variable of sort 1]. We use the underlining under c in $x = \underline{c}$ to make it clear that c is not a variable here.

3. Given a global element $x \in |X|$ of the L-set X, and a morphism $X \xrightarrow{f} Y$ with Y a 'sheaf', then there is a well determined value of f at x: it is the global element of Y which corresponds to the composite 'global section' $1 \xrightarrow{x} X \xrightarrow{f} Y$ as in 1. We'll denote this 'value' by f(x), and by $|f(x)|$ if we absolutely have to. - It is easy to see that an identity of two terms which is valid in $\mathcal{E}$ gives rise to true equalities between values: if e.g. $\models^{\mathcal{E}}_{x,y} g(f(x)) = h(y)$, then $g(f(x)) = h(y)$ for global elements x and y of the right kind. [Note that since the notion of 'global section' makes sense in any category with 1, this last remark also applies in this generality.]

To define $\Gamma\colon \underset{\sim}{\text{cHa}}(\mathrm{Sh}(L)) \to L\text{-}\underset{\sim}{\text{cHa}}$, let $(\mathcal{H},\leq,1,\wedge,\bigvee) \in |\underset{\sim}{\text{cHa}}(\mathrm{Sh}(L))|$; without loss of generality, $\mathcal{H}$ is a 'sheaf' (why?). The cHa $H = (H,\leq_H,1_H,\wedge_H,\bigvee_H)$ is defined as follows:

$H = \Gamma(\mathcal{H})$ (with $\Gamma\colon \mathrm{Sh}(L) \to \mathrm{SET}$ the global section functor)

$\leq_H = \Gamma(\leq) \hookrightarrow H\times H$ [more precisely: $\leq\colon \mathcal{H}\times\mathcal{H} \to \Omega$ gives rise to its kernel, also denoted by $\leq\colon \leq \hookrightarrow \mathcal{H}\times\mathcal{H}$; applying Γ and noting that it preserves product and monomorphism, we obtain the subobject $\leq_H \hookrightarrow H\times H$]

$\wedge_H = \Gamma(\wedge)\colon\ H\times H \to H$

$1_H = \Gamma(1)\colon\ \underbrace{\mathbb{1}}_{\text{terminal object in SET.}} \to H$

The only slightly involved part is the description of $\bigvee_H$: let $\Phi \subset H$ be any subset of $H = \Gamma(\mathcal{H})$; define the predicate $\langle\Phi\rangle$ as follows: $[\![x \in \langle\Phi\rangle]\!] = \langle\Phi\rangle(x) = [\![\bigvee\!\!\!\bigvee_{c\in\Phi} x = \underline{c}]\!]$ [$\bigvee\!\!\!\bigvee$: (infinite) disjunction sign in place of $\bigvee$, since the latter is becoming overused; the elements of Φ are global sections, hence $\bigvee\!\!\!\bigvee_{c\in\Phi} x = \underline{c}$ is a well-formed formula of the (infinitary) language L_1]; now, $\langle\Phi\rangle \in |\mathcal{P}(\mathcal{H})|$ and in fact, it is a global element of $\mathcal{P}(\mathcal{H})$ [remember the definition of $\mathcal{P}(\mathcal{H})$!], so we can put

$$\bigvee_H \Phi \underset{\mathrm{df}}{=} \bigvee\langle\Phi\rangle$$

with actually $|\bigvee\langle\Phi\rangle|$ in mind. - We now have to verify that H is $\underset{\sim}{\text{cHa}}$; once we have done that, we'll also define a cHa-morphism $L \to H$; and, finally, this will be the value of the functor $\Gamma\colon \underset{\sim}{\text{cHa}}(\mathrm{Sh}(L)) \to L\text{-}\underset{\sim}{\text{cHa}}$ at $\mathcal{H} = (\mathcal{H},\ldots) \in |\underset{\sim}{\text{cHa}}(\mathrm{Sh}(L))|$.

We leave it to the reader to check that $\leq_H$ is a partial ordering, $\wedge_H$ is the g.l.b. operation with respect to $\leq_H$ and 1_H is the greatest element in H. We verify that $\bigvee_H \Phi$ is indeed the sup of Φ: let $x \in H$; then $x \geq \bigvee_H \Phi \Leftrightarrow [\![x \geq \bigvee \langle\Phi\rangle]\!] = 1_L \Leftrightarrow [\![\forall_{y\in H}(y \in \langle\Phi\rangle \to x \geq y)]\!] = 1_L$

$$\underset{\substack{\uparrow\\ \text{by intuitionistic logic}}}{\Leftrightarrow} [\![\forall_{y\in H}(\bigvee_{c\in\Phi} y = \underline{c} \to x \geq y)]\!] = 1 \Leftrightarrow [\![\bigwedge_{c\in\Phi} x \geq \underline{c}]\!] = 1 \Leftrightarrow \text{for all } c\in\Phi,$$

$[\![x \geq \underline{c}]\!] = 1 \Leftrightarrow$ for all $c\in\Phi$, $x_H \geq c$, as required.

We next show that H satisfies the local identity $(\bigvee_H \Phi) \wedge_H x = \bigvee_H (\Phi(\wedge_H)x)$. First of all, we claim that for a global element $x \in H$

$$\langle\Phi(\wedge_H)x\rangle = \underbrace{\langle\Phi\rangle(\wedge)x}_{\substack{\text{the value of the corresponding}\\ \text{term at the global arguments}\\ \langle\Phi\rangle \text{ and } x, \text{ with } \wedge \text{ the}\\ \text{internal intersection of } H.}}$$

Writing $\equiv$ for valid bientailment (equality of truthvalues; $\varphi \equiv \psi$ meaning $[\![\varphi]\!] = [\![\psi]\!]$), we have

$$\begin{aligned} z \in \langle\Phi\rangle(\wedge)x &\equiv \exists_{y\in H}(y \in \langle\Phi\rangle \;\&\; z = y \wedge x) \\ &\equiv \exists_{v\in H}((\bigvee_{c\in\Phi} y = \underline{c}) \;\&\; z = y \wedge x) \\ &\equiv \bigvee_{c\in\Phi} (z = (\underline{c} \wedge x)) \quad . \end{aligned}$$

by (infinitary) intuitionistic logic

On the other hand

$$\begin{aligned} z \in \langle\Phi(\wedge_H)x\rangle &\equiv \bigvee_{d\in\Phi(\wedge_H)x} (z = \underline{d}) \\ &\equiv \bigvee_{c\in\Phi} (z = \underline{c} \wedge x). \end{aligned}$$

From these, our <u>claim</u> above is immediate ('<u>extensionality</u>'). The local identity now follows:

$$(\bigvee_H \Phi)\wedge_H x = (\bigvee\langle\Phi\rangle)\wedge x \underset{\substack{\uparrow \\ \text{the internal} \\ \text{local identity} \\ \text{applied to the} \\ \text{global arguments} \\ \langle\Phi\rangle \text{ and } x.}}{=} \bigvee(\langle\Phi\rangle\wedge x) \underset{\substack{\uparrow \\ \text{claim}}}{=} \bigvee\langle\Phi(\wedge_H)x\rangle$$

$$\underset{\substack{\uparrow \\ \text{def. of } \bigvee_H}}{=} \bigvee_H(\Phi(\wedge_H)x).$$

Having verified that $H(=\Gamma(\mathcal{H}))$ is a cHa, we now define the requisite cHa-morphism $i: L \to H$. Given $U \in L$, let $\{1|U\}$ be an abbreviation for $\{x \in \mathcal{H}: x = 1_{\mathcal{H}} \ \& \ U\}$ (here U is used as a zero-ary predicate, a <u>proposition</u>; it is not a variable), and define i to be the function such that

$$i(U) = \underbrace{\bigvee \underbrace{\{1|U\}}_{\text{a global element of } \mathcal{P}(\mathcal{H})}}_{\therefore \text{ a global element of } \mathcal{H}, \text{ hence an element of } H.}$$

[$\{1|U\}$ is the predicate for which $\{1|U\}(x) = [\![x = 1_{\mathcal{H}}]\!] \ U$] .

Let us verify that i preserves 1, $\leq$, $\wedge$ and $\bigvee$. Leaving 1 and $\leq$ to the reader, we first note that

$$(\bigvee\Sigma_1) \wedge (\bigvee\Sigma_2) = \bigvee(\Sigma_1(\wedge)\Sigma_2)$$

with $x \in \Sigma_1(\wedge)\Sigma_2 \Longleftrightarrow x = y_1 \wedge y_2$ for some $y_i \in \Sigma_i (i = 1,2)$, is a general fact about cHa's, true internally as well (<u>exercise</u>).

$$i(U_1)\wedge_H i(U_2) = (\bigvee\{1|U_1\}) \wedge (\bigvee\{1|U_2\}) = \bigvee(\{1|U_1\}(\wedge)\{1|U_2\})$$

$$= \bigvee\{1|U_1\wedge_L U_2\} = i(U_1\wedge U_2),$$

since $x\in\{1|U_1\}(\wedge)\{1|U_2\} \equiv (\exists_{y_1,y_2\in H}\ x = y_1\wedge y_2\ \&\ \underbrace{y_1\in\{1|U_1\}}_{y_1=1\ \&\ U_1}\ \&\ \underbrace{y_2\in\{1|U_2\}}_{y_2=1\ \&\ U_2}) \equiv$

$\equiv x\in\{1|U_1\wedge_L U_2\}$; this shows that i preserves $\wedge$.

To show that i preserves $\bigvee$, we first note the equivalence:

$$\models_x (x \geq \bigvee\{1|U\} \leftrightarrow (U \to x = 1_H))$$

$(U\in L)$ (<u>exercise</u>).

Then we have: for a family $\langle U_j : j\in I\rangle$ of elements of L, and

$$U = \bigvee^{(L)}\{U_j : j\in L\},$$

$$x \geq i(\bigvee_j U_j) \Leftrightarrow x \geq i(U) \Leftrightarrow x \geq \bigvee\{1|U\} \Leftrightarrow \models U \to x = 1_H$$

above equivalence
and the fact that
now

$$x \geq \bigvee\{1|U\} \Leftrightarrow \models \underline{x} \geq \bigvee\{1|U\}$$

$$\Leftrightarrow \models U_j \to x = 1_H \text{ for all } j\in I \Leftrightarrow x \geq i(U_j) \text{ for all } j\in I$$

elementary
intuitionistic logic; alternatively
$(\bigvee_{i\in I} U_i \leq [\![x = 1_H]\!]) \Leftrightarrow U_i \leq [\![x = 1_H]\!]$ for all $i\in I$

which shows that $i(U)$ answers the description of $\bigvee_j i(U_j)$, i.e.
$i(\bigvee_j U_j) = \bigvee_j i(U_j)$ as claimed.

We have defined the functor Γ: cHa(Sh(L)) $\to$ L-cHa.

Theorem 5.1.1. Γ: cHa(Sh(L) $\to$ L-cHa is an equivalence of categories.

In the next 3 lemmas, let $H = (H, \ldots) \in$ cHa(Sh(L)); assume H is a 'sheaf'.

Lemma 5.1.2. Every $x \in |H|$ is the restriction of some global section, in fact $x \cdot = \cdot (\bigvee\{x\})|Ex$.

[Note that every $x \in |H|$ gives rise to the predicate P on H s.t. $P(y) = [\![y = x]\!]$; denoting this predicate by $\{x\}$, we have that $\{x\}$ is global element of $P(H)$; so $\bigvee\{x\}$ is a global element of H].

Proof: Our assertion is that $[\![x = \bigvee\{x\}]\!] = Ex$.

$$\begin{aligned}
\text{But } y \geq \bigvee\{x\} &\equiv \forall_{z \in H}(z \in \{x\} \to y \geq z) \equiv \\
&\equiv \forall_{z \in H}(z = x \to y \geq z) \\
&\equiv y \geq x
\end{aligned}$$

$$\begin{aligned}
\text{hence } [\![x = \bigvee\{x\}]\!] &= [\![x \geq \bigvee\{x\}]\!] \cdot [\![\bigvee\{x\} \geq x]\!] = \\
&= [\![x = x]\!] [\![\bigvee\{x\} \geq \bigvee\{x\}]\!] \\
&= Ex.1_L = Ex, \text{ as required.} \quad \square
\end{aligned}$$

Notation: For any $x \in |H|$, and $U \in L$, let us write $\{x|U\}$ for the predicate such that $\{x|U\}(z) = [\![z \in \{x|U\}]\!] = [\![z = x]\!] \cdot U$ (check that this is a predicate); $\bigvee\{x|U\}$ then is a global element of H, denote this last by $[x,U]$.

Lemma 5.1.3. $[x,U] = x \wedge_H i(U)$ whenever $x \in H$. (x is global)

Proof: $x \wedge i(U) = x \wedge \bigvee\{z \in H : z = 1_H \ \& \ U\} =$

$$= \underset{\substack{\uparrow \\ \text{by (internal) local} \\ \text{identity in } H}}{} \bigvee(\{z \in H : z = 1_H \ \& \ U\}(\wedge)x) \underset{\substack{\uparrow \\ \text{by logic}}}{=} \bigvee\{u \in H : u = x \ \& \ U\} \underset{\substack{\uparrow \\ \text{def}}}{=} \bigvee\{x|u\} =$$

$$= [x,U]. \quad \square$$

Lemma 5.1.4. Let x,y be global elements in $|H|$; $x,y \in H$; and let $U \in L$. Then

$$U \leq [\![x = y]\!] \text{ if and only if } [x,U] \cdot{=}\cdot [y,U].$$

Proof: We have $[x,U] \cdot{=}\cdot [y,U] \Leftrightarrow$

$\Leftrightarrow [\![\bigvee\{x|U\} = \bigvee\{y|U\}]\!] = 1_L$; on the other hand:

$$\bigvee\{x|U\} = \bigvee\{y|U\} \equiv \forall_{z\in H}(z \geq \bigvee\{x|U\} \leftrightarrow z \geq \bigvee\{y|U\});$$

note that $z \geq \bigvee\{x|U\} \equiv \forall_{v\in H}[v \in \{x|U\} \to z \geq v]$

$$\equiv \forall_{v\in H}[(v = x \ \&\ U) \to z \geq v]$$

$$\equiv U \to z \geq x$$

by logic

therefore $(\bigvee\{x|U\} = \bigvee\{y|U\}) \equiv \forall_{z\in H}((U \to z \geq x) \leftrightarrow (U \to z \geq y))$

$$\equiv U \to \forall_{z\in H}(z \geq x \leftrightarrow z \geq y)$$

by logic

$$\equiv U \to [x = y]$$

since $\geq$ is an (internal) partial order.

Clearly $[\![U \to (x = y)]\!] = 1_L \Leftrightarrow U \leq [\![x = y]\!]$.

Hence $[x,U] \cdot{=}\cdot [y,U] \Leftrightarrow [\![\bigvee\{x|U\} = \bigvee\{y|U\}]\!] = 1$

$\Leftrightarrow U \leq [\![x = y]\!]$ as claimed. □

Claim 5.1.5. The functor Γ: cHa(Sh(L)) $\to$ L-cHa is faithful.

Proof: Let $h\colon \mathcal{H} \to \mathcal{K}$ be in cHa(Sh(L)); let $h_0 = \Gamma(h)\colon H \to K$. For any $x \in |\mathcal{H}|$, we have $[\![x = \bar{x}]\!] = Ex$ for $\bar{x} = \bigvee\{x\}$ (by 5.1.2.); also $E\bar{x} = 1_L$; it follows that $\underset{(y \in |\mathcal{K}|)}{[\![y \ominus h(x)]\!]} = [\![y \ominus h(\bar{x})]\!].Ex$ (why?); on the other hand $|h(\bar{x})| = (\Gamma h)(\bar{x}) \in K$ and $[\![y \ominus h(\bar{x})]\!] = [\![y =_{\mathcal{K}} |h(\bar{x})|]\!]$, hence $[\![y \ominus h(x)]\!] = [\![y = (\Gamma h)(\bar{x})]\!] \cdot Ex$; so, if $h_1, h_2\colon \mathcal{H} \rightrightarrows \mathcal{K}$ are such that $\Gamma h_1 = \Gamma h_2$ (both cHa(Sh(L))-maps), then the last equality applied to both h_1, h_2 as h yields that

$$[\![y \ominus h_1(x)]\!] = [\![y \ominus h_2(x)]\!]$$

i.e. $h_1 = h_2$. $\square$

Claim 5.1.6. The functor Γ is full.

Proof: Let h_0 be a morphism in L-cHa; i.e. we have a commutative diagram

$$\begin{tikzcd} H \arrow[rr, "h_0"] & & K \\ & L \arrow[ul, "i"] \arrow[ur, "i'"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

also, let $L \xrightarrow{i} H$ be the Γ-image of $\mathcal{H} \in$ cHa(Sh(L)); $L \xrightarrow{i'} H = \Gamma(\mathcal{K})$. In particular, assume that $\mathcal{H}$ and $\mathcal{K}$ are sheaves. To show the fullness of Γ, we have to show the existence of $h\colon \mathcal{H} \to \mathcal{K} \in$ cHa(Sh(L)) such that $\Gamma(h) = h_0$. - The following fact that we now claim to hold is clearly necessary for the existence of such an h:

$$(5.1.7.)\quad \left\{ \begin{array}{l} \text{for } x, y \in H = \Gamma(\mathcal{H}), \\ \\ [\![x =_{\mathcal{H}} y]\!] \le [\![h_0(x) =_{\mathcal{K}} h_0(y)]\!] \end{array} \right.$$

To prove this, let $U = [\![x = y]\!]$. Since $U \le [\![x = y]\!]$, by 5.1.4 and 5.1.3 we have $x \wedge_H i(U) \cdot{=}\cdot y \wedge_H i(U)$. Let us apply h_0 to both sides and recall that h_0 preserves $\wedge$ as well as $h \circ i = i'$; we get $h_0(x) \wedge_K i'(U) \cdot{=}\cdot h_0(y) \wedge_K i'(U)$; applying 5.1.4 and 5.1.3 again (in the opposite direction and to $\mathcal{K}$ in place of $\mathcal{H}$), we get $U \le [\![h_0(x) =_{\mathcal{K}} h_0(y)]\!]$ as required.

For an arbitrary $x \in |\mathcal{H}|$, let $\bar{x} = \bigvee\{x\}(\in H)$ and define $\bar{h}_0(x) = h_0(\bar{x})$; we have $\bar{h}_0\colon |\mathcal{H}| \to |\mathcal{K}|$; $\bar{h}_0$ represents a morphism $h\colon \mathcal{H} \to \mathcal{K}$ in $\mathrm{Sh}(L)$ since

$$\underset{(\text{why?})}{[\![x =_{\mathcal{H}} y]\!] \le} [\![\bar{x} = \bar{y}]\!] \underset{\text{by } 5.1.7}{\le} [\![h_0(\bar{x}) = h_0(\bar{y})]\!] = [\![\bar{h}_0(x) = \bar{h}_0(y)]\!]$$

We have $[\![y \in h(x)]\!] = [\![y =_{\mathcal{K}} h_0(\bar{x})]\!].Ex$ (as we know). – It is obvious that $\Gamma(h) = h_0$ ($\mathcal{H}$ & $\mathcal{K}$ are sheaves) (why?); it remains to verify that $h\colon \mathcal{H} \to \mathcal{K}$ is a $\mathrm{cHa}(\mathrm{Sh}(L))$-morphism.

Showing that h preserves $\le$, 1 and $\wedge$ is easy – and it can be left safely an exercise. Preservation of $\bigvee$ is more complicated; somehow we have to get a hold of $\bigvee\Sigma$ for an arbitrary predicate $\Sigma \in |P(\mathcal{H})|$ through sups like $\bigvee_H \Phi$ of ordinary subsets $\Phi \subset H$. We now make some crucial definitions.

Let, for any predicate $\Sigma \in |P(\mathcal{H})|$ and any $x \in H$, $x_\Sigma \underset{\mathrm{df}}{=} [x, [\![x \in \Sigma]\!]] = x \wedge_H i([\![x \in \Sigma]\!]) = \bigvee\{x \mid [\![x \in \Sigma]\!]\}$ (see notation and 5.1.3. above; this definition can be made with any $\mathcal{H} \in \mathrm{cHa}(\mathrm{Sh}(L))$; we will use it for $\mathcal{K}$ as well). Next, the set Φ_Σ of global elements, $\Phi_\Sigma \subset H$, is defined as $\Phi_\Sigma \underset{\mathrm{df}}{=} \{x_\Sigma\colon x \in H\}$. The first step in getting a hold of $\bigvee\Sigma$ is

Lemma 5.1.7'. $\bigvee \Sigma \cdot = \cdot \bigvee \langle \Phi_\Sigma \rangle$.

[Of course, both Σ and $\langle \Phi_\Sigma \rangle$ (see bottom of p. 5.1.5) are global elements of $P(H)$; hence the two sides of the equality in 5.1.7' are global elements of $|H|$].

Proof of 5.1.7'. Let $x \in H = \Gamma(H)$. We have $[x,U] = \bigvee\{z \in H: (z = x) \,\&\, \mathbb{U}\}$; hence it is easy to see that $U \leq [\![[x,U] = x]\!]$ (exercise). Applied to $U = [\![x \in \Sigma]\!]$, this gives us $[\![x \in \Sigma]\!] \leq [\![x = x_\Sigma]\!]$; clearly (look at the definition!) $[\![x = x_\Sigma]\!] \leq [\![x \in \langle \Phi_\Sigma \rangle]\!]$; this shows $[\![x \in \Sigma]\!] \leq [\![x \in \langle \Phi_\Sigma \rangle]\!]$ holds for all global x. For an arbitrary $x \in |H|$, we have $\bar{x} \in H$ such that $Ex = [\![x = \bar{x}]\!]$ (by 5.1.2); so

$$\underbrace{[\![x \in \Sigma]\!] \leq [\![\bar{x} \in \Sigma]\!]}_{\substack{\uparrow \\ \text{because } \Sigma \text{ is a} \\ \text{predicate}}} .Ex \leq [\![\bar{x} \in \langle \Phi_\Sigma \rangle]\!].Ex \underbrace{=}_{\substack{\uparrow \\ \text{because } \langle \Phi_\Sigma \rangle \text{ is} \\ \text{a predicate}}} [\![x \in \langle \Phi_\Sigma \rangle]\!] \quad .$$

What we have shown is $[\![\Sigma \subset \langle \Phi_\Sigma \rangle]\!] = 1_L$; it immediately follows that the left-hand-side in 5.1.7' is $\leq$ r.h.s.

Next, look at any $x \in H$ again. Clearly, $[\![\{x | [\![x \in \Sigma]\!]\} \subset \Sigma]\!] = 1_L$ (membership in $\{x | [\![x \in \Sigma]\!]\}$ implies, by definition, membership in Σ; exercise), so $x_\Sigma = \bigvee \{x | [\![x \in \Sigma]\!]\} \leq \bigvee \Sigma$ (with value 1_L); looking at the definition of Φ_Σ: $[\![y \in \langle \Phi_\Sigma \rangle]\!] = [\![\bigvee\!\!\!\bigvee_{x \in H} y = \underline{x_\Sigma}]\!]$, we immediately see now that the r.h.s. in 5.1.7' is $\leq$ l.h.s.

$\square$ for 5.1.7

We need another lemma before we can prove that h preserves $\bigvee$ (internally).

Lemma 5.1.8. With our previous notation

$$\bigvee \langle h_0[\Phi_\Sigma] \rangle = \bigvee \langle \Phi_{h[\Sigma]} \rangle$$

[<u>explanation</u>: $h_0[\Phi]$ means the ordinary image of the set Φ under h_0: $h_0[\Phi] = \{h_0(c): c\in\Phi\}$. $h[-]$ is taking the image internally; hence $h[\Sigma]$ is the predicate such that $[\![z\in h[\Sigma]]\!] = [\![\exists_{y\in H}(y \in \Sigma \;\&\; z = h(y))]\!]$. For the meaning of $\Phi_{(-)}$ and <->, see bottom of p. 5.11 and p.4. The sup signs on both sides refer, of course, to the sup morphism $\bigvee^K: P(K) \to K$].

<u>Proof</u>: We make two preparatory remarks:

(5.9) $$z \in h[\Sigma] \equiv \exists_{y\in H}(y \in \Sigma \;\&\; z = h(y))$$

$$\equiv \exists_{y\in H}(y \in \Sigma \;\&\; z = \underline{h_0(\bar{y})})$$

(recall: $Ey = [\![y = \bar{y}]\!]$, $[\![z \circleddash h(y)]\!] = [\![z = h_0(\bar{y})]\!].Ey$)

$$\equiv \bigvee_{c\in H} (\underline{c} \in \Sigma \;\&\; z = \underline{h_0(c)}).$$

(reasons are the same)

(5.10) for any $x\in H$, $[\![x\in\Sigma]\!] \leq_L [\![h_0(x) \leq_K h_0(x_\Sigma)]\!]$:

to show this, first of all recall that

$$i'([\![x\in\Sigma]\!]) = \bigvee\{v\in K: v = 1_K \;\&\; [\![x\in\Sigma]\!])$$

hence $[\![x\in\Sigma]\!] \leq [\![i'([\![x\in\Sigma]\!]) = 1_K]\!]$; (*)

now, $x_\Sigma = x \wedge i([\![x\in\Sigma]\!])$, hence

$$h_0(x_\Sigma) = h_0(x) \wedge i'([\![x\in\Sigma]\!])$$

(since h_0 preserves $\wedge$, and $h\circ i = i'$).

By (*), $[\![x\in\Sigma]\!] \leq [\![h_0(x) \leq_K i'([\![x\in\Sigma]\!])]\!]$.

From the expression for $h_0(x_\Sigma)$, now the assertion is clear.

Now, to show that l.h.s. $\leq$ r.h.s. in 5.1.8, let $x \in H$ be arbitrary, $y \underset{df}{=} h_0(x)$. Since $x_\Sigma = x \wedge i([\![x \in \Sigma]\!])$, we have $h_0(x_\Sigma) = h_0(x) \wedge i'([\![x \in \Sigma]\!])$ (since h_0 preserves $\wedge$, and $h \circ i = i'$). On the other hand, $y_{h[\Sigma]} = y \wedge i'([\![y \in h[\Sigma]]\!]) = h_0(x) \wedge i'([\![h_0(x) \in h[\Sigma]]\!])$.

Now, clearly $[\![x \in \Sigma]\!] \leq_L [\![h_0(x) \in h[\Sigma]]\!]$, hence

$$i'([\![x \in \Sigma]\!]) \leq_K i'([\![h_0(x) \in h[\Sigma]]\!]);$$

and so $h_0(x_\Sigma) \leq_K y_{h[\Sigma]}$;

clearly, $y_{h[\Sigma]} \leq \bigvee \langle \Phi_{h[\Sigma]} \rangle$ (why?)

hence $h_0(x_\Sigma) \leq_K \bigvee \langle \Phi_{h[\Sigma]} \rangle$

hence $\models \forall_{z \in K}(z \in \langle h_0[\Phi_\Sigma] \rangle \to z \leq_K \bigvee \langle \Phi_{h[\Sigma]} \rangle)$ (why?)

and finally, $\models \bigvee \langle h_0[\Phi_\Sigma] \rangle \leq_K \bigvee \langle \Phi_{h[\Sigma]} \rangle$

as we promised.

Next, we show that r.h.s. $\leq$ l.h.s. in 5.1.8. Denoting the l.h.s. in 5.1.8 by z_0, by (5.1.10) and the definition of z_0, it is clear that for any $x \in H$

$$[\![x \in \Sigma]\!] \leq_L [\![h_0(x) \leq_K z_0]\!];$$

Now, let $z \in K$ be arbitrary.

We conclude by the last line that for any $x \in H$

$$[\![z = \underline{h_0}(x) \ \& \ \underset{-}{x} \in \Sigma]\!] \leq_L [\![z \leq_K z_0]\!];$$

hence

$$[\![\bigvee\!\!\!\bigvee_{x \in H} (z = \underline{h_0}(x) \wedge \underset{-}{x} \in \Sigma)]\!] \leq_L [\![z \leq_K z_0]\!];$$

By (5.1.9), we obtain

$$[\![z \in h[\Sigma]]\!] \leq_L [\![z \leq_K z_0]\!];$$

i.e.

$$\models (z \in h[\Sigma] \to z \leq_K z_0) .$$

Since $\{z|[\![z \in h[\Sigma]\} = \{z' \in K\colon z' = z \;\&\; z \in h[\Sigma]\}$, we obtain

$$\models_{z'} z' \in \{z|[\![z{\in}h[\Sigma]\} \to z' \leq_K z_0.$$

Now, $z_{h[\Sigma]} = \bigvee\{z|[\![z{\in}h[\Sigma]]\!]\}$; it follows that

$$\models z_{h[\Sigma]} \leq_K z_0.$$

This last inequality being true for all $z{\in}K$, a glance at the definition of the r.h.s. in 5.1.8 shows that r.h.s. $\leq_K z_0$, as required; this completes the proof of 5.1.8.

Having the two Lemmas 5.1.7 & 5.1.8, we can now finish the proof of h preserving $\bigvee$ (internally) as follows: we show that for any $\Sigma \in |\mathcal{P}(H)|$, we have $\bigvee^{(K)}(h[\Sigma]) = h(\bigvee^{(H)}(\Sigma))$ (remember, Σ is a global element of $\mathcal{P}(H)$, here we mean the literal equality of global elements of K, or equivalently, that the equality has value 1_L); this will be clearly sufficient (why?).

$$\bigvee^{(K)}(h[\Sigma]) \underset{\substack{\uparrow \\ \text{by 5.1.7}}}{=} \bigvee^{(K)}\langle\Phi_{h[\Sigma]}\rangle$$

$$\underset{\substack{\uparrow \\ \text{by 5.1.8}}}{=} \bigvee^{(K)}\langle h_0[\Phi_\Sigma]\rangle$$

$$\underset{\substack{\uparrow \\ \text{by the definition of} \\ \bigvee^K \text{ derived from } K}}{=} \bigvee^K(h_0[\Phi_\Sigma])$$

$$= h_0\left(\bigvee^H(\Phi_\Sigma)\right)$$

$\uparrow$ since h_0 preserves $\bigvee$

$$= h_0\left(\bigvee^{\mathcal{H}}\langle\Phi_\Sigma\rangle\right)$$

$\uparrow$ def. of $\bigvee^H$

$$= h_0\left(\bigvee^{\mathcal{H}}(\Sigma)\right) = h\left(\bigvee^{\mathcal{H}}(\Sigma)\right),$$

$\uparrow$ by 5.1.7 $\qquad$ $\uparrow$ since $h_0 = \Gamma(h)$

$\square$ for 5.1.6

The final task in the proof of 5.1.1 is to show

Claim 5.1.11. For every $L \xrightarrow{i} H$ in L-cHa, there is $\mathcal{H}$ such that $\Gamma(\mathcal{H}) \simeq (L \xrightarrow{i} H)$.

Proof: Let $L \xrightarrow{i} H$ in L-cHa be given. We define $(\mathcal{H}, \leq, 1, \wedge, \bigvee)$ as follows. $\mathcal{H}$ is the L-set for which $|\mathcal{H}| = H$; we define $[\![x \leq y]\!] = \bigvee\{U \in L: x \wedge_H i(U) \leq y \wedge_H i(U)\}$, and of course $[\![x=y]\!] = [\![x \leq y]\!].[\![y \leq x]\!] = \bigvee\{U \in L: x \wedge_H i(U) = y \wedge_H i(U)\}$. It follows that $[\![x \leq y]\!]$ is the maximal element U in L such that $x \wedge_H i(U) \leq y \wedge_H i(U)$, similarly for '='. Namely, if $U_0 = [\![x \leq y]\!]$, then

$$\begin{aligned}
x \wedge_H i(U_0) &= x \wedge_H i\left(\bigvee\{U \in L: x \wedge_H i(U) = y \wedge_H i(U)\}\right) \\
&= x \wedge_H \bigvee^{(H)}\{i(U): x \wedge_H i(U) = y \wedge_H i(U)\} \\
&= \bigvee^{(H)}\{x \wedge_H i(U): x \wedge_H i(U) = y \wedge_H i(U)\} \\
&= \bigvee^{(H)}\{y \wedge_H i(U): x \wedge_H i(U) = y \wedge_H i(U)\} \\
&= \ldots = y \wedge_H i(U_0).
\end{aligned}$$

We denote (as before) $x \wedge_H i(U)$ by $[x,U]$. - It is easy to see that $\mathcal{H}$ is an L-set, every element of $|\mathcal{H}|$ is global, and $(\mathcal{H},\leq)$ is a partially ordered set (internally in L-sets).

A few more remarks: What we said above about $[\![x\leq y]\!]$ being the maximal element U such that ... can be rephrased by saying:

$$x \wedge_H i(U) \leq_H y \wedge_H i(U) \Leftrightarrow U \leq_L [\![x\leq y]\!]$$

$$x \wedge_H i(U) = y \wedge_H i(U) \Leftrightarrow U \leq_L [\![x=y]\!] .$$

It immediately follows that

$$[\![x\in\Sigma]\!] \leq [\![x=x_\Sigma]\!]$$

and also, that $x \leq_H y \Leftrightarrow [\![x\leq y]\!] = 1_L$.

It is easy to see that the morphism $\wedge\colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ represented by $\wedge_H\colon H \times H \to H$ is the (internal) intersection for $(\mathcal{H},\leq)$; $1_{\mathcal{H}}$ is 1_H. To describe $\bigvee^{(\mathcal{H})}$, for $\Sigma \in |\mathcal{P}(\mathcal{H})|$ and $x\in H$ define $x_\Sigma \underset{df}{=} x \wedge_H i([\![x\in\Sigma]\!])$ (as before), $\Phi_\Sigma \underset{df}{=} \{x_\Sigma\colon x\in H\}$ and $\bigvee^{(\mathcal{H})}\Sigma \underset{df}{=} \bigvee^{(H)}\Phi_\Sigma$. We next verify that, indeed, $\bigvee^{(\mathcal{H})}\Sigma$ so defined is the (internal) sup of Σ.

Let $y \in H$ be arbitrary, $x_0 \underset{\text{df}}{=} \bigvee^{(H)}(\Phi_\Sigma)$; $U \underset{\text{df}}{=} [\![\forall_{x\in H}(x\in\Sigma \to x \le y)]\!]$. Our assertion is equivalent to saying that $U = [\![x_0 \le y]\!]$.

We have $[\![x\in\Sigma]\!] \le_L [\![x=x_\Sigma]\!]$ and $[\![x_\Sigma \le x_0]\!] = 1_L$; this clearly shows that $[\![x_0 \le y]\!] \le_L U$.

To show the reverse inequality, start with $x \wedge_H i([\![x\le y]\!]) \le_H y$ (see above). Intersecting with $i(U)$, and using $i([\![x\le y]\!].U) = i([\![x\le y]\!]).i(U)$, we obtain

$$x \wedge_H i([\![x\le y]\!].U) \le_H y \wedge_H i(U).$$

Since $[\![x\in\Sigma]\!].U \le_L [\![x\le y]\!].U$ by the definition of U, we obtain

$$x \wedge_H i([\![x\in\Sigma]\!].U) \le_H y \wedge_H i(U)$$

or

$$\underbrace{x \wedge_H i([\![x\in\Sigma]\!])}_{x_\Sigma} \wedge_H i(U) \le_H y \wedge_H i(U)$$

$$\therefore \bigvee^{(H)}\{x_\Sigma \wedge_H i(U) : x\in H\} \le_H y \wedge_H i(U)$$

$$\underbrace{\bigvee^{(H)}\{x_\Sigma : x\in H\}}_{x_0} \wedge i(U) \le_H y \wedge_H i(U)$$

$$\therefore U \le [\![x_0 \le y]\!], \text{ as required; thus}$$

we have verified $\bigvee^{(H)}$.

Next, we verify the internal local identity for H. This is equivalent to saying that, for any $\Sigma \in |P(H)|$ and $x\in H$, we have $(\bigvee^{(H)}\Sigma) \wedge x \le \bigvee^{(H)}(\Sigma(\wedge)x)$ (remember that the opposite inequality is automatic, and that 'all elements are global' now), or equivalently

(!)⇒ $$(\bigvee^{(H)}\Phi_\Sigma) \wedge x \le \bigvee^H \Phi_{\Sigma(\wedge)x} .$$

We have $(\bigvee^{(H)}\Phi_\Sigma) \wedge x = \bigvee^{(H)}(\Phi_\Sigma(\wedge)x)$ by the local identity for H. Now, let $y \in \Phi_\Sigma(\wedge)x$ be arbitrary, i.e.

$$y = z_\Sigma \wedge x = z \wedge i[\![z\in\Sigma]\!] \wedge x.$$

Let $y' \underset{df}{=} (z\wedge x) \wedge i[\![(z\wedge x) \in \Sigma(\wedge)x]\!]$; clearly, $y' \in \Phi_{\Sigma(\wedge)x}$. Since $[\![z\in\Sigma]\!] \le [\![(z\wedge x) \in \Sigma(\wedge)x]\!]$, we conclude $y \le_H y' \le_H \bigvee^H \Phi_{\Sigma(\wedge)x}$. Since $y \in \Phi_\Sigma(\wedge)x$ was arbitrary, it follows that $(\bigvee^H \Phi_\Sigma) \wedge x = \bigvee^H(\Phi_\Sigma(\wedge)x) \le \bigvee^H \Phi_{\Sigma(\wedge)x}$ as required.

Having verified that $\mathcal{H}$ is an (internal) cHa in $\mathcal{E}$, now we check that $\Gamma(\mathcal{H}) \simeq (L \xrightarrow{i} H)$.

Lemma 5.1.12. The correspondence $x \longmapsto$ the arrow $1 \longrightarrow \mathcal{H}$ in L-sets represented by the function $* \longmapsto x$ is a 1-1 correspondence of H with $\mathrm{Hom}_{L\text{-sets}}(1,\mathcal{H})$.

Proof: Let $f\colon 1 \longrightarrow \mathcal{H}$ be any arrow in L-sets; define $|f| \underset{df}{=} \bigvee^{(H)}(x\wedge i[\![x \ominus f]\!])$ [we write $[\![x \ominus f]\!]$ instead of $[\![x \ominus f(*)]\!]$, with $*$ the unique element of $|1_{L\text{-sets}}|$]. We claim: $[\![|f| \ominus f]\!] = 1_L$. Indeed, for any $x, x' \in H$, we have: $[\![x' \ominus f]\!].[\![x \ominus f]\!] \le [\![x = x']\!]$ hence: $x' \wedge i([\![x' \ominus f]\!][\![x \ominus f]\!]) = x \wedge i([\![x' \ominus f]\!][\![x \ominus f]\!])$ (by using the 'basic equivalences', p. 5.17). Since the right hand side is $(x \wedge i[\![x \ominus f]\!]) \wedge i[\![x' \ominus f]\!]$ we get $x' \wedge i([\![x' \ominus f]\!]) \wedge i([\![x \ominus f]\!]) \le x \wedge i([\![x \ominus f]\!])$ hence

$$\Big(\bigvee_{x'\in H}(x' \wedge i[\![x' \ominus f]\!])\Big) \wedge i[\![x \ominus f]\!] \le x \wedge i[\![x \ominus f]\!]$$

or

$$|f| \wedge i[\![x \ominus f]\!] \le x \wedge i[\![x \ominus f]\!] .$$

By the 'basic equivalences",

$$[\![x \ominus f]\!] \leq_L [\![\,|f| \leq x]\!].$$

On the other hand, it is easy to see that

$$[\![x \ominus f]\!] \leq_L [\![x \leq |f|\,]\!];$$

hence

$$[\![x \ominus f]\!] \leq [\![\,|f| = x]\!] .$$

Since $[\![\,|f| = x]\!][\![x \ominus f]\!] \leq [\![\,|f| = f]\!]$, we obtain $[\![x \ominus f]\!] \leq [\![\,|f| \ominus f]\!]$;
taking sup on the left

$$1_L = \bigvee_{x \in H}^{(L)} [\![x \ominus f]\!] \leq [\![\,|f| \ominus f]\!], \text{ as}$$

claimed. - This means, as it is immediate, that $f\colon 1 \longrightarrow H$ is represented by $* \longmapsto |f|$; showing that the described correspondence is onto; one-oneness is left to the reader.

$\square$ for 5.1.12.

Now, we remind the reader of the definition of $\Gamma(H)$. $\Gamma(H) = L \xrightarrow{i'} H'$, where H' is the partially ordered set $\langle H',\leq'\rangle$ with $H' = \Gamma(H) = \mathrm{Hom}_{L\text{-sets}}(1,H)$, $\leq' = \Gamma(\leq)$, with $\leq$ understood as the subobject $\leq \hookrightarrow H \times H$; moreover, i' is defined as follows: with any $U \in L$, we associate the predicate $\{1|u\} \in |P(H)|$ (such that $[\![x \in \{1|u\}]\!] = [\![x = 1_H]\!].U$), then we consider the corresponding arrow $1 \xrightarrow{\{1|U\}} P(H)$ (represented by $* \longmapsto \{1|U\}$), and finally, we consider the composite $1 \xrightarrow{\{1|U\}} P(H) \xrightarrow{\bigvee^{(H)}} H$; this is the arrow $1_{L\text{-sets}} \longrightarrow H$ $(\in \Gamma(H))$ that we designate as $i'(U)$.

Let us call the function defined in 5.1.12 by $g\colon H \longrightarrow H'$; 5.1.12 says that g is one-one and onto; we leave it to the reader to check that it is actually an isomorphism of the partially ordered sets $(H,\leq_H)$ and $(H',\leq')$; hence, of course, g is also a cHa-morphism, and an isomorphism in the category cHa. We claim that the triangle

$$\begin{tikzcd} & H \arrow[d, "g"] \\ L \arrow[ur, "i"] \arrow[r, "i'"'] & H' \end{tikzcd}$$

commutes; this will finish proving that, in the category L-cHa, $(L \xrightarrow{i} H)$ is isomorphic to $\Gamma(H)$.

Let $U \in L$. $i'(U)$, i.e. the composite arrow $1 \xrightarrow{\{1|U\}} P(H) \xrightarrow{\bigvee^{(H)}} H$, is represented by

$$* \longmapsto \bigvee{}^{(H)}(\{1|U\});$$

according to our description of $\bigvee^{(H)}$,

$$\bigvee{}^{(H)}\{1|U\} = \bigvee{}^{(H)}\Phi_{\{1|U\}} = \bigvee_{x \in H}{}^{(H)}\{x \wedge i[\![x \in \{1|U\}]\!]\}$$

$$= \bigvee_{x}{}^{(H)}\{x \wedge i[\![x = 1_H]\!] \wedge i(U)\} \underset{df}{=} x_0 .$$

We will be finished once we see that this last quantity equals $i(U)$. It is trivial that, for each $x \in H$,

$$x \wedge i[\![x = 1_H]\!] \wedge i(U) \leq i(U), \quad \text{hence}$$

$$x_0 \leq i(U);$$

On the other hand, taking $x = i(U)$, we see that the term $x \wedge i[\![x = 1_H]\!] \wedge i(U)$ in x_0 becomes equal $i(U)$ (exercise). This completes the proof.

We have finished the proof of Claim 5.1.11, and thereby the proof of Theorem 5.1.1 as well. □

5.2 Restating matters in the sheaf theoretic language

Let L be a cHa, and consider the category of sheaves, Sh(L), over L. With Γ: Sh(L) $\longrightarrow$ SET the global section functor (see), Γ induces an equivalence

$$\Gamma:\ \mathrm{cHa}(\mathrm{Sh}(L)) \xrightarrow{\sim} L\text{-cHa};$$

this is a consequence of 5.1.1, and the existence of a (natural) isomorphism between Γ: Sh(L) $\longrightarrow$ SET and the composite $\mathrm{Sh}(L) \xrightarrow[\text{[-]}]{\sim} L\text{-sets} \xrightarrow{\Gamma} \mathrm{SET}$. In this subsection, we restate some of the detailed results of the last section. It turns out that the picture obtained that way is more appealing geometrically than the one in 5.1. No proofs will be given; the proofs are routine by actually examining the above isomorphism of functors.

$\boxed{(1)}$ Here is how $\Gamma(\mathcal{H}) = L \xrightarrow{i} \mathrm{H}$ looks for a cHa-object $\mathcal{H}$ in Sh(L). [We write $\mathcal{H}$ for a cHa, in $\mathcal{E}$, using then $\mathcal{H}$ for underlying set-object; similarly for H.]

$$\mathcal{H} = \langle \mathcal{H}, \le, \ldots\rangle;\qquad \mathrm{H} = \langle H, \le_H, \ldots\rangle$$

$$H = \Gamma(\mathcal{H}) \qquad\qquad (\Gamma:\ \mathrm{Sh}(L) \to \mathrm{SET})$$

$$\le \hookrightarrow \mathcal{H}\times\mathcal{H}, \quad \text{so} \quad \Gamma(\le) \text{ is a subset of } H\times H;$$

$$\le_H = \Gamma(\le)$$

$i: L \longrightarrow \mathrm{H}$ is defined as follows. We have the supremum morphism (of $\mathcal{H}$)

$$\bigvee : \mathcal{P}(\mathcal{H}) \longrightarrow \mathcal{H}$$

(for $\mathcal{P}(\mathcal{H})$, etc., see). Let $U \in L$.

Define $h^{(U)} \in (\mathcal{P}(\mathcal{H}))(1_L)$: $h^{(U)}$ is a nat. transf. $\mathcal{H} \to \Omega$; for $x \in \mathcal{H}(V)$,

$$(h^{(U)})_V(x) = \bigvee\{W \le V : x \restriction W = 1_{\mathcal{H}} \restriction W\}$$

$$= \text{the largest } W \le V \text{ such that}$$
$$x \restriction W = 1_{\mathcal{H}} \restriction W \in \Omega(V);$$

put $i(U) = \bigvee_{(1_L)}(h^{(U)})$.

$\boxed{(2)}$ Let $L \xrightarrow{i} H$ be an object of L-cHa. We construct $\mathcal{H}$, denoted $sh(L \xrightarrow{i} H)$ (or: $sh(i)$, or even $sh(H)$), as follows: define $\mathcal{H}$: $L^{op} \longrightarrow SET$ by

$$\underline{\mathcal{H}(U) = H|i(U)} \qquad (= \{x \in H : x \le i(U)\});$$

for $x \in \mathcal{H}(U)$, $V \le U$

$$x \restriction V = x \wedge_H i(V) \qquad (\in \mathcal{H}(V)).$$

$\mathcal{H}$ turns out to be a <u>sheaf</u>.

The subpresheaf $\le \hookrightarrow \mathcal{H} \times \mathcal{H}$ is defined as follows: for $x, y \in \mathcal{H}(U)$,

$$\underline{\langle x,y\rangle \in (\le(U)) \Longleftrightarrow x \le_H y}$$

Now, examining the description of Γ: cHa(Sh(L)) $\to$ L-cHa given in $\boxed{(1)}$, we see that

$$\Gamma(\mathrm{sh}(L \xrightarrow{i} H)) = (L \xrightarrow{i} H)$$

literally (and not just up to iso.).

$\boxed{(3)}$ Let us look at a morphism in L-cHa, i.e. a commutative triangle

$$\begin{tikzcd} H \arrow[rr, "h"] & & K \\ & L \arrow[ul, "i"] \arrow[ur, "i'"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

We have, by the main theorem, a unique

$$H \xrightarrow{\tilde{h}} K$$

(where $\mathcal{H} = \mathrm{sh}(L \xrightarrow{i} H)$, $\mathcal{K} = \mathrm{sh}(L \xrightarrow{i'} K)$ such that $\Gamma(\tilde{h}) = h$.
$\tilde{h}$ is a natural transformation

$$\tilde{h}\colon \mathcal{H} \longrightarrow \mathcal{K}$$

(with additional properties). - Notice that due to the special definition of $\mathcal{H}$, we have that each $\mathcal{H}(U)$ is a subset of $H = \Gamma(\mathcal{H})$; $\mathcal{H}(U) = H|i(U) \subset H$; similarly for $\mathcal{K}$. We claim that $\tilde{h}$ is given as follows: for $U \in L$,

$$\tilde{h}_U = h \restriction \mathcal{H}(U).$$

Let $x \in \mathcal{H}(U)$; then also $x \in H = \mathcal{H}(1)$; also $x \restriction U = x \wedge i(U) = x$ since $x \leq i(U)$; since the triangle

$$\begin{tikzcd} & H \arrow[dd, "h"] \\ L \arrow[ur, "i"] \arrow[dr, "i'"'] & \\ & K \end{tikzcd}$$

commutes, and h preserves $\leq$, from $x \leq i(U)$ it follows that $h(x) \leq i'(U)$, i.e. $h(x) \in \mathcal{K}(U)$; so $\tilde{h}_U(x) = \tilde{h}_U(x \upharpoonright U) = \tilde{h}_1(x) \upharpoonright U = h(x) \upharpoonright U \underset{\uparrow}{=} h(x)$

since $h(x) \in \mathcal{K}(U)$

proving our claim.

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(4) <u>Summary</u>: (i) every internal $cHa \in sh(L)$ is isomorphic to one of the form $sh(L \xrightarrow{i} H)$;

(ii) for $\mathcal{H} = sh(L \xrightarrow{i} H)$,

$\mathcal{K} = sh(L \xrightarrow{i'} K)$

we have $H = \Gamma(\mathcal{H})$, $K = \Gamma(\mathcal{K})$ and Γ induces a bijection between

$$\mathrm{Hom}_{\underset{\sim}{cHa}(Sh(L))}(\mathcal{H}, \mathcal{K}) \quad \text{and}$$

$$\mathrm{Hom}_{L\text{-}\underset{\sim}{cHa}}((L \xrightarrow{i} H), L \xrightarrow{i'} K);$$

(iii) in fact, given any h with

$$\begin{tikzcd}
 & H \arrow[dd, "h \ \in \mathrm{Hom}(i,i')"] \\
L \arrow[ur, "i"] \arrow[dr, "i'"'] \arrow[r, phantom, "\circlearrowright"] & \\
 & K
\end{tikzcd}$$

then the $\tilde{h} \in \mathrm{Hom}(\mathcal{H}, \mathcal{K})$ with $\Gamma(\tilde{h}) = h$ is given "by restriction":

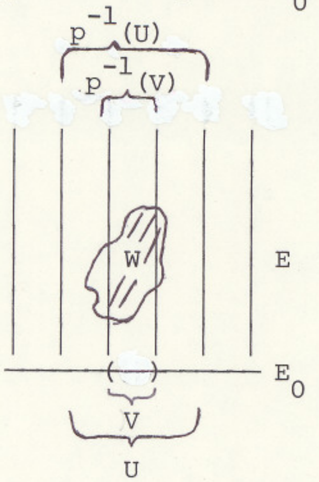
$$\tilde{h}_U = h \upharpoonright (H | i(U)).$$

Consider the special case of spaces

$$E \xrightarrow{p} E_0$$

$$E' \xrightarrow{p'} E_0$$

over a fixed top-space E_0; let $L = \mathcal{O}(E_0)$, $H = \mathcal{O}(E)$, $K = \mathcal{O}(E')$, $i = p^{-1}$, $i' = (p')^{-1}$. Then $\mathcal{H} = \mathrm{sh}(L \xrightarrow{i} H)$ is the sheaf whose U-sections are those open sets of E that are contained in the 'cylinder' $p^{-1}(U)$ (U open $\subset E_0$)



If $W \in \mathcal{H}(U)$, then $W \mid V$ (for $V \leq U$) is the intersection $W \wedge p^{-1}(V)$.

Given a continuous map $E' \xrightarrow{f} E$, this induces $h = f^{-1}\colon H \longrightarrow K$; if f is a <u>fiber map</u> in the sense that $y \in \underbrace{p'^{-1}(\{x\})}_{\text{fiber over } x \in E_0}$ implies $f(y) \in p^{-1}(\{x\})$

then the diagram

$$\begin{tikzcd} H \arrow[rr, "h"] & & K \\ & L \arrow[ul, "i"] \arrow[ur, "i'"'] & \end{tikzcd}$$

will commute, and therefore will give rise to an internal cHa-morphism $\mathcal{H} \to \mathcal{K}$.

5.3. The object of cHa-morphisms

Let $\mathcal{E}$ be an arbitrary topos, $(H), (K) \in \mathrm{cHa}(\mathcal{E})$. We have defined the notion of morphism $(H) \longrightarrow (K)$; but we have not yet introduced the natural notion of the <u>object</u> of cHa-morphisms, $\mathrm{Hom}^*((H),(K))$; this should be a subobject of the exponential K^H (a <u>subsheaf</u> if $\mathcal{E} = \mathrm{Sh}(L)$). - We have a formula

$$M(z) = M_{(H),(K)}(z)$$

in the language of $\mathcal{E}$, with variable z of sort the object K^H that is abbreviated

"z is a map $H \to K$ that preserves $1, \leq, \wedge$ and $\bigvee$"

(the reader can write this out in detail); we define $\mathrm{Hom}^*((H),(K))$ as the extension of this formula:

$$\mathrm{Hom}^*((H),(K)) := [z\colon M(z)] \hookrightarrow K^H.$$

The connection with the ordinary notion of morphism is as follows. Let $h\colon H \to K$, an arrow in $\mathcal{E}$; let $\underline{h}$ be its exponential transpose $1 \longrightarrow K^H$. Then the sentence $M(\underline{h})$ (substitution for z) is grammatically correct. Now, h belongs to $\mathrm{Hom}_{\mathrm{cHa}(\xi)}((H),(K))$ just in case

$$\models^{\xi} M(\underline{h})$$

holds; inspection of our original definition will show this.

We want to point out a vivid description of the "object of cHa-morphisms" in $\mathcal{E} = \mathrm{Sh}(L)$; we will apply this later to identify the <u>real number object</u> (and other similar things) in such toposes.

Let $\mathcal{E} = \mathrm{Sh}(L)$, L a cHa; let $L \xrightarrow{i} H$, $L \xrightarrow{i'} K$ L-cHa's; form

$$\mathcal{H} = \mathrm{sh}(L \xrightarrow{i} H)$$

$$\mathcal{K} = \mathrm{sh}(L \xrightarrow{i'} K)$$

(see 5.2). Now, form the following sheaf M over L: for $U \in L$, $M(U)$ is the set of cHa-morphisms

$$h: H|i(U) \xrightarrow{h} K|i'(U)$$

such that the triangle

$$\begin{tikzcd} H|i(U) \arrow[rr, "h"] & & K|i'(U) \\ & L|U \arrow[ul, "i|U"] \arrow[ur, "i'|U"'] & \end{tikzcd}$$

commutes; in other words,

$M(U)$ is the set of functions

$$h: H|i(U) \longrightarrow K|i'(U)$$

satisfying: (i) $h(i(V)) = i'(V)$ (for $V \le U$)

(ii) h preserves binary $\wedge$, and $\vee$, of H.

For $h \in M(U)$, $V \le U$, let

$$h \restriction V \underset{\mathrm{df}}{=} h \restriction (H|i(V))$$

M is clearly a presheaf; we'll show below that M is (isomorphic to) the object of cHa-morphisms $\mathrm{Hom}^*(\mathcal{H}, \mathcal{K})$. But before we do that, we consider the case when the cHa's comes from spaces:

$$L = \mathcal{O}(E_0)$$

$$H = \mathcal{O}(E)$$

$$K = \mathcal{O}(E')$$

$E \xrightarrow{p} E_0$ gives rise to $i = p^{-1}: L \longrightarrow H$, $E' \xrightarrow{p'} E_0$ gives rise to $i' = (p')^{-1}: L \longrightarrow H'$. A <u>fiber map</u> $E' \xrightarrow{f} E$ is one for which the diagram

$$\begin{tikzcd} E' \arrow[rr, "f"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & E_0 & \end{tikzcd}$$

commutes. The <u>sheaf</u> M' (over E_0) <u>of fiber maps</u> from $\begin{matrix} E' \\ \downarrow \\ E_0 \end{matrix}$ to $\begin{matrix} E \\ \downarrow \\ E_0 \end{matrix}$ is the sheaf whose U-sections (U open $\subset E_0$) are the fiber maps:

$$\begin{tikzcd} (p')^{-1}(U) \arrow[rr] \arrow[dr, "p' \text{ restricted}"'] & \arrow[d, phantom, "\circlearrowright"] & (p)^{-1}(U) \arrow[dl, "p \text{ restricted}"] \\ & U & \end{tikzcd} \tag{1}$$

with induced topology (sober if E' is!)

with induced top.

with restriction defined in the obvious way.

To compare M and M', let us assume that E_0, E and E' are all sober. Then, by 3.5, the commutative triangles (1) are in $|-|$ correspondence with those of the form

$$\begin{tikzcd} H|i(U) \arrow[rr, "h"] & & K|i'(U) \\ & L|U \arrow[ul, "i|U"] \arrow[ur, "i'|U"'] & \end{tikzcd}$$

i.e. with the elements of $M(U)$. - So, the object of cHa-morphisms $\mathcal{H} \to \mathcal{K}$ is identified with the sheaf of fiber maps from $(E' \to E_0)$ to $(E \to E_0)$.

Let's show that M is (essentially) the object of cHa-morphisms. Let $h \in M(U)$ as described above; h gives rise to the natural transformation

$$\tilde{h}: \mathcal{H}|U \to \mathcal{K}|U$$

by defining

$$(\tilde{h}_{V})(\underbrace{x}_{x \in \mathcal{H}(V) = H|i(V)}) = h(x)$$

(compare (iii) in (4), 5.2).

Consider the presheaf $\tilde{M}$ obtained by replacing h by $\tilde{h}$, for all $h \in M(U)$;

$$\tilde{M}(U) = \{\tilde{h}: h \in M(U)\};$$

$$\tilde{h} \upharpoonright U = \widetilde{h \upharpoonright U}\ .$$

Notice that $\tilde{M}$ so defined is a <u>sub</u> <u>presheaf</u> of the exponential sheaf $\mathcal{H}^{\mathcal{K}}$; clearly also $M \simeq \tilde{M}$.

<u>Proposition 5.3.1</u>. $\tilde{M}$ <u>is</u> the object of cHa-morphism

$$\mathrm{Hom}^*(\mathcal{H}, \mathcal{K}) \quad (\mathcal{H} = sh(L \xrightarrow{i} H),\ \mathcal{K} = sh(L \xrightarrow{i'} K)).$$

<u>Proof:</u> We have to show that for $g \in \mathcal{K}^{\mathcal{H}}$:

$$g \in \underbrace{[M(z)]}_{\mathrm{Hom}^*(\mathcal{H}, \mathcal{K})}(U) \Leftrightarrow g \in \tilde{M}(U).$$

Let first $U = 1 = 1_L$. Then $g \in [M(z)](1) = \Gamma[M(z)] \Leftrightarrow \models^{Sh(L)} M(\underline{g})$ by 4.5.3.2.

By what we said in the second paragraph of p. 5.27,

$$\models^{Sh(L)} M(\underline{g}) \Leftrightarrow g \in \mathrm{Hom}_{\underset{\sim}{cHa}(\xi)}(H, K).$$

By (ii) & (iii) in (4), 5.2, $g \in \mathrm{Hom}_{\underset{\sim}{cHa}(E)}(H, K)$ iff $g = \tilde{h}$ for some (obviously unique) $h \in \Gamma(M)$

$$\therefore\ g \in \mathrm{Hom}(H, K) \Leftrightarrow g \in \Gamma(\tilde{M}).$$

The three displayed equivalences show:

$$[M(z)](U) = \tilde{M}(U)$$

$$\text{for } U = 1 = 1_L;$$

the general case is obtained by the special case applied in the topos $Sh(L|U)$ to the $\underset{\sim}{cHa}$-objects $(H|U) = sh(L|U \xrightarrow{i} (H|i(U)))$, $(K|U) = \ldots$ (exercise).

$\square$

As an application, let's see how the notion of 'point' is internalized. Given $H \in \underset{\sim}{cHa}(E)$ (E an arbitrary topos), we have the notion of the object of points of H

$$Pt^*(H)$$

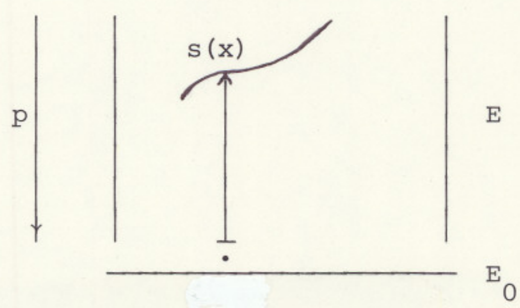
defined in the natural way:

$$Pt^*(H) \underset{df}{=} \mathrm{Hom}^*_{\underset{\sim}{cHa}}(H, \Omega),$$

with $\Omega = P(1)$ the initial $\underset{\sim}{\text{cHa}}$ in $\mathcal{E}$. - Let $\mathcal{E} = \mathrm{Sh}(L)$. We can assume that $H = \mathrm{sh}(L \xrightarrow{i} H)$. Clearly, $\Omega = \mathrm{sh}(L \xrightarrow{\mathrm{Id}} L)$ (<u>check!</u>). Therefore, the object of points of H is the sheaf whose U-sections are those $h\colon H|i(U) \longrightarrow L|U$ for which the diagram

$$H|i(U) \underset{h}{\overset{i|U}{\leftrightarrows}} L|U \circlearrowleft \mathrm{Id}_{L|U}$$

commutes. - In case L, H come from the sober spaces E_0, E, respectively, as usual, $i = p^{-1}$, then what we get is precisely the <u>sheaf of continuous sections of p</u>



$$s\colon U \to E;\quad p \cdot s = \mathrm{Id}_U .$$

The moral is: the points of the internal $\underset{\sim}{\text{cHa}}$ (locale) derived from $E \xrightarrow{p} E_0$ are internally the same as the sheaf of continuous sections of p.

5.4. The real numbers in intuitionism; the real number object in toposes.

First, we'll theorize about the real numbers in informal basic intuitionistic mathematics; afterwards we will state some conclusions concerning toposes.

We restate the well known notion of Dedekind real; we choose a formulation that is classically the same as the usual one, but intuitionistically is one of more possibilities (to be sure, the best one).

A Dedekind cut is a pair $C = (C_\ell, C_u)$ of subsets C_ℓ, C_u of the rationals $\mathbb{Q}$, satisfying the following conditions:

(1) C_ℓ, C_u are inhabited: there is p such that $p \in C_\ell$ and there is q such that $q \in C_u$;

(2) $C_\ell \cap C_u = \emptyset$;

(3) C_ℓ is closed downward, C_u is closed upward
($p \in C_\ell,\ p' < p \Rightarrow p' \in C_\ell$)

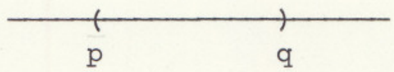
[Remark: (2) & (3) implies: $p \in C_\ell,\ q \in C_u \Rightarrow p < q$];

(4) for every $p \in C_\ell$ there is $p' > p,\ p' \in C_\ell$ ("no last element" formulated in a positive way); for every $q \in C_u$ there is $q' < q,\ q' \in C_u$;

(5) whenever $p < q$ are rationals, then either $p \in C_u$ or $q \in C_\ell$.

End of def. of Ded. real

<u>Remark</u>: The real number C is being specified by declaring that for rationals: $p < C \Longleftrightarrow p \in C_\ell$, $q > C \Longleftrightarrow q \in C_u$. So, (5) is says that given $p < q$



we should be able to tell either that C is to the right of p, or to the left of q; clearly, this says that in some sense we can "locate C arbitrarily precisely". In fact, consider the following condition:

(5)* Given any $n \in \mathbb{N} - \{0\}$, we can find $p \in C_\ell$, $q \in C_u$ such that $(0 <) q - p < \frac{1}{n}$.

To present a typical intuitionistic proof, let's prove that every Dedekind real satisfies (5)*; it is easy to show that (1) - (4) & (5)* imply (5) back. [This shows the equivalence of the definitions in FS and TT].

By (1), find $p \in C_\ell$, $q \in C_u$. Divide the interval (p,q) into short pieces: find rationals

$$p_0 = p < p_1 < \cdots < p_{m-1} < p_m = q$$

such that $p_{i+1} - p_i < \frac{1}{2n}$. Now, construct the following subset X of $\{p_0,\ldots,p_m\}$: for each pair of indices $i < j$ (in $[0,m]$), find that <u>either</u> (Case 1) p_i is in C_ℓ, <u>or</u> (Case 2) that p_j is in C_u (be careful: we cannot <u>decide</u> of a given p if it in in C_u!); in Case 1, throw p_i into the set X, in Case 2, throw p_j into X (if both Cases hold, drop both p_i, p_j into X); do this for all pairs $i < j$; at the end, you have your set X. [- The good thing about X is that, because of its definition, it

($-\infty <$ any finite rational $< \infty$) together with the symbol $\emptyset$ ('empty set')).

Define $\emptyset \leq$ every element of P;

$$<p,q> \leq <p',q'> \underset{df}{\Longleftrightarrow} p' \leq p < q \leq q',$$

$$<p,q> \nleq \emptyset.$$

It is easy to see that P is a '<u>finitely</u> <u>complete</u>' poset;

$$<p,q> \cap <p',q'> = <\min(p,p'), \max(q,q')>$$

if $\min(p,p') < \max(q,q')$; otherwise it is $\emptyset$; the maximal element is $<-\infty, \infty>$.

[One has in mind that $<p,q>$ is the open interval (p,q).]

We define the following system of prescribed coverings:

1. for $\emptyset$, the empty family is a covering of $\emptyset$:

$$\emptyset \in \mathrm{Cov}_0(\emptyset)$$

2. for $<-\infty, \infty>$, the family of intervals

$$\{<n,n+1>: \; n \in \mathbb{Z}\} \text{ is a prescribed}$$

$$\text{covering, } \in \mathrm{Cov}_0(<-\infty, \infty>)$$

3. for finite $p < q$: (a) for any $n \in \mathbb{N} - \{0\}$, and write

$p_i = p + i\,\frac{q-p}{2n}$ $(i = 0,1,\ldots,2n)$, the family

$$\{<p_i,p_{i+2}>: \; i = 0,\ldots, 2n-2\} \text{ is}$$

$$\text{a prescribed covering } \in \mathrm{Cov}_0(<p,q>)$$

(b) the family $\{<p + \frac{1}{n}, q - \frac{1}{n}>: n \in \mathbb{N} - 0,$

$n > \frac{1}{2(q-p)}$ } is a prescribed covering of $<p,q>$.

is decidable: we can actually see if a rational has been dropped into X or not; if we let $X' = \{p_i : i = 1,\ldots,m$ and either $p_i \in C_\ell$, or $p_i \in C_n\}$, this is not clear of X'.] We claim that there is at most one $i \in [1,m]$ such that $p_i \notin X$; otherwise we'd have $i < j$ $p_i, p_j \notin X$; but then we have not done our job for i and j ! Now, remember that every element p_i of X comes with our finding out about it that it is in C_ℓ, or that it is in C_u. Let $X_\ell = \{p_i \in X: p_i$ is in $C_\ell\}$, $X_u = \{p_j \in X: p_j$ is in $C_u\}$. Clearly, $p_i \in X_\ell$, $p_j \in X_n$ imply $p_i < p_j$. Let p_{i_0} be the largest element of X_ℓ, p_{j_0} the smallest element of X_n; then for any p_k in (p_{i_0}, p_{j_0}), we must have $p_k \notin X$; but there is at most one such p_k; conclusion: $0 < p_{j_0} - p_{i_0} < 2.\ \frac{1}{2n} = \frac{1}{n}$. This finishes the proof.

The set $\mathbb{R}$ of Dedekind reals is made into a topological space, by the following definition: for rationals $p < q$, define

$$(p,q)^* = \text{the open interval } (p,q) =$$

$$= \{C \in \mathbb{R}: \ p \in C_\ell \ \& \ q \in C_u\} .$$

One verifies easily that the collection of all "open intervals" $(p,q)^*$ form a basis of a topology, which, by the definition, is the topology on $\mathbb{R}$. Henceforth, $\mathbb{R}$ is the topological space so defined [for emphasis, we might write $\mathbb{R}$ however.]

We now point out a p-site, actually a pre-p-site (see end of subsection 3.1), whose points are exactly the reals.

Let P be the following partially ordered set: the elements of P are ordered pairs $\langle p,q \rangle$ of extended rationals, $p,q \in \mathbb{Q} \cup \{-\infty, \infty\}$, with $p < q$

There are no more prescribed coverings other than those listed under 1.-3. - Let P also denote the site generated by the presite given so far; and let $L(P)$ denote the cHa generated by P (see § 3.)

<u>Theorem 5.4.1.</u> $\mathbb{R}$ is homeomorphic to $Pt(P) = Pt(L(P))$.

<u>Proof</u>: Let's start with a few remarks of a general nature (that actually would belong to an earlier place).

Define, for a p-site P, $Pt(P)$ = the set of p-site morphisms $P \to \mathbb{P}(1)$; by the universal property of $P \xrightarrow{\varepsilon} L(P)$ (see § 3.1), $Pt(P)$ is in a bijective correspondence with $Pt(L(P))$:

$$\tilde{x} \in Pt(L(P)) \longmapsto x = \tilde{x} \circ \varepsilon \in Pt(P).$$

Defining the topology on $Pt(P)$ by the <u>basis</u> consisting of the sets

$$U^* = \{x \in Pt(P) : \ x(U) = \mathbb{1}\} \qquad (U \in P)$$

(<u>check</u>), this correspondence actually becomes a homeomorphism (see the description of $L(P)$; <u>check</u>). - So, we can restrict attention to the top. space $Pt(P)$.

On the other hand, if P is defined by the presite P_0, and $Pt(P_0)$ denotes the set of left exact functors $P \longrightarrow \mathbb{P}(\mathbb{1})$ preserving P_0-coverings, then by 3.2', we have that

$$Pt(P_0) = Pt(P)$$

as sets; the topology on $Pt(P_0) = Pt(P)$ is as above.

Returning to our concrete situation, we therefore show that spaces $Pt(P_0)$ and $\mathbb{R}$ are homeomorphic.

1. Define the map $\mathbb{R} \xrightarrow{h} Pt(P_0)$ as follows. Given $C \in \mathbb{R}$, let $u(= h(c)$, in the "prime ficter" style) be the following subset of P:

1. for $p < q$ both in $\mathbb{Q}$,

$$(p,q) \in u \Leftrightarrow p \in C_\ell \ \& \ q \in C_u;$$

2. for $(-\infty, q)$, (p, ∞), $(-\infty, \infty)$:

$$(-\infty, \infty) \in u,$$

$$(-\infty, q) \in u \Leftrightarrow q \in C_u,$$

$$(p, \infty) \in u \Leftrightarrow p \in C_\ell;$$

3. $\emptyset \notin u$.

We claim that u is a 'prime filter': it contains 1_P, is closed under $\wedge$, and whenever $\{U_i \le U: i \in I\} \in Cov_0(U)$, $U \in u$, then for some $i \in I$ we have $U_i \in u$. [such a u obviously defines an element of $Pt(P_0)$, and vice versa]. These are routine to check, using the definition of Dedekind cut. E.g. the last condition with $j = \{U_i \le U: i \in I\}$ is true: for $j \in Cov_0(\emptyset)$, by the third clause for u; for $j \in Cov_0(\langle -\infty, \infty \rangle)$, by the inhabitedness of C_ℓ and C_u and an additional argument resembling one below; and for $j \in Cov_0(\langle p,q \rangle)$, under (a), $p,q \in \mathbb{Q}$, as follows:

using (5)*, find r', s' such that $r' \in C_\ell$, $s' \in C_u$ and $s' - r' < \frac{1}{4n}(q-p)$; let $r = \max(p,r')$, $s = \min(q,s')$; then it is clear that for some $i = 0,\dots, 2n-2$ (see 3. in the def. of Cov_0), we have $p_i \le r < s \le p_{i+2}$; it follows that $p_i \in C_\ell$ and $p_{i+2} \in C_u$, hence $\langle p_i, p_{i+2} \rangle \in u$ as required. For j under (b): use condition (4) for Ded. cuts.

2. Define the map $h'\colon Pt(P_0) \longrightarrow \mathbb{R}$ as follows. Given $u \in Pt(P_0)$, let $C = (C_\ell, C_u)$ be defined as follows (we'll put $C = h'(u)$):

$$p \in C_\ell \Leftrightarrow \text{there is } q \in \mathbb{Q} \text{ such that } \langle p,q\rangle \in u$$

$$q \in C_u \Leftrightarrow \text{there is } p \in \mathbb{Q} \text{ such that } \langle p,q\rangle \in u.$$

Since by $(-\infty, \infty) \in u$ we must have some $\langle n,n+1\rangle \in u$, it follows that C_ℓ, C_u are inhabited. If $r \in C_\ell \cap C_u$ were the case, we'd have $(p,r) \in u$ and $(r,q) \in u$, hence $\emptyset = (p,r) \cap (r,q) \in u$, contradiction. Condition (3) for Dedekind cuts is clear. - For (4), use the covering "of type (b)", and for (5), for a covering "of type (a)".

3. Next we check that $h \circ h' = Id_{Pt(P)}$, $h' \circ h = Id_{\mathbb{R}}$ - easy. - The topologies on both $\mathbb{R}$ and $Pt(P)$ are defined by certain bases; to show e.g. that $h\colon \mathbb{R} \longrightarrow Pt(P)$ is continuous, it suffices to check that $h^{-1}([U])$ is open in $\mathbb{R}$ ($U \in P$, $[U] = \{u \in Pt(P)\colon U \in u\}$); and similarly for h'. But we find that

$$h^{-1}([U]) = U^*$$

$$(h')^{-1}([U^*]) = [U];$$

here for finite $U = \langle p,q\rangle$, U^* was defined before; $\langle -\infty, \infty\rangle^* = \mathbb{R}$, $\langle -\infty, q\rangle^* = \{C \in \mathbb{R}\colon\ q \in C_u\}$, etc.

$\square$ for 5.4.1.

Now, let $\mathcal{E}$ be a topos. By formalizing the above, we obtain internal notions in $\mathcal{E}$. E.g., by imitating the usual construction, one has the object of rational numbers $Q = Q^{(\mathcal{E})}$ in $\mathcal{E}$; then $\mathrm{Ded}^{(\mathcal{E})} = \mathcal{R}$ is given as a certain subobject $\mathcal{R} \hookrightarrow \mathcal{P}(Q) \times \mathcal{P}(Q)$. Also, from Q one derives the 'real pre-site' P introduced above; P (as an object) is a certain subobject of $Q^{(i)} \times Q^{(i)}$ ($Q^{(i)} = Q \cup \{-\infty, \infty\} = Q \amalg 2$); Cov_0 is given as a certain subobject

$$\mathrm{Cov}_0 \hookrightarrow \mathcal{P}(\mathrm{P}) \times \mathrm{P}$$

$$(\text{meaning: } \langle\gamma, x\rangle \in \mathrm{Cov}_0 \Leftrightarrow \gamma \in \mathrm{Cov}_0(x));$$

finally, $\mathrm{Pt}^*(\mathrm{P})$ is the internal space of points of P; the set $\mathrm{Pt}^*(\mathrm{P}) \cong \mathrm{Pt}^*(\mathrm{L}(\mathrm{P}))$ was described in 5.3; the topology on $\mathrm{Pt}^*(\mathrm{P})$ is defined by the obvious process of formalization, and it consists in specifying a subobject

$$\mathcal{O}(\mathrm{Pt}^*(\mathrm{P})) \hookrightarrow \mathcal{P}(\mathrm{Pt}^*(\mathrm{P})),$$

e.t.c.

The theorem proved above is valid in $\mathcal{E}$; it gives us an internal homeomorphism

$$\mathrm{Pt}^*(\mathrm{P}) \simeq \mathbb{R}.$$

On the account of this, we declare that for us the real number object <u>is</u> $\mathrm{Pt}^*(\mathrm{P})$.