The theory of abstract sets based on first-order logic with dependent types

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Abstract

Since the mid 1990's, the author has been working on a foundational project involving higher dimensional categories on the one hand, and model-theoretical methods used for typed versions of first-order logic on the other. In this paper, the project is called the *Type-Theoretic Categorical Foundation of Mathematics*, and it is referred to by the acronym TTCFM. The present paper is a discussion of the first level of TTCFM, abstract set theory. To display the close relationships, as well as the differences, of abstract set theory and topos theory, I will base my discussion on a review of parts of F. William Lawvere's paper [Lawvere 1976].

At the invitation of Professor Andras Mate, the author gave a series of – rather technical – lectures, entitled "First Order Logic with Dependent Sorts", at the Department of Logic of the Eotvos University in early March, 2013. This paper was inspired by the lively and thorough-going interest for the subject by Professor Mate and the audience of the lectures. The paper is dedicated to Professor Andras Mate on the occasion of his 60th birthday.

Introduction

I must start with a polemic with Solomon Feferman's views on categorical foundations of mathematics, CFM's, in general. Below, I cite a passage from [Feferman 1977], page 154. Jean-Pierre Marquis used the same passage as the leading quote in his paper [Marquis 2012]. The breaking up of the statement, inserting the numbers and italicizing, are my additions.

> "To avoid misunderstanding, let me repeat that I am not arguing for accepting current set-theoretical foundations of mathematics. Rather, it is that on the platonist point of view of mathematics
>
> [1] something like present systems of set theory must be prior to any categorical foundations. More generally,
>
> [2] on any view of abstract mathematics priority must lie with notions of operation and collection."

My brief reply is that I agree with part [2], but not with [1]. Certainly, "collection" and "operation" are fundamental; thus, I would not argue with [2]. However, I cannot go along with [1]. The implied inference that any systematic view of abstract mathematics for which priority lies with notions of operation and collection has, *necessarily*, "something like present systems of set theory prior to it" is un-argued, and, I think, incorrect. Even if Feferman finds that the various categorical foundational systems proposed in the existing literature are, in his judgment, all based on, or are variants of, "something like present systems of set theory", he cannot conclude, and, in fact, he has not given any argument for concluding, that a newly proposed such system, even though it proclaims to be "categorical" and is, therefore, "suspect", will depend *metaphysically* on "something like present systems of set theory". If, instead of "something like present systems of set theory", "naive set theory" had been used, I would not have any objection. In other words, my objection is to the identification of naive set theory, an inherently *non-systematic* complex of sometimes contradictory, but still very fertile, ideas, with the codified *systems* of existing formalized set-theories.

One more paragraph about the general context of this paper and TTCFM. The adjective "foundational" in talking about the project of TTCFM is misleading. I consider that mathematics in the usual sense has been "well founded" by all that has happened in the last two centuries in the rigorization of mathematics, mathematical logic, set theory and category theory, and

I have no intention to improve on the *foundations* in a narrow sense of mathematics thus arrived at. I am interested in the world of abstract concepts: collections, operations, identity, numbers, logic, proofs, formal languages, all that one thinks of when one mentions *naive set theory* – which is, again, a misnomer, since somehow "set" gets all the emphasis in it. It happens that this "naive set theory" – "naive concept theory" would be better – is a *mathematical* subject if judged by its methods – but on a closer look, mathematics is the one that *uses* the methods of (naive) concept theory; that is, mathematics is concept-theoretical, and not the other way around.

In the main part of the paper, I will discuss the concept of set, called *abstract set*, that underlies the ground level of TTCFM. TTCFM has other, in fact infinitely many, "higher" types of collection as well, ones that do *not* have *underlying* (abstract) sets. In the last section, very briefly I will go beyond sets to categories, n-categories, and omega-categories. With the omega-categories, there is an entirely different metaphysic associated, one that I talked about at the Octoberfest in Montreal in 2004, where I used as a leading quote one from Bertrand Russell's "Logical Atomism". Simply put, now the atoms (urelements) of the abstract sets entirely disappear, and instead, we have an infinite, never-ending analysis of existents into "more-and-more atomic" ingredients.

Let me mention in passing that in its pure version, TTCFM has only collections, no operations distinct from collections.

The idea of abstract set comes from naive set theory, and it was conceived of in the 19th century, notably by Georg Cantor. The higher-type collections of TTCFM are partially given, and further suggested (through the intuitive idea of "categorification"), by *category theory*, introduced in the 1940's by S. Eilenberg and S. Mac Lane's. Today, category theory is an established field of mathematics, and the *metatheory* of TTCFM makes a full use of it. However, it is not the case that the higher-type collections of TTCFM are all themselves categories; in fact, they are, from level 2 on, not even categories with additional structure: for example, a *bicategory* (a concept due to J. Benabou) is not a category with additional structure; it has no *underlying* category.

The main novel ingredient of TTCFM is its theory of identity, based on the concept of *FOLDS-identity*. FOLDS-identity replaces Gottlob Frege's notion of global identity (equality), the one that is universally accepted in (model-theoretical) logic, hence in axiomatic set theory as well, as part of *first-order logic with identity* in its pure (as-yet-un-applied) form. FOLDS-identity is also part of pure FOLDS, that is, FOLDS-identity is taken over as given from the general theory of FOLDS by every application of FOLDS. Identity seems to disappear as something minor, or as something derived rather than basic, in the conceptual apparatus of abstract mathematics beside "collection" and "operation" – as conceived of by "something like present systems of set theory" referred to by Feferman. Therefore, the new theory of identity *distinguishes* TTCFM from "present systems of set theory", as well as from all other systems I am aware of.

It is important to note that the *meta-theory* of TTCFM does depend on established set theory. However, its formalized version, being explicit and *elementary*, that is, *first-order*, stands on its own, without anything "prior to it". There is no contest intended with established set theory: I am proposing an alternative, not an exclusive, foundational scheme. The system of TTCFM, even in a partial form, can also serve as the basis of significant metatheoretical considerations.

2. Lawvere's concept of abstract sets

F. William Lawvere, in section 2 of [Lawvere 1976], discusses *abstract sets*, and the way the concept of (elementary) topos constitutes a mathematical formalization of the theory of abstract sets. He writes: "An abstract set X has elements each of which has no internal structure whatsoever"; in other words, X is a set of "urelements" in the accepted terminology of present-day set-theory. Urelements have played a limited role in previous systems of set theory; notably, in an early version due to E. Zermelo of ZFC, in the variant NFU due to R. Jensen of W. v. O. Quine's New Foundations, and in J. Barwise's version of the Kripke-Platek axiom system. However, in all these systems, urelements lie at the bottom of a *cumulative* structure in which one undertakes an imaginary and more-or-less limitless repeated *construction* (via a formal or informal application of Gottlob Frege's principle of comprehension) of collections of already available entities, some of the latter being urelements that have no elements at all, and some others being already constructed sets. In the topos of abstract sets, the "standard model", discussed by Lawvere in *loc. cit.*, of the (first-order!) axioms of defining the notion of topos, as well as in the theory of abstract sets I am proposing here, the *only* kind of *collection* is "abstract set": *all* elements of *all* sets are *urelements*. On the other hand, in both theories we also have a primitive notion of *operation*, a *mapping from* the set X to the set Y, $f : X \to Y$. A topos is a *category*, satisfying certain further first-order axioms, ones that form a surprisingly compact axiom system producing a very rich set of consequences.

The reader who is new to the subject should consider how surprising it is that mathematics can be developed using only the apparently meager resources of abstract sets and functions (mappings, also considered a primitive notion) between them. In fact, the best way to understand topos theory is by making a serious attempt *from scratch* at such a development, without any preconceived idea about axioms of topos, or even axioms of category, and arriving at those axioms as the ones that are *necessary*. For obvious reasons of space, the discussion of abstract set-theory that follows is not doing full justice to the *power* of that theory.

In discussing "abstract set theory", instead of following the (purely) categorical language, in the next section I will use a type-theoretic language, one that points to the way I intend to extend topos theory to TTCFM. Although Lawvere's aim is to introduce and empower the purely *categorical* language, his paper contains many points of reference that help in the exposition of the *type-theoretic* metaphysic pursued here.

In [Lawvere 1976], on page 119, last two paragraphs, we read:

"I believe the conclusion is that

[1] membership-as-primary entails membership as global and absolute whereas in practice membership is local and relative; [. . .].
These considerations lead one to formulate the following "purified" concept of (constant) *abstract set* as the one actually used in naive set-theoretical practice of modern mathematics:

[2] An abstract set X has elements each of which has no internal structure whatsoever;

[3] X has no internal structure except for equality and inequality of pairs of [its] elements,

[4] and has no external properties save its cardinality;

[5] still an abstract set is more refined (less abstract) than a cardinal number in that it does have elements while a cardinal number does not. The latter feature makes it possible for abstract sets to support the external relations known as

[6] *mappings*, which constitute the second fundamental concept of naive set theory (cardinal numbers would admit only the less refined external relations by one being less than another or not). Thus "mapping" is too fundamental to be formally defined, although we remark that a mapping satisfies the familiar $\forall x \exists ! y$ condition [...]. The third concept is that of

[7] *composition* of mappings [...]. Of course, composition of mappings is

[8] *associative*, and there is an *identity mapping* for each set."

I feel that there is a need for a disclaimer here. There are many references in the Lawvere article at hand to "(mathematical) practice"; there is one right at the start of the above quote, in line [1]. I must dissociate myself with the implications (there is one right there, in line [1]) of his statements contrasting categorical methods *in mathematical practice* with usual-set-theoretical ones. It is my opinion that those implications amount to an unrealistic view of global superiority of categorical methods over set-theoretical ones. It is important to add to this that I do not at all regard Lawvere's *concept-theoretic* views the less interesting, or the less important, because of our disagreement concerning *mathematical practice*. In fact, William Lawvere has had the most profound influence on my thinking after classical mathematical logic.

In what follows, I show a first illustration of FOLDS both in an "impure" form, with the addition of some "syntactic sugar", and in its "pure form", by formalizing some of Lawvere's notions.

I will depart from Lawvere's context, by explicitly introducing *elements* of the sets. This is something Lawvere wants to avoid, since in the notion of "topos" he is aiming at, "elements" of a "set" are represented by map(ping)s into the "set". I intend to show the flexibility of FOLDS: it is not the case that it works only in strictly categorical contexts.

3. The type-theoretic view of abstract sets

Here is our interpretation of passage [1]. The interpretation will, first, be purely *grammatical*: a description what we can and what we cannot say *meaningfully*; after which we venture cautiously into metaphysics.

In conceiving "membership as global", we use a variable, say x, to denote an arbitrary entity; another, say A, for a set; and consider as meaningful the proposition $x \in A$, " "x belongs to A". The proposition "$x \in A$" can be true or false depending on what x and A are. Syntactically (grammatically), it is meaningful to negate the proposition and write $\neg(x \in A)$; it is meaningful to use the proposition for unconstrained sentence building by the logical operators, including (unrestricted) quantifiers.

On the other hand, "membership as local and relative" has A, the set, as an ordinary variable as before, whereas x is a further variable *declared* as being of the *variable type* $\mathrm{El}(A)$, "element of A", in symbols: "$x{:}\,\mathrm{El}(A)$". The declaration means that the variable x may be used exclusively as one ranging over elements of the set A. Therefore, "there exists x such that "$\ldots x \ldots$" holds" will mean that there is an element *in* A such that "$\ldots$" holds; "for all x, $\ldots x \ldots$" holds will mean that for all elements x of A, "$\ldots x \ldots$" holds. Note that the type of x, $\mathrm{El}(A)$, is a *dependent* type: the type itself *depends* on the variable A – which can itself be considered as a variable declared being of the *constant* type Set.

Note further that the declaration "$x{:}\,\mathrm{El}(A)$" does not function as a proposition any more; for instance, we cannot negate it; in our grammar, we cannot write $\neg(x \in \mathrm{El}(A))$. This raises the question about the possible uses of the "statement" "$x : \mathrm{El}(A)$". The answer is: it is used in the quantificational phrases such as "$\exists x : \mathrm{El}(A)$ such that $\ldots x \ldots$" and "$\forall x : \mathrm{El}(A)\ \ldots x \ldots$" explained above, and *in no other situation*.

Metaphysically, we imagine, for an abstract set A, the elements x of A as having no absolute existence, independently of A. The questions: "what is x?" , "what properties does x have?", "what relations do x and y have to each other?" make no sense without first having declared or assumed that x is an element of A, y is an element of B, for some A and B that are "given", contemplated prior to contemplating x and y. The entity x can have some property only *as an element* of A. A itself can be said to have any property only after A being *declared* or recognized or assumed that it is a *set*. On the other hand, when we say that *something* is true for all elements of A, or for some element of A, that *something* must be meaningful for elements of A, *but not necessarily for other things*.

In FOLDS, this kind of relative existence will be universal. Certain (in fact, most) entities will *depend* on more than one *previous* entity (above, x depends on the single previous entity A). In fact, *all* entities will depend on an organized system (which, in the "ground case", may be empty), a *context*, of *previous* entities.

This metaphysic of abstract existents may look strange at first sight, but it should be recognized that it is in fact one that modern *abstract* mathematics (group theory, point-set topology, and many more "advanced" subjects) uses all the time – without claiming that it, the metaphysic of abstract sets, is the *only* one that mathematics "uses all the time". When we say: "let G be a solvable group", we imagine the elements of G to be urelements, and understand what it means that G is solvable without difficulty; but when we say: "consider the group A of automorphisms of (the former) G", the elements of A are no longer urelements. It is an important point in understanding abstract set theory that it can deal with entities such as this A, given the fact that an abstract set can have only urelements as elements. This dealing-with not-withstanding, we cannot claim that mathematical practice uses abstract sets *only*!

We have deployed Bertrand Russell's idea of *types* of mathematical objects. The original idea of type theory was to avoid the paradoxes by disallowing newly constructed objects of (necessarily) new types being brought into "questionable" *relations* with old objects of old types, *relations* that, originally, were defined for old objects of old types *only*, the prime example of such a *relation* being *membership* itself. What will happen now is that we will not be allowed to bring meaningfully elements of different sets into the relation of *equality* even! Of course, in the Russellean context, and in its more modern variants such as the simple theory of types, the issue of equality of elements of *distinct* types is a trivial one, since we may say that elements of distinct types are necessarily *unequal*; in other words, assuming a global identity is harmless. Not so in set theory – since we do not have a distinct idea of two sets being *distinct* as we do for the non-variable types of simple type theory.

As we said above, part [2] of the quote is saying that *all* elements of *all* abstract sets are urelements.

Passage [3] says that for each set A, given to us we have a relation of *equality*, $=_A$, or simply $=$ when A is understood, such that for elements x and y of A, it is meaningful to write $x =_A y$, a proposition that can be true or false, and that can be used in the process of sentence-building by the logical operations. We will find it natural to take it as an *axiom* that $=_A$ is a reflexive, symmetric and transitive relation, an equivalence relation in short, which is our first opportunity to write down some formulas and adopted axioms in our formal theory, albeit only in a preliminary form.

Reflexivity: $\forall A{:}\mathrm{Set}\ .\ \forall x{:}\mathrm{El}(A)\ .\ x = x.$

Symmetry: $\forall A{:}\mathrm{Set}\ .\ \forall x,\, y{:}\mathrm{El}(A)\ .\ x = y \longrightarrow y = x.$

Transitivity: left to the reader.

This is also the opportunity to make the break with the metaphysic of *all* "present systems of set theory". In "usual" set-theory, let it be ZF, GB, MK, NF, or any such system using global membership, it is meaningful and eminently reasonable to define *equality* of two sets A and B by declaring that $A = B$ iff $\forall x \,.\, x \in A \longleftrightarrow x \in B$. This definition – or axiom of extensionality, if you wish – is not available in abstract set theory, since the sentence is ungrammatical in at least two ways.We have no untyped variable; therefore, we cannot write "$\forall x$" . On the other hand, "$x \in A$", or rather, its available substitute "$x{:}\,\mathrm{El}(A)$", is not amenable to sentence-construction: it cannot be used as it is done in the biconditional in extensionality. We may try to repair the statement like this: $\forall x{:}\,\mathrm{El}(A) \,.\, x{:}\,\mathrm{El}(B)$ & $\forall x{:}\,\mathrm{El}(B) \,.\, x{:}\,\mathrm{El}(A)$. The latter formulation uses typed variables only, but uses a variable declaration such as $x{:}\,\mathrm{El}(B)$ as a proposition, by *asserting* it, something we don't have the permission to do. Finally, we may try this: $\forall x{:}\,\mathrm{El}(A) \,.\, \exists y{:}\,\mathrm{El}(B) x = y$ & $\forall y{:}\,\mathrm{El}(B) \,.\, \exists x{:}\,\mathrm{El}(A) x = y$ – which fails because of the use of *global equality*: variables of distinct types, x and y, are asserted equal, a concept not available in abstract set theory. It is in fact the most characteristic feature of abstract set theory that it does not use Fregean global equality. I rhetorically ask the critics whose prime representative is Solomon Feferman: am I still within the realm of the metaphysic of "present systems of set theory"? Those critics may doubt that I would be able to do anything reasonable after thus depriving myself of the use of such basics as the axiom of extensionality, and they may declare that I have abandoned civilization in favor of an existence in the desert, but they cannot at the same time maintain that I am still operating under the rules of that civilization!

It is important for us to remember that, in abstract set theory, equality (identity) of sets cannot be defined via extensionality. This will lead us to abandoning equality for sets altogether.

Some words about FOLDS in general. In pure FOLDS, there are no relations such as the typed equality $=_A$; relations are imitated by further dependent types. For instance, we may have the variable declaration $A{:}\,\mathrm{Set} \,.\, x, y{:}\,\mathrm{El}(A) \,.\, e{:}\,\mathrm{Equ}(A, x, y)$, with Equ a new *kind* (type-head), giving rise to the correctly formed type $\mathrm{Equ}(A, x, y)$. Under the stated variable declaration, the assertion $\exists e \,.\, \mathrm{TRUE}$ stands for "x equals y". The reformulation has an intuitive (metaphysical) meaning: e is a *witness* to the fact that x equals y.

Later, I will have the operation of *application* used directly in the "sugared" version of FOLDS, which can be treated and eliminated in pure FOLDS in a similar fashion, by introducing a new kind and corresponding new types.

In pure FOLDS, there are only

1) dependent types "consistently formed",

2) variables declared being of such types,

3) quantifiers each ranging over a variable declared (see also above on quantifiers!) such that the "quantifier does not leave any variable dangling" (I cannot write, after the declaration $A : \mathrm{Set}\ \ .\ x : \mathrm{El}(A)\ .\ e : \mathrm{Equ}(A, x, x)$, the quantified statement $\forall A\ .\ \exists e\ .$ TRUE, because this leaves x dangling (the correct formula $\exists e\ .$ TRUE has the free variables A and x, thus the incorrect formula $\forall A\ .\ \exists e\ .$ TRUE would have free x only, which depends on A); whereas I am allowed to write $\forall x\ .\ \exists e\ .$ TRUE, since the only free variable left is A, and A does not depend on x), and

4) the propositional constants TRUE and FALSE, and the usual propositional connectives used without restrictions.

This much should be enough to get a first idea of the – simple! – syntax of *pure* FOLDS and, I repeat, pure FOLDS is enough, that is, it can be made enough, possibly at the cost of extending the underlying *signature*. The eventual complexity of FOLDS formulations come from the complexity of the *signatures* used: the rules that tell us how one can form consistent (grammatically correct) dependent types from already declared variables.

Returning to the point of view of naive set theory, we see that FOLDS uses many "sets", collections, even bare collections, albeit only in (sometimes very) restricted manners. The collection of all entities e that are witnesses of the statement $x =_A y$, a collection *depending* on the entities A, x and y, is an example. However, we do not call this and other similar ones "sets". The word "set" is restricted to the entities of the type Set . The metaphysic of FOLDS consists in a relative, restricted acceptance of collections and entities of many different kinds, with each entity accepted only with a specific dependence declared, and to be used grammatically only in constrained circumstances. This is in very sharp contrast of set (class) theory when we think of it as formalized by Gottlob Frege, the first and still highly relevant formalization – all others, including the present one, having been arrived at by attempts of eliminating the paradoxes! Of course, this feature itself is not new with FOLDS: it appears first in traditional type theory. In FOLDS, we have multiplied the collections relative to previous theories, but we (still) exercise tight control over the use of them.

Although it is obvious, it is important to emphasize the fact that the semantics of FOLDS, in particular, the reading of the quantifiers, outlined by the above remarks as part of naïve set theory, can be turned in a natural manner into a *formal* semantics, formulated in an established system of set theory such as ZFC. For the development of TTCFM, it is important that the semantics of FOLDS is adequately formulated in any elementary topos, in particular, in a Grothendieck topos. We have formal theories in classical FOLDS as well as in intuitionistic FOLDS, and there are completeness theorems for both kinds: see [Makkai 1995]. The classical version implies that a proof based on the set-theoretical semantics of a FOLDS proposition can always be converted to a formal proof entirely within the formalism of FOLDS. The intuitionistic version implies a similar conclusion with respect to intuitionistic set theory.

4. The development of the type-theoretic language of abstract sets: the concrete category structure

Statement [4] in our Lawvere quote is a very important one, although it may be a bit obscure as it is, due to the lack of a formalization. Let us take the meaning of "A and B have the same cardinality" for granted at this point (of course, it is the familiar meaning). [4] is a *meta*-statement, one *about* the *language* of the theory, and it says: "if the sets A and B have the same cardinality, then B shares all *grammatically correctly formulated* properties of A: in other words, for such a property $P(-)$, $P(A)$ if and only if $P(B)$". Lawvere's statement is, roughly, that the cardinality of a set tells you *all* you can say about that set *alone*, *if you use the correct language only*. I maintain that this is the passage that *demands* the formulation of an appropriate formal language, one that will satisfy the requirement implicit in statement [4]. In "ordinary set theory", if we take as the property $P(-)$ to be "5 is an element of A", this property of A will not be shared by all (other) sets having the same cardinality as A. Thus, in "ordinary set theory" with its ordinary formal (first-order) language, statement [4] becomes patently false. Lawvere could not have made statement [4] without having in mind some kind of restricted language, necessarily (I think) excluding global equality, in which he imagines abstract set theory being formulated. However, there is no indication of an attempt to codify *such* a language in Lawvere's paper – or for that matter, in related work by Lawvere. On the other hand, since I am in the business of proposing such a language, I am obligated to answer the question whether Lawvere's requirement is satisfied in my proposal. By "Lawvere's imperative" I will refer to the statement that the answer to the question is "yes". Below, I will formally state, and sketch the proof of, Lawvere's imperative, and give it also in a more general and more interesting form.

Passage [5] is the statement that it is impossible to reduce the identity theory for sets (or that for groups, for instance) to the mere relation of "having the same cardinality" (isomorphism).

I turn to the rest: [6], [7] and [8]. These constitute the *category structure*, in fact, *concrete* category structure on the abstract sets. The most important new kind is Map for "mapping"; it serves to form types $\mathrm{Map}(A, B)$, where A and B are sets (set variables). In our version of abstract set theory, as distinct from Lawvere's (no claim is implied that "ours is better" than Lawvere's!), there is the important operation app of *application*; when $A : \mathrm{Set}$, $f : \mathrm{Map}(A, B)$, $x : A$, then we have $\mathrm{app}(f, x) : B$. There is some cheating in this last statement written in "sugared FOLDS", since, actually, an existence statement is implied, rather than a declaration – namely, that something called $\mathrm{app}(f, x)$ *exists* as an element of B. The pure-FOLDS treatment removes the cheat: we have a new kind, App, to form types of the shape $\mathrm{App}(A, B, f, x, y)$, where A, f, x are as above, $y : B$. Intuitively, $a : \mathrm{App}(A, B, f, x, y)$ means that a is a witness to – or even: a cause of – the fact that $y : B$ is $\mathrm{app}(f, x)$, the value of f when applied to x. Of course, we adopt the existence axiom that says that such witness (cause, fact) exists:

$$\begin{aligned}\textbf{\textit{Ax1}}\quad &\vdash \forall A, B : \mathrm{Set}\ .\ f : \mathrm{Map}(A, B)\ .\ x : \mathrm{El}(A)\ .\\ &\exists y : \mathrm{El}(B)\ .\ \exists a : \mathrm{App}(A, B, f, x, y)\ .\ \mathrm{TRUE}\end{aligned}$$

The symbol $\vdash$ is the assertion symbol here reserved for axioms; we will also use $\vDash$ for assertion in case we are asserting a theorem, a statement proved by using the previously stated axioms and theorems (if any).

(This last example exhibits a very important circumstance for which I have no explanation: all the important existence axioms in the theory of (even) higher-dimensional categories, when properly expressed, have exactly two existential quantifiers in them: not one, not three, not four, ..., but two (when we had only one existential quantifier before, for instance in the case of the equality axioms, we neglected certain higher dimensional background that is really there)).

Let me introduce an abbreviated notation for the last display, one that I will imitate and extend without further explanation later.

$$\textbf{\textit{Ax}1} \quad \vdash f : A \to B\ .\ x : A \quad \Longrightarrow \quad y : B\ .\ a \in \mathrm{App}(f, x, y)$$

I have omitted the Set-typings of A and B. I have omitted the quantifiers, the universal ones on the left of "$\Rightarrow$", existential ones on the right; I have abbreviated the notation of the type $\mathrm{App}(\dots)$, and omitted the propositional constant TRUE. If, in a similar abbreviation, there is no "$\Rightarrow$", then all quantifiers are universal.

What I called "sugared FOLDS" is J. Cartmell's language [Cartmell 1986] augmented by quantification (Cartmell does not have quantifiers). This is a very *useful* language: it is expressive and intuitive; however, it departs fundamentally from the pure metaphysic underlying FOLDS. For instance, one can use an equality on sets, and if $A = B$ and $f : A \to B$, then we also have $f : B \to B$, $f : B \to A$, $f : A \to B$. The full formulation of sugared FOLDS is extremely complicated because "everything is allowed" via equality; see the original article [Cartmell 1986].

From the point of view of the metaphysic of pure FOLDS, operations in the ordinary sense (such as our $\mathrm{app}(f, x)$) are fully meaningful only in the presence of an equality concept under which we can say that the value of the operation at any given argument is *uniquely* determined – and equality is available in FOLDS only in very restricted cases. For instance, it is not available for sets; therefore an operation whose value is a set is not "pure FOLDS". FOLDS augmented with operations is still interesting from the point of view of the foundationalist: he would want to show that operations are an *ideal* addition to pure FOLDS that can be *metatheoretically eliminated*, it resulting in a conservative extension of pure FOLDS. (The idea in the last sentence comes from David Hilbert's metamathematics.) I have thereby come to an area of the current mathematical problems in the subject: the essential equivalence or otherwise of pure FOLDS formulations with more naive-set-theoretical, and therefore easier-to-handle concepts.

The sugaring of FOLDS may continue, and, following the algebraic language, we may write $f'x$, or $f(x)$ or even fx for $\mathrm{app}(f, x)$.

To be economical, we will use abbreviations for well-formed formulas of

FOLDS and call them *defined concepts* as usual in logic. Just as in ordinary logic, we are allowed to change free variables by *proper* substitutions in defined concepts, and expect to be understood. As before, A, B, C, D denote variables declared as being of type Set . $f : A \to B$ is the same as the type declaration $f : \mathrm{Map}(A, B)$. In each of the defined concepts that follow, first a context of typed variables appears; the variables are all to be counted as the free variables of the concept. The standard uniform notation for the first, for instance, would be something like $= (A, x, y)$; what we actually use, $x = y$, is a drastic abbreviation.

$$x : A \,.\, y : A \quad :: \quad x = y \quad :=: \quad \exists\, e : \mathrm{Equ}(A, x, y) \,.\, \mathrm{TRUE}$$

$$x : A \,.\, y : B \,.\, f : A \to B \quad :: \quad y = f(x) \quad :=: \quad \exists\, a : \mathrm{App}(A, B, f, x, y) .\ \mathrm{TRUE}$$

$$f : A \to B \,.\, g : A \to B \quad :: \quad f = g \quad :=: \quad \forall\, x : A, y : B \,. \quad [y = f(x) \ \&\ y = g(x)]$$

$$f : A \to B \,.\, g : B \to C \,.\, h : A \to C \quad :: \quad h = gf \quad :=: \quad \forall x : A, y : B, z : C\,,$$

$$[z = h(x) \exists\, y : B[(y = f(x) \ \&\ z = g(y))]]$$

[Remark: this last definition echoes the final part of part [6] of our Lawvere quote.]

$$f : A \to A \quad :: \quad f = \mathrm{id}_A \quad :=: \quad \forall x : A,\ y : A \,.\, [y = f(x) \ \&\ y = x]$$

$$f : A \to B \quad :: \quad \mathrm{Iso}(f) \quad :=: \quad \exists g : B \to C \,.\, i : A \to A \,.\, j : B \to B$$

$$[i = \mathrm{id}_A \ \&\ j = \mathrm{id}_B \ \&\ i = gf \ \&\ j = fg]$$

$$A \cong B \quad :=: \quad \exists f : A \to B \,.\ \mathrm{Iso}(f).$$

In the notations of these defined concepts, we used the ordinary equality symbol, =. But that is merely a reminder of an analogy, rather than a direct identification with the ordinary concept. The defined concept $x = y$ makes sense under the type declarations to the left, and its free variables are, in fact, A, x, y rather than just x and y. The fact that A does not appear in the notation is a more serious abuse of language than the use of the symbol =. In the concept "$y = f(x)$", the free variables are A, B, f, x and y. $y = f(x)$ is not obtained by substitution from $y = w$; in FOLDS, we do not have *terms* such as $f(x)$.

We use the $\exists!$ phrase as an abbreviation in the expected manner; it can be applied prefixed to any typed variable.

Next, we list further axioms

$\boldsymbol{Ax}$2 Equality is an equivalence relation:

$$\vdash x, y, z : A \;.\; [x = x \;\&\; (x = y \to y = x) \;\&\; ((x = y \;\&\; y = z) \to x = z)]$$

$\boldsymbol{Ax}$3 Function-application is well-defined:

$$\vdash f : A \to B \;.\; x, u : A \;.\; y : B \;.\; [(x = u \;\&\; y = f(x)) \to y = f(u)]$$

$\boldsymbol{Ax}$4 Function-application is *operational*, the value is uniquely determined:

$$\vdash f : A \to B \;.\; x : A \;.\; y, z : B \; [(y = f(x) \;\&\; z = f(x)) \to y = z]$$

$\boldsymbol{Ax}$5 Function-application is *invariant* under equality:

$$\vdash f : A \to B \;.\; x : A \;.\; y, z : B \; [(y = f(x) \;\&\; y = z) \to z = f(x)]$$

$\boldsymbol{Ax}$6 Existence of the identity morphism:

$$\vdash \; \text{TRUE} \quad \Longrightarrow \quad f : A \to A \;.\; [f = \mathrm{id}_A]$$

$\boldsymbol{Ax}$7 Existence of the composite of composable arrows:

$$\vdash f : A \to B . g : B \to C \quad \Longrightarrow h : A \to C . \; [h = gf]$$

Axioms $Ax1$ to $Ax7$ are the axioms of the *minimal theory of abstract sets*. I will denote said theory by $T_{\mathrm{abss\,min}}$. A sample theorem of $T_{\mathrm{abss\,min}}$ is the associative law; in one possible form as follows:

$$\vDash \; f : A \to B \;.\; g : B \to C \;.\; h : C \to D \;.\; i : A \to C \;.\; j : B \to D \;.\; k : A \to D.$$

$$[(i = gf \;\&\; j = hg \;\&\; k = hi) \longrightarrow k = jf]$$

The informal semantics given above should be enough for the reader to construct a proof of the theorem, showing that it follows from the axioms. I still find it worth while elaborating on the formal semantics of the language.

I start with re-stating the complete type-structure of L_{abss}, the underlying language of a variety of possible axiomatic theories of abstract sets, among them the $T_{\text{abss min}}$.

$A : \text{Set}$

$A : \text{Set} \quad :: \quad x : \text{El}(A)$

$A, B : \text{Set} \quad :: \quad f : \text{Map}(A, B)$

$A : \text{Set} \;\; . \; x, y : \text{El}(A) \quad :: \quad e : \text{Equ}(A, x, y)$

$A, B : \text{Set} \;\; . \; f : \text{Map}(A, B) \; . \; x : \text{El}(A) \; . \; y : \text{El}(B) \quad :: \quad a : \text{App}(A, B, f, x, y)$

Each line is a rule on how one can introduce a new variable on the basis of variables previously introduced. For instance, the type $\text{Equ}(A, x, y)$, and thus the variable declaration $e : \text{Equ}(A, x, y)$ is grammatically correct if and only if $A{:}\text{Set}$ (A is a variable of type Set), and $x, y : \text{El}(A)$ (x and y are variables of type $\text{El}(A)$). It is to be noted that declarations such as $\{A : \text{Set} \;\; . \; x : \text{El}(A) \; . \; e : \text{Equ}(A, x, x)\}$, $\{A : \text{Set} \;\; . \; f : \text{Map}(A, A)\}$, with repeated variables inside a type, are correct, since they follow the rules.

The rules for the formation of (well-formed) formulas were described before in general for FOLDS.

The *ensemblist*, or set-theoretical, semantics of the language $L = L_{\text{abss}}$ can be given by turning the above typing rules into the specification of an interpretation of L, an L-structure, say M. This specification is that of a system of sets as follows. M consists of:

the set $M(\text{Set})$;

for (each element) A in $M(\text{Set})$, the set $M(\text{El})(A)$;

for A and B in $M(\text{Set})$, the set $M(\text{Map})(A, B)$;

for A in $M(\text{Set})$, x and y in $M(\text{El})(A)$, the set $M(\text{Equ})(A, x, y)$;

for A and B in $M(\text{Set})$, f in $M(\text{Map})(A, B)$, x in $M(\text{El})(A)$ and y in $M(\text{El})(B)$, the set $M(\text{App})(A, B, f, x, y)$.

This description of "L-structure" is well-suited to define the formal semantics of our language. When reading a formula in an L-structure M, for each variable, one reads an element of the appropriate type in M: e.g., for a variable $t : \text{App}(A, B, f, x, y)$, one reads an element in the set $M(\text{App})(A, B, f, x, y)$, where A, B, f, x, y are appropriate elements, that is, A, B are in $M(\text{Set})$, f is in $M(\text{Map})(A, B)$, x is in $M(\text{El})(A)$, y is in $M(\text{El})(B)$. The quantifier $\exists t{:}\text{App}(A, B, f, x, y)$ has a clear meaning when read in M: we mean the existence of some t in the set $M(\text{App})(A, B, f, x, y)$.

Let me note that in almost all of my previous writings and lectures I used a simplified kind of semantics, which, classically, is equivalent to the one given above. In the simplified version, an L-structure is a Set-valued *functor* from a certain category associated with the language. In the original write-up of FOLDS, the monograph [Makkai 1995], I also describe the "family approach" for FOLDS in general, the one adopted here in the special case at hand; see pp. 22 and 23 in *loc.cit.* From the point of view of the metaphysic of abstract sets, the "family approach" is more germane.

We can talk about *models* of a sentence of L, models of a theory T and a theorem of T in the expected senses. Using this semantics, it is not hard to verify that the associative law as stated is a theorem of $T_{\min}$.

$T = T_{\mathrm{abss\,min}}$ is a very rudimentary theory. The models of T are essentially the same as the concrete categories, that is, a small category C equipped with a faithful functor $F : C \to \mathrm{Set}$.

More precisely, let (C, F) be a concrete category. Define the L-structure M as follows:

$M(\mathrm{Set}) = \mathrm{Ob}(C)$;

for A in $M(\mathrm{Set})$, $M(\mathrm{El})(A) = F(A)$;

for A and B in $M(\mathrm{Set})$, $M(\mathrm{Map})(A, B) = \{f \in \mathrm{Arr}(C) : f : A \to B\}$;

for A in $M(\mathrm{Set})$, x and y in $M(\mathrm{El})(A)$, $M(\mathrm{Equ})(A, x, y) = \{0\}$ if $x = y$, otherwise $M(\mathrm{Equ})(A, x, y)$ is the empty set;

for A and B in $M(\mathrm{Set})$, f in $M(\mathrm{Map})(A, B)$, x in $\mathrm{El}(A)$ and y in $\mathrm{El}(B)$, $M(\mathrm{App})(A, B, f, x, y) = \{0\}$ if $(Ff)x = y$, otherwise $M(\mathrm{App})(A, B, f, x, y)$ is the empty set.

Let us call any L-structure obtained thus from a concrete category a *concrete* L-structure. We have:

(i) Every concrete L-structure is a model of T.

(ii) There are enough concrete L-structures: any statement in FOLDS over L which is a true in every concrete L-structure is true in every model of T.

The proof of (i) is a direct verification. For the proof of (ii), one has to prove properties of an arbitrary model M, for instance the associative law, and perform an operation involving the taking equivalence classes of elements and of arrows, to arrive at a concrete L-structure $[M]$ that is *equivalent* in a very strong sense (FOLDS equivalence) to M, in particular, M and $[M]$ will satisfy the same L-sentences.

We can state and prove the formal way of saying what we called Lawvere's imperative, "[an abstract set] has no external properties save its cardinality".

Proposition. Lawvere's imperative. *For any formula $\Phi(A)$ over the signature L_{abss} with the single free* Set *-variable A, the following is a theorem of $T_{\text{abss}\,\text{min}}$:*

$$\vDash [\Phi(A)\ \&\ A \cong B] \to \Phi(B)$$

In particular, in a concrete category, if an object A has a property expressible in the FOLDS language L for concrete categories, then any other object B that is isomorphic to A is going to have the same property.

It is natural that the proof should involve a structural induction on Φ, which involves formulas with more than one free variable.

Let X be a *context* of variables, that is, a finite set of variables obtained by successively adjoining variables declared according to the rules stated above. Let Y be an isomorphic copy of X disjoint from X, briefly a *copy* of X, with the isomorphism x in X mapped to $\bar{x}$ in Y. Construct the new context $I(X,Y)$ extending $X \cup Y$ thus. For each x in X of type Set, let $i(x)$ be a new variable of type $\mathrm{Map}(x,\bar{x})$; we add each $i(x)$ to $X \cup Y$. We define $\mathrm{Iso}(X,Y)$ to be the conjunction of the following formulas:

for $x : \mathrm{Set}$ in X, the formula "$\mathrm{Iso}(i(x))$";

for x, $y : \mathrm{Set}$, $f : x \to y$ all in X, the formula expressing that the square

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "i(x)"'] & y \arrow[d, "i(y)"] \\ \bar{x} \arrow[r, "\bar{f}"] & \bar{y} \end{tikzcd}$$

commutes: the formula "there is $k : x \to \bar{y}$ such that $k = i(y)\ .\ f$ and $k = \bar{f}\ .\ i(x)$";

for $x : \mathrm{Set}$, $u : \mathrm{El}(x)$ all in X, the formula "$\bar{u} = (i(x))(u)$".

Proposition. *Let Φ be a formula with the set of free variables the context X. With the formula $\mathrm{Iso}(X,Y)$ defined above, we have that the following is a theorem of $T_{\text{abss}\,\text{min}}$:*

$$\vDash \Phi(X)\ \&\ \mathrm{Iso}(X,Y) \quad \Longrightarrow \quad \Phi(Y)\,.$$

Suppose X and X' are contexts, X a subcontext of X' such that $X'-X$ is a singleton $\{u\}$. According to the five ways a variable can be declared in our language, u can be of five kinds of types. For instance, the fifth possibility is that $u : \mathrm{App}(A, B, f, x, y)$ with A, B, f, x, y appropriate variables in X. Let, further, Y' be a copy of Y extending a copy X' of X, let the context $I(X', Y')$ be constructed to extend $I(X, Y)$.

Consider the context $I(X', Y')$. It has the following new elements with respect to $I(X, Y) : u$ and $\bar{u}$, and if $u : \mathrm{Set}$, one more, the arrow $i(u) : u \to \bar{u}$. We have the formula $\mathrm{Iso}(X', Y')$ in the context $I(X', Y')$. We write $\mathrm{Iso}^*(X', Y')$ for $\exists\, i(u) : u \to \bar{u}.\ \mathrm{Iso}(X', Y')$ when $u : \mathrm{Set}$; otherwise, we let $\mathrm{Iso}^*(X', Y')$ be the same as $\mathrm{Iso}(X', Y')$. The set of free variables of the formula $\mathrm{Iso}^*(X', Y')$ is $I(X, Y) \cup \{u, \bar{u}\}$ in all cases.

Lemma. *With the above notation, the (universal closure of the) following is a theorem of $T_{\mathrm{abss\,min}}$:*

$$\vDash \mathrm{Iso}(X, Y) \to \forall\, u : \tau\ .\ \exists\, \bar{u} : \bar{\tau}\, \mathrm{Iso}^*(X', Y')\,.$$

"$u : \tau$" is the typing of u (an example of which we gave above); $\bar{u}$ is the copy of u in Y'.

The lemma expresses that, given any isomorphism f of the "diagrams" X and Y, and any additional element u "based on X", we can find an element $\bar{u}$ "based on Y", and an isomorphism extending f of $X' = X \cup \{u\}$ onto $Y' = Y \cup \{\bar{u}\}$. For a category theorist, this is a triviality, except that here "diagram" has a slightly different and extended sense compared to that in category theory. The proof of the lemma is of no difficulty; one has to distinguish the "five cases" mentioned above.

The proof of the proposition using the lemma is also straightforward.

The last-stated proposition extends "Lawvere's imperative" in an interesting manner, of significance to structuralism; we may call it "Benaceraff's imperative" (see [Benaceraff 1965], [McLarty 1993], [Makkai 1999]). Let us take the notion of *group* in abstract set theory to be the same as the standard one adopted in category theory, a *group-object* in a category. A group-object is a *diagram* consisting of an object G – set for us– ; another set, the (categorical) product $G \times G$, equipped with projections π_0 and $\pi_1 \colon G \times G \to G$; the group-operation $m \colon G \times G \to G$; the unit map $1 \to G$ (with 1 the terminal object, which is a one-element set in abstract set-theory); these data being subject to conditions specifying, in the first place, that $G \times G$ with the given projections is indeed a categorical product, 1 is a one-element set, and then, and mainly, the usual group laws. From our point of view the conditions mentioned do not matter at all: the notion of isomorphism of two groups is the same as that of an isomorphism of two diagrams as used in the Proposition, actually involving only two kinds of variables, set-types and arrow-types. The upshot is that every property of a group expressible in *any* FOLDS theory T of abstract sets, (but) *in the given language* L_{abss}, satisfying the minimal condition that T extends $T_{\mathrm{abss\,min}}$, is invariant under isomorphisms of groups.

5. More on sets, and beyond sets

The exposition of abstract set theory in the previous sections may be called the purely analytic part of the theory. There are further considerations of a synthetic nature: judgments concerning the existence of abstract sets and more generally, the existence of *abstract structures* consisting of abstract sets and other entities named in the language with prescribed properties. (We may see the abstract structures as *realized contexts*, with reference to the syntactic idea of *context of variables*. Groups above are an example of abstract structure.) The set-existence axioms of ZFC set theory are synthetic judgments in the context of the metaphysic of first-order axiomatic set theory (by first-order logic, I mean first-order logic with equality in the usual sense). Below, I will point out that the analytic framework of abstract set theory is capable of articulating first-order set-theoretical judgments *provided we adopt the axiom of regularity*. The reason is that it is possible to adequately treat *pure sets*, sets for which the axiom of regularity holds, as abstract structures.

On the other hand, TTCFM goes beyond abstract set theory. The problems of set-existence for pure-set theory, embodied by the Burali-Forti paradox, encourage a new way, the category-theoretical way, of construing the totality of abstract sets as a *particular* category, *the category of sets*. We introduce the *language of categories*, in which there is a single ground kind, CATEGORY. The variable declaration $C : \mathrm{CATEGORY}$ (possibly with another letter for C) starts every context (system of variable declarations) in abstract category theory, just like $A : \mathrm{SET}$ (with another letter for A, perhaps) starts every context in abstract set theory. Just like the language of abstract sets has led in a natural way to the concept of category, and then to the language of categories, the language of categories paves the way to considering the totality of categories as a particular case of *two-dimensional category*, the *two-dimensional category of categories.* In that, the category of sets appears as a particular ground object, characterized by certain specifications, characterized not uniquely in a Fregean absolute sense, but up to categorical *equivalence*, the notion that takes the place of "having the same cardinality" ("isomorphic") for sets.

Just like a category is not a set (it has no underlying set since there is no equality assumed on its objects), a two-dimensional category is not a category any more, not even a category with additional structure; for instance, the composition of 1-cells (arrows) is not (strictly) associative.

It is now natural that there should be n-categories for every non-negative integer n ($n = 0$ is the case of sets) so *that the totality of n-categories is a an* $(n+1)$*-category.* And thereby lies a story, the story of higher (dimensional) categories, a story that is far from having reached an end, due to the(interesting!) mathematical complexities arising. The story is driven by what I call the Fregean imperative: *construe as an object* the totality of entities of a definite species established at an earlier stage of the synthetic process – *if necessary* as a particular instance of a new kind of totality. I note that the totality of ω-categories is again an ω-category; here, the "new" totality is not of a new kind!

It seems necessary to stress some grammatical clarifications here. As we pointed out in the previous sections, the language of abstract sets (plural!) can be considered as the language of a *single concrete* category. We could have used instead a FOLDS language of a *single* category (without "concrete"), and indeed in that way we would have gotten closer to Lawvere's intention, the introduction of the notion of (elementary) topos. The pure FOLDS language L_{cat} of a single category is given explicitly in [Makkai 1998]; I have used it on innumerable occasions of talking about FOLDS. The axioms for a topos appear as well-formed FOLDS sentences over L_{cat}. Like L_{abss} in the previous sections, L_{cat} also has five "kinds" , but of course, those kinds are different; the main difference being that in L_{cat} composition is primitive as in Lawvere's metaphysic. L_{cat} can be *interpreted* in L_{abss}, and in fact, much of that interpretation has been carried out in the previous section. Said interpretability is essentially the same as saying that every concrete category has an underlying category. The language of categories (plural!) mentioned in the previous paragraph is a "higher" language, in which the plurality of categories serves as *the range of a variable.*

What I said in the previous paragraphs is very sketchy. For instance, I suppressed the adjective "weak" from "higher category". If the reader wants to relate to the literature what was said above, he/she has to include that adjective with "two-dimensional category", "n-category", etc.

Let me return to pure sets. Let us put ourselves into a first-order theory (with equality) of sets such as ZFC, although only a small fragment of that theory is necessary, and more importantly, only intuitionistic logic is needed. Let x be any set, let $\mathrm{tr}(\{x\})$ be the transitive closure of the singleton $\{x\}$, that is the least set y such that y is transitive ($u \in v \in y$ imply that $u \in y$) and x belongs to y. Consider the structure X whose underlying set is $\mathrm{tr}(\{x\})$, has the distinguished element x, and is equipped with the binary relation $\in$ restricted to the underlying set. It turns out, *assuming the axiom of regularity* which will be assumed from now on, that the structures $X[x] = (\mathrm{tr}(\{x\}); x, \in\restriction \mathrm{tr}(\{x\}))$ determined by an arbitrary (pure) set x (under the axiom of regularity, all sets are pure) can be *characterized up to isomorphism* as the *pure-set structures*, the structures $X = (|X|; x, E)$, with E a binary relation on $|X|$, x an element of $|X|$, that satisfy the following conditions:

1) E is transitive: $uEvEw$ imply uEw;

2) E is well-founded: Call the subset A of $|X|$ inductive if for all u in $|X|$, the fact that $\{v : vEu\}$ is a subset of A implies that u is in A. The condition is: if A is inductive, then $A = |X|$.

3) E is extensional: $u = v$ iff for all w, wEu iff wEv;

4) $|X|$ is the downward E-closure of $\{x\}$: for all subsets A of $|X|$, if x is in A, and for all v in A and u such that uEv, we have that u is in A, then $A = |X|$.

The pure-set structures $X[x]$ defined above are called the standard pure-set structures. We have:

For every pure-set structure X, there is a unique pure set x such that X is isomorphic to $X[x]$; moreover, the isomorphism $X \to X[x]$ is uniquely determined.

In fact, the distinguished element x can be taken out of the naming of the structure, and instead its existence satisfying condition 4) required, since if such x exists, it is necessarily unique.

The just-stated assertion is a variant of what is called Mostowski's collapsing lemma. For the *transitive structures* $X = (|X|, E)$, the pure-set structures without the distinguished element x and condition 4), we have that for any transitive structures X and Y, there is at most one structure preserving morphism from X to Y, and if $f : X \to Y$ is such, then f maps X to an initial subset of Y, that is, a subset B of $|Y|$ such that $u(E_Y)v$ and $v \in B$ imply that u is in B. In particular, there is but a single map $X \to X$, the identity. Moreover, for every transitive structure X, there is a unique transitive set y such that X is isomorphic to $(y, \in\restriction y)$; usually this latter statement is the one that is called the Mostowski lemma.

Let us denote the unique x as in the displayed statement by $x[X]$. From now on, X and Y will mean pure-set structures; $X = (|X|, x, E_X), Y = (|Y|, y, E_Y)$. We write $X = Y$ for the statement that X is isomorphic Y, and $X \in Y$ for the following statement: there is $f : (|X|, E_X) \to (|Y|, E_Y)$ such that $f(x)(E_Y)y$. We have the following two facts provable in set theory:

For pure-set structures X and Y, $X = Y$ iff $x[X] = x[Y]$ and $X \in Y$ iff $x[X]$ is an element of $x[Y]$.

The reader will probably see that the above facts add up to a complete abstract-set theoretical statement of pure-set theory. In L_{abss}, consider the variable declaration

$$A, E : \text{Set} \;\; . \; l, \; r : E \to A \; . \; x \; : \; \text{El}(A).$$

The intention of the two arrows $E \to A$ is to make up a binary relation on the set A. We can write down a FOLDS formula Φ with exactly the free variables mentioned that *expresses* that (A, x, E) is a pure-set structure. $\Phi(A, E, l, r, x)$ says that A is a set, (E, l, r) is a relation on A meaning that for every a and b in A, there is at most one e in E such that $l(e) = a$ and $r(e) = b$ (we will say that aEb if there is such e), and the conditions 1) to 4) are satisfied. For the formulation of Φ in L_{abss}, we need to talk about subsets of A in abstract set theory; we do this as told to us by topos theory. (Lawvere's [Lawvere 1976], the starting point of the present paper, has much information about the both the motivation and the mathematics of topos theory.)

The axioms of pure set theory can be stated within abstract set theory. For instance, the axiom of extensionality becomes

$$\forall X \text{ and } Y[\Phi(X) \,\&\, \Phi(Y) \quad \Longrightarrow \quad [X = Y \leftrightarrow \forall Z \,.\, [\Phi(Z) \to (Z \in X \text{ iff } Z \in Y)]]]\,.$$

Of course, here X and Y are abbreviations for tuples like (A, E, l, r, x) above appropriately declared. We will have the easy meta-theorem that states that the thus-obtained theory over L_{abss} is deductively equivalent to the pure-set theory we started with.

I have spelled out the above details about pure sets to emphasize that from the point of view of abstract set theory, usual axiomatic set theory (in which the axiom of regularity is assumed) is a theory of a particular kind of *structure*, similarly to groups or topological spaces. Topological spaces are the more relevant here, since their definition is not first-order as that of "group" is. Pure sets are not first-order-defined structures: witness the universal quantifiers on subsets in their definition.

Developing pure-set theory within abstract-set theory is a natural thing to do only when it is done on the bases of a *natural* axiom system for abstract sets. The development would use the definition $\Phi(X)$ given above for "pure set". This definition would be treated in the theory as, for instance, the definition of topological space is treated in topology.

A natural axiom system for abstract sets is obtained by extending $T_{\text{abss min}}$ and adopting the topos axioms formulated in our language; let us call the resulting theory T_{topos}; it is a FOLDS theory over our original language L_{abss}. To give examples of theorems in T_{topos}: the "axiom of extensionality" stated above, and the translation of the usual power-set axiom are provable. Other, stronger, systems can also be contemplated; for instance we can consider an unrestricted subset comprehension axiom scheme. Compare [Makkai 2010].

The point of view of abstract set theory will maintain that it is unnatural and unnecessary to assume that the underlying set of every group also carries a pure-set structure, quite unrelated to the group structure itself – but note that ordinary axiomatic set theory (although not naive set theory!) is making just this "superfluous" assumption!

Historical remarks

For FOLDS, see [Makkai 1995] and [Makkai 1998]. For further aspects of TTCFM, see [Hermida et al. 2000/2001/2002] and [Makkai 1999/2004].

In the summer of 2003, after the Chicago meeting of the ASL, I have posted two communications on the FOM list (fom@cs.nyu.edu), the first on June 22nd, the second on July 18th, in which I outlined the system for abstract sets discussed in the present paper; see [Makkai, 2003]. The language L_{abss} was given in identical terms on page 5 of the second posting. I did not state the propositions that now I call Lawvere's and Benaceraff's imperatives, but on page 3 of the second posting I say "we will have metamathematical results to the effect that what we do is precisely what is necessary to maintain the right invariance properties of all statements and constructions", for instance, concerning groups. The treatment of pure sets given in some detail here is hinted at in the second posting.

An important forerunner to this paper is Colin McLarty's paper [McLarty 1993]. He bases his discussion on [Lawvere 1964] and claims that Lawvere's Elementary Theory of the Category of Sets (ETCS) is a theory that satisfies what I called Benaceraff's imperative. On page 494, we find the statement of the important special case of Benaceraff's imperative for the natural number object as abstract structure, and on page 495 of the paper, we find the statement of Lawvere's imperative. There are outlines of proofs of the statements being theorem schemas of ETCS. However, the reservations I had about Lawvere's statement in the passage of [4] above apply here too: there is no specification in McLarty's paper, just as there is none in Lawvere's paper, of an adequate language underlying the statements that are supposed to be invariant under isomorphism. Thus, my only, but essential, contribution in the present paper and it's earlier versions [Makkai 2003] is such a language, the FOLDS language of abstract set theory.

The paper [Lawvere 2005] is the reprint of the full version, already in existence in 1965, of [Lawvere 1964]. [Lawvere 2005] is a detailed exposition of the theory of abstract sets in classical (Boolean) first-order logic. Although there is no indication in [Lawvere 2005] that first order logic can be, or should be, restricted in the way L_{abss} does it, the theory in [Lawvere 2005] can be translated without any essential change in the content of the theory into an exposition of abstract set theory within our FOLDS language L_{abss}.

Our discussion of pure sets versus abstract sets is related to section 9.2 of [Johnstone 1977].

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