§11 Regular cardinals

In what follows, κ, λ, μ, ν, ρ always denote cardinals.

A cardinal κ is said to be *regular* if κ is infinite, and the union of fewer than κ sets, each of whose cardinality is less than κ, is of cardinality less than κ. In symbols: κ is regular if κ is infinite, and

for any set I with $|I| < \kappa$ and any family $\langle A_i \rangle_{i \in I}$ of sets such that $|A_i| < \kappa$ for all $i \in I$, we have $|\bigcup_{i \in I} A_i| < \kappa$.

(Among finite cardinals, only 0, 1 and 2 satisfy the displayed condition; it is not worth including these among the cardinals that we want to call "regular".)

To see two examples, we know that $\aleph_0$ is regular: this is just to say that the union of finitely many finite sets is finite. We have also seen that $\aleph_1$ is regular: the meaning of this is that the union of countably many countable sets is countable.

For future use, let us note the simple fact that

the ordinal-least-upper-bound of any set of cardinals is a cardinal; for any set I, and cardinals λ_i for $i \in I$, $\operatorname{lub}_{i \in I} \lambda_i$ is a cardinal.

Indeed, if $\alpha = \operatorname{lub}_{i \in I} \lambda_i$, and $\beta < \alpha$, then for some $i \in I$, $\beta < \lambda_i$. If we had $\beta \backsim \alpha$, then by $\beta < \lambda_i \leq \alpha$, and Cantor-Bernstein, we would have $\beta \backsim \lambda_i$, contradicting the facts that λ_i is a cardinal and $\beta < \lambda_i$. Thus, if $\beta < \alpha$, then $\beta \not\backsim \alpha$, which means that α is a cardinal.

The condition of regularity can be stated in the following equivalent manner: the infinite cardinal κ is regular iff

(1) for each set I with cardinality less than κ, and for each family $\langle \alpha_i \rangle_{i \in I}$ of ordinals α_i, each less than κ, we have that $\operatorname{lub}_{i \in I} \alpha_i < \kappa$.

Indeed, assuming that κ is regular, and assuming that $|I| < \kappa$ and $\alpha_i < \kappa$ for all $i \in I$, we have that $|\alpha_i| \leq \alpha_i < \kappa$, thus, $|\operatorname{lub}_{i\in I} \alpha_i| = |\bigcup_{i\in I} \alpha_i| < \kappa$; but, for any cardinal κ, and any ordinal α, $|\alpha| < \kappa$ implies that $\alpha < \kappa$ (why?); thus, $\operatorname{lub}_{i\in I} \alpha_i < \kappa$ follows.

Conversely, assume that (1) holds, and show that κ is regular. In particular, κ is a limit ordinal. Let $|I| < \kappa$ and $|A_i| < \kappa$ for each i. Let $\lambda_i = |A_i|$; λ_i is an ordinal less than κ. Hence, by (1), $\lambda \operatorname{d\bar{e}f} \operatorname{lub}_{i\in I} \lambda_i < \kappa$. As we noted above, λ is a cardinal. Let $\mu = \max(\lambda, |I|)$; since λ, $|I|$ are both cardinals less than κ, μ is a cardinal $< \kappa$. But then $|I| \leq \mu$ and $|A_i| = \lambda_i \leq \lambda \leq \mu$. Hence, if μ is infinite, we have

$$|\bigcup_{i\in I} A_i| \leq |\sum_{i\in I} A_i| \leq \mu \cdot \mu = \mu < \kappa;$$

(§10)

and if μ is finite, then $\bigcup_{i\in I} A_i$ is finite, and so again

$$|\bigcup_{i\in I} A_i| < \kappa.$$

This shows that κ is regular.

In fact, we can formulate regularity in terms of ordinals, without (overtly) referring to cardinals. Let us say of a limit ordinal δ that it is *regular* if it satisfies

(1') for any ordinal β less than δ, and for each family $\langle \alpha_\iota \rangle_{\iota<\beta}$ of ordinals α_ι, each less than δ, we have that $\operatorname{lub}_{\iota<\beta} \alpha_\iota < \delta$.

Note that if κ is an infinite cardinal, then it is a limit ordinal, and if it satisfies (1), then it satisfies (1'), since $\beta < \kappa$ implies that $|\beta| < \kappa$. Conversely, I claim that if the limit ordinal δ satisfies (1'), then it is a regular cardinal. δ is infinite. Assume (1') for such a δ. Let $\beta < \delta$. If we had $\beta \sim \delta$, then we would have an indexing $\langle \alpha_\iota \rangle_{\iota<\beta}$ of *all* ordinals less than δ, and so, also using that δ is limit, $\operatorname{lub}_{\iota<\beta} \alpha_\iota = \operatorname{lsub}_{\iota<\beta} \alpha_\iota = \delta$, contradicting (1'). This shows that δ is a cardinal. But now looking at formulation (1), if we have an arbitrary set I with

$|I|<\delta$, we can index I by an ordinal $\beta<\delta$, and transform any instance of (1) into one of (1').

We have shown that an equivalent definition of "regular cardinal" is given by the condition (1'), together with the condition that δ is a limit ordinal.

We have a large class of regular cardinals. Let us write κ^+ for the least cardinal greater than κ. If $\kappa=n\in\omega$, then $\kappa^+=n+1$, and if $\kappa=\aleph_\alpha$, then $\kappa^+=\aleph_{\alpha+1}$. A *successor cardinal* is one of the form κ^+. The fact is that all infinite successor cardinals are regular. The reason is that if $|I|<\kappa^+$ and $|A_i|<\kappa^+$ for $i\in I$, then $|A_i|\le\kappa$, and $|I|\le\kappa$, thus

$$\left|\bigcup_{i\in I}A_i\right|\le\sum_{i\in I}|A_i|\le|I|\times\kappa\le\kappa\cdot\kappa=\kappa<\kappa.$$
$$\uparrow$$
$$\S10$$

On the other hand, e.g. $\aleph_\omega=\operatorname{lub}_{n<\omega}\aleph_n$ is not regular (we say *singular* for not regular). Namely, now $|\omega|=\omega=\aleph_0<\aleph_\omega$, and for each $n\in\omega$, $|\aleph_n|=\aleph_n<\aleph_\omega$, but $\left|\bigcup_{n\in\omega}\aleph_n\right|=|\aleph_\omega|=\aleph_\omega$; thus, the family $\langle\aleph_n\rangle_{n\in\omega}$ is a counter-example to the regularity condition for $\aleph_\omega$.

We have now seen that the cardinals $\aleph_0$ and $\aleph_{\alpha+1}$ are regular. We just saw that for $\alpha=\omega$, $\aleph_\omega$ is singular. Are there *limit cardinals,* that is, cardinals of the form $\aleph_\delta$ for a limit ordinal δ that are regular? One sees that if so, then we must have $\aleph_\delta=\delta$: the reason is that the set $\{\aleph_\alpha:\alpha<\delta\}$ is *cofinal* in $\aleph_\delta$: $\operatorname{lub}_{\alpha<\delta}\aleph_\alpha=\aleph_\delta$ (why?); therefore, if $\delta<\alpha$ were the case, we would have a contradiction to formulation (1') of the regularity of $\aleph_\delta$. However, if we take the *smallest* δ for which $\aleph_\delta=\delta$, $\kappa=\aleph_\delta$ turns out to be singular; thus, the above does not have a converse! Consider the following sequence:

$$\kappa_0=\aleph_0$$

$$\kappa_{n+1}=\aleph_{(\kappa_n)}\qquad(n<\omega)$$

defined recursively. We have $\kappa_0<\kappa_1$ since $0<\aleph_0$. By an easy induction, we show that $\kappa_n<\kappa_{n+1}$ for all $n<\omega$. Let us define $\lambda=\operatorname{lub}_{n<\omega}\kappa_n$. I claim that $\aleph_\lambda=\lambda$. We have

$\aleph_\lambda = \operatorname{lub}_{\alpha<\lambda} \aleph_\alpha = \operatorname{lub}_{n<\omega} \aleph_{\kappa_n}$ since $\lambda=\operatorname{lub}_{n<\omega} \kappa_n$ (**exercise**: show that if $\alpha_\beta \le \alpha_{\beta'}$ for $\beta \le \beta' < \gamma$, and $\gamma=\operatorname{lub}_{i\in I} \beta_i$, then $\operatorname{lub}_{i\in I} \alpha_{\beta_i} = \operatorname{lub}_{\beta<\gamma} \alpha_\beta$); but $\aleph_{\kappa_n} = \kappa_{n+1}$, and so $\aleph_\lambda = \operatorname{lub}_{n<\omega} \kappa_{n+1} = \lambda$. However, $\lambda=\aleph_\lambda$ is not regular: we have $\lambda = \operatorname{lsub}_{n<\omega} \kappa_n$ and $\omega<\lambda$.

It turns out that, using the usual axioms of set-theory, one cannot prove that there are regular limit cardinals. A regular limit cardinal is called a *weakly inaccessible* cardinal. A *strongly inaccessible*, or simply *inaccessible*, cardinal κ has, by definition, the additional property that for all $\lambda<\kappa$, we also have that $2^\lambda<\kappa$. It is easy to see that limit cardinals with this latter property are exactly the ones of the form $\operatorname{beth}_\delta$ for a limit ordinal δ; these latter are called *strong limit cardinals*. Thus, an inaccessible cardinal is the same as a regular strong limit cardinal. In section 13, we will take a look at inaccessible cardinals -- despite the fact that we cannot prove their existence with the axioms so far listed. In fact, the existence of inaccessible cardinals will be seen as a reasonable *new* axiom of set theory.

There is an important calculation of the cardinality of a particular kind of set involving regularity. We consider the following situation. We have $(B, \mathcal{F})$ where

(i) B is a set

and

(ii) $\mathcal{F}$ is a family $\mathcal{F} = \langle f_i \rangle_{i\in I}$ of *partial operations* on B, that is, each f_i is a function with domain contained in B^{J_i}, $\operatorname{dom}(f_i) \subset B^{J_i}$ for a particular "arity" J_i (an arbitrary set, given for each $i\in I$; usually, J_i is a natural number) and $\operatorname{range}(f_i) \subset B$. As a reminder,

$B^J = \{f\colon\ f$ is a function, $\operatorname{dom}(f)=J$, $\operatorname{range}(f)\subset B\}$.)

We are going to write $f\colon\subset A \longrightarrow B$ to mean that $f\colon\operatorname{dom}(f) \longrightarrow B$ and $\operatorname{dom}(f) \subset A$. Thus, the partial J_i-ary operation f_i may be displayed as $f_i\colon\subset B^{J_i} \longrightarrow B$.

Examples. (A) The notion of *group* is a very important one in mathematics. A group is a set B together with a binary operation $\cdot : B^2 \to B$, a unary operation $(\,)^{-1} : B \to B$, and a 0-ary operation $e \in B$, where these data are required to satisfy the identities $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $e \cdot x = x \cdot e = x$, $x^{-1} \cdot x = x \cdot x^{-1} = e$. This fits our setup above, with $I = 3$, $J_0 = 2$, $f_0 = \cdot$, $J_1 = 1$, $f_1 = (\,)^{-1}$, $J_2 = 0$, $f_2 = e$ (note that a 0-ary operation $f : A^0 \to A$ is basically the same as a distinguished element of A : $A^0 = \{\emptyset\} = 1$, and to give a function $f : \{\emptyset\} \to A$ is the same as to give the element $e = f(\emptyset) \in A$). In this case, all operations are *total*, that is, $\mathrm{dom}(f_0) = B^2$, etc.

A similar example is provided by each of a number of other kinds of algebraic structure: lattice, Boolean algebra (these we will see in some detail later), ring, field, etc. (In fact, in the case of a field, division is a genuinely *partial* operation.)

(B) The notion of limit in analysis can be considered as a partial operation. If $\vec{x} \in \mathbb{R}^\omega$ is an infinite (ω-type) sequence of reals, $\vec{x} = \langle x_n \rangle_{n<\omega}$, then $\lim \vec{x} = \lim_{n \to \infty} x_n$ may not exist, but if it does, it is uniquely determined. Let $\lim : \subset \mathbb{R}^\omega \longrightarrow \mathbb{R}$ be the partial operation whose domain is the set of those sequences of which the limit exists, and whose value, at such a sequence, is the limit of that sequence. In this case, the arity of $\lim$ is the infinite ordinal ω.

Lim can also be defined on $(\mathbb{R}^{\mathbb{R}})^\omega$. That is, if $\vec{f} \in (\mathbb{R}^{\mathbb{R}})^\omega$, i.e., if $\vec{f}$ is a sequence $\vec{f} = \langle f_n \rangle_{n<\omega}$ of functions $f_n : \mathbb{R} \to \mathbb{R}$ from the reals to the reals, then $\lim \vec{f}$ is the function $f : \mathbb{R} \to \mathbb{R}$ for which $f(x) = \lim_{n \to \infty} f_n(x)$; $\lim \vec{f}$ is defined iff the limit $\lim_{n \to \infty} f_n(x)$ exists for all $x \in \mathbb{R}$ ("pointwise limit").

Below, we will consider the systems $(\mathbb{R}, \lim)$, $(\mathbb{R}^{\mathbb{R}}, \lim)$ as examples of $(B, \mathcal{F})$ of the general situation; in both cases, $I = 1 = \{0\}$ and $f_0 = \lim$.

Let us return to a general "system" $(B, \mathcal{F})$ as above. Relative to $(B, \mathcal{F})$, a *closed set* is any subset X of B for which

$$\vec{x} \in X^{J_i} \cap \mathrm{dom}(f_i) \implies f(\vec{x}) \in X$$

whenever $i \in I$, $\vec{x} \in B^{J_i}$. For instance, if $(B.\mathcal{F})$ is a group, then a closed set is what is usually called a *subgroup* of the given group: a subset of the (underlying set of the) group which is closed under $\cdot$, $(\ )^{-1}$ and contains e. In the "limit" examples, closed sets are what are usually called *closed*: closed under limits (in the function-space example, this is only one possible kind of closedness; there are also other "topologies" on that space).

It is clear that the intersection of any family of closed sets is closed (**exercise**; the empty intersection is taken to be B itself). Therefore, for any subset A of B, we may take the intersection of all closed sets containing A, and call it the *closure*, or *$\mathcal{F}$-closure* of A; notation: $\bar{A}$, or $\bar{A}^{(\mathcal{F})}$.

In the example of type (A) (groups, Boolean algebras, ...), $\bar{A}$ is called the *subalgebra generated by* A; in the examples under (B), it is called the *closure* of the set A.

What we want to do is calculate the cardinality of $\bar{A}$, in terms of some cardinalities given with the situation. Let us denote:

$$\lambda \underset{\mathrm{def}}{=} \max(\aleph_0, |I|) ,$$

$$\mu \underset{\mathrm{def}}{=} \max(\aleph_0, |A|)$$

(that is, $\lambda = |I|$ unless I is finite, in which case $\lambda = \aleph_0$; similarly for μ),

and

$$\kappa \underset{\mathrm{def}}{=} \text{the least infinite } \textit{regular} \text{ cardinal for which } \kappa > |J_i| \text{ for every } i \in I .$$

We use the notation

$$\nu^{<\kappa} \overset{\text{def}}{=} \text{lub}\{\nu^{\rho}: \rho < \kappa\,,\ \rho \text{ any cardinal}\}$$

We have, with these notations, the estimate formulated in the next

Theorem. With $(B, \mathcal{F})$ any system of partial operations, we have, for the cardinality $|\bar{A}|$ of the closure $\bar{A}$ of a subset A of B, that

$$|\bar{A}| \leq \max(\lambda, \mu)^{<\kappa} \qquad (2)$$

with κ, λ and μ as determined above.

Of course, we always have that $\mu \leq |\bar{A}|$ (why?); the two estimates determine $|\bar{A}|$ quite sharply (see also more on this below).

Before we turn to the proof of (2), we note three properties of the operation $\nu^{<\kappa}$. One is

$$\nu \leq \nu^{<\kappa} \quad (\text{for } \kappa \geq 2), \qquad (3)$$

which is obvious; the other is

$$\kappa \leq \nu^{<\kappa} \quad (\text{for } \nu \geq 2); \qquad (4)$$

the third is

$$(\nu^{<\kappa})^{\rho} = \nu^{<\kappa} \quad (\kappa \text{ infinite regular}, \ \rho<\kappa\,, \ \nu\geq 2). \qquad (5)$$

For (4), note that $\rho < 2^{\rho} \leq \nu^{\rho} \leq \nu^{<\kappa}$ for every $\rho < \kappa$; but κ is the least cardinal $>$ all
$\rho < \kappa$; it follows that $\kappa \leq \nu^{<\kappa}$.

(5) is slightly more involved. We claim that

$$^{\rho}(\nu^{<\kappa}) = \bigcup \{^{\rho}(\nu^{\mu}) : \mu < \kappa\};$$

here, we have made a distinction between exponentiation "of sets" and that of "cardinals"; $^{\rho}\sigma$ means the set of all functions $\rho \to \sigma$; ν^{μ} is a cardinal (in particular, an ordinal), the result of cardinal exponentiation applied to the cardinal-arguments ν and μ. In this display, on the left side, we have the set of all ρ-type sequences of ordinals $< \nu^{<\kappa}$; on the right, we have a union of sets of ρ-type sequences.

The claim is clearly the same as to say that for any $\langle \alpha_{\xi} \rangle_{\xi<\rho} \in {}^{\rho}(\nu^{<\kappa})$, there is $\mu < \kappa$ such that $\langle \alpha_{\xi} \rangle_{\xi<\rho} \in {}^{\rho}(\nu^{\mu})$. But, for each $\xi<\rho$, $\alpha_{\xi} \in \nu^{\mu_{\xi}}$ for some $\mu_{\xi} < \kappa$, since $\nu^{<\kappa}$ is the union of the sets ν^{μ}, $\mu < \kappa$. Since κ is regular, $\rho < \kappa$ and each $\mu_{\xi} < \kappa$, we have that $\operatorname{lub}_{\xi<\rho} \mu_{\xi} < \kappa$ (see the second definition of regularity). With $\mu = \operatorname{lub}_{\xi<\rho} \mu_{\xi}$, μ is a cardinal, $\nu^{\mu_{\xi}} \leq \nu^{\mu}$ for all $\xi < \rho$, and thus $\alpha_{\xi} \in \nu^{\mu}$ for all $\xi < \rho$. We have that $\langle \alpha_{\xi} \rangle_{\xi<\rho} \in {}^{\rho}(\nu^{\mu})$ as we wanted.

On the basis of the last claim, we have

$$(\nu^{<\kappa})^{\rho} = |{}^{\rho}(\nu^{<\kappa})| = |\bigcup \{^{\rho}(\nu^{\mu}) : \mu < \kappa\}|$$
$$\leq \sum_{\mu<\kappa} (\nu^{\mu})^{\rho} = \sum_{\mu<\kappa} \nu^{\mu\cdot\rho} \leq \kappa \cdot \nu^{<\kappa} = \nu^{<\kappa}$$
$$\uparrow \qquad \uparrow$$
$$\mu\cdot\rho=\max(\mu,\rho)<\kappa \quad (4)$$

as required for (5).

To prove (2), we give a formula for the set $\bar{A}$ itself first. Define, by recursion on the ordinal

$\alpha < \kappa$, the sets A_α as follows:

$$A_0 = A$$

$$A_\alpha = \bigcup_{\beta<\alpha} A_\beta \quad \text{if } \alpha < \kappa \text{ is a limit ordinal; and}$$

$$A_{\beta+1} = A_\beta \cup \{f_i(\vec{a}) : i \in I,\ \vec{a} \in \mathrm{dom}(f_i) \cap A_\beta^{J_i}\}\ .$$

Clearly, $A_\beta \subset A_\alpha \subset B$ whenever $\beta < \alpha < \kappa$. We claim that

$$\bar{A} = \bigcup_{\alpha<\kappa} A_\alpha\ . \qquad (6)$$

For one thing, the right-hand-side is contained in the left-hand-side: by induction on $\alpha < \kappa$, we show that $A_\alpha \subset \bar{A}$. The induction-steps for $\alpha = 0$ and α limit are obvious. For $\alpha = \beta+1$, note that since $A_\beta \subset \bar{A}$ (induction hypothesis), and since $\bar{A}$ is closed, $f_i(\vec{a}) \in \bar{A}$ whenever $i \in I$ and $\vec{a} \in \mathrm{dom}(f_i) \cap A_\beta^{J_i}$; thus, by the definition of $A_{\beta+1}$, $A_{\beta+1} \subset \bar{A}$ as desired.

Secondly, we need to show that the union $\bigcup_{\alpha<\kappa} A_\alpha$ is closed; since $\bar{A}$ is the least closed set containing A, and the union contains A, it will follow that $\bar{A} \subset \bigcup_{\alpha<\kappa} A_\alpha$, which is the other containment we need.

To show that the union is closed, let $i \in I$, and $\vec{a} = \langle a_j \rangle_{j \in J_i} \in (\bigcup_{\alpha<\kappa} A_\alpha)^{J_i} \cap \mathrm{dom}(f_i)$; we want that $f_i(\vec{a}) \in \bigcup_{\alpha<\kappa} A_\alpha$. Since $a_j \in \bar{A}$, there is $\alpha_j < \kappa$ such that $a_j \in A_{\alpha_j}$. We now have the family $\langle \alpha_j \rangle_{j \in J_i}$ of ordinals less than κ. By the definition of κ, we have that $|J_i| < \kappa$; also, κ is regular. Therefore, by the second version of the definition of "regular",

$$\beta \underset{\mathrm{def}}{=} \mathrm{lub}\{\alpha_j : j \in J_i\} < \kappa$$

(second use of regularity of κ !).

Then, of course, $a_j \in A_\beta$ for all $j \in J_i$, since $A_{\alpha_j} \subset A_\beta$. Thus, $\vec{a} \in \mathrm{dom}(f_i) \cap A_\beta^{J_i}$, and by the definition of $A_{\beta+1}$, $f_i(\vec{a}) \in A_{\beta+1}$, and as a consequence, $f_i(\vec{a}) \in \bigcup_{\alpha<\kappa} A_\alpha$.

Having the formula (6), it is easy to estimate $|\bar{A}|$. By induction on $\alpha < \kappa$, we show that $|A_\alpha| \le \max(\mu,\lambda)^{<\kappa}$. By (3), this is true for $\alpha = 0$. If the assertion holds for all $\beta < \alpha < \kappa$, and α is a limit ordinal, then

$$|A_\alpha| \le \sum_{\beta<\alpha} |A_\beta| \le \sum_{\beta<\alpha} \max(\mu,\lambda)^{<\kappa} = |\alpha| \cdot \max(\mu,\lambda)^{<\kappa}$$

$$\le \kappa \cdot \max(\mu,\lambda)^{<\kappa} \underset{\S 10}{=} \max(\kappa, \max(\mu,\lambda)^{<\kappa}) \underset{(4)}{=} \max(\mu,\lambda)^{<\kappa}.$$

Finally, for $\alpha = \beta+1$, the set $U \underset{\mathrm{def}}{=} \{f_i(\vec{a}) : i \in I,\ \vec{a} \in \mathrm{dom}(f_i) \cap A_\beta^{J_i}\}$ has cardinality

$$\le |I| \cdot (\max(\lambda,\mu)^{<\kappa})^{|J_i|} \underset{\text{ind. hyp}}{\le} \lambda \cdot (\max(\lambda,\mu)^{<\kappa})^{\rho} \underset{\rho \overset{\mathrm{def}}{=} |J_i|}{}$$

$$\underset{(5)}{=} \lambda \cdot \max(\lambda,\mu)^{<\kappa} = \max(\lambda, \max(\lambda,\mu)^{<\kappa}) \underset{(3)}{=} \max(\lambda,\mu)^{<\kappa}.$$

Since $A_\alpha = A_\beta \cup U$, $|A_\alpha| \le \max(\lambda,\mu)^{<\kappa} + \max(\lambda,\mu)^{<\kappa} = \max(\lambda,\mu)^{<\kappa}$ as desired. Our induction is complete.

Finally,

$$|\bar{A}| \le \sum_{\alpha<\kappa} |A_\alpha| \le \kappa \cdot \max(\lambda,\mu)^{<\kappa} =$$

$$\max(\kappa, \max(\lambda,\mu)^{<\kappa} = \max(\lambda,\mu)^{<\kappa}$$

(4)

as wanted for (2). **This completes the proof of the Theorem**.

If, in particular, $\lambda \le \mu$ and μ is of the form $\nu^{<\kappa}$, then $\max(\lambda,\mu)^{<\kappa} = \mu^{<\kappa} = (\nu^{<\kappa})^{<\kappa} = \sup_{\rho<\kappa} (\nu^{<\kappa})^{\rho} = \sup_{\rho<\kappa} \nu^{<\kappa} = \nu^{<\kappa} = \mu$. Thus, we obtain:

(5)

if $|A| = \nu^{<\kappa}$ *for some* $\nu \ge 2$, *and* $|I| \le |A|$, *then* $|\bar{A}| = |A|$.

Note that for all infinite ν, $\nu^{<\aleph_0} = \sup_{n<\omega} \nu^n = \sup_{n<\omega} \nu = \nu$. Thus, if $\kappa = \aleph_0$, then for all infinite A, we have that $|A| = |A|^{<\kappa}$, and we conclude that

if $\kappa = \aleph_0$, A *is infinite, and* $|I| \le |A|$, *then* $|\bar{A}| = |A|$.

Let us turn to the examples above. In the example of type (A), we have that $\kappa = \aleph_0$, because all operations are finitary, that is, the arities J_i are finite cardinals. Also, $\lambda=\aleph_0$. Thus, if $(B, \mathcal{F})$ is an infinite group (say), and $A \subset B$ is any infinite subset of B, then $\bar{A}$, the subgroup generated by A, is of the same cardinality as A. If $|B| = \nu$, then for any $\mu \le \nu$, we may choose a subset $A \subset B$ such that $|A| = \mu$ (why?); we thus obtain that

an infinite group has subgroups in each infinite cardinality less than or equal the cardinality of the group.

Of course, the same conclusion holds for Boolean algebras, etc.

In the examples under (B), we have $\kappa = \aleph_1$. Now, $\nu^{<\kappa} = \nu^{\aleph_0}$. Also, now

$\lambda = \max(\aleph_0, 1) = \aleph_0$. With $(\mathbb{R}, \lim)$, we get a trivial conclusion, namely that $|\bar{A}| \le \max(\aleph_0, |A|)^{\aleph_0} \le (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ (trivial since $\bar{A} \subset \mathbb{R}$, and $|\mathbb{R}| = 2^{\aleph_0}$). But, with $(\mathbb{R}^{\mathbb{R}}, \lim)$ we obtain, similarly, that $A \subset \mathbb{R}^{\mathbb{R}}$, $|A| \le 2^{\aleph_0}$ imply that $|\bar{A}| \le 2^{\aleph_0}$ as well, which is "non-trivial" now since $|\mathbb{R}^{\mathbb{R}}| = 2^{2^{\aleph_0}}$ (for the cardinalities related to $\mathbb{R}$, see later ...)

Here is another example. Let us fix a positive integer n. The class (in fact, set) $\mathcal{B}$ of *Borel* sub*sets* of $\mathbb{R}^n$ is the least subset X of $\mathcal{P}(\mathbb{R}^n)$ which contains all open sets, and which is closed under countable union and complementation:

whenever $A_i \in X$ for all $i \in \omega$, then $\bigcup_{i \in I} A_i \in X$;

$A \in X \Longrightarrow \mathbb{R}^n - A \in X$

(Borel sets are important in measure theory). We have that $|\mathcal{B}| = 2^{\aleph_0}$: the number of Borel sets is $2^{\aleph_0}$. (It immediately follows that there are non-Borel sets; why?) This result comes out of the above general theorem in the following way. Let A be the set of all open subsets of $\mathbb{R}^n$. Let $B = \mathbb{R}^n$, $I = \{0, 1\}$; $J_0 = \omega$; $J_1 = 1$; $f_0 : B^{\omega} \to B$ is the operation of countable union: $f_0(\langle A_i \rangle_{i \in \omega}) = \bigcup_{i \in I} A_i$; $f_1 : B \to B$ is complementation: $f_1(A) = B - A$. Then it is clear that $\mathcal{B}$ is nothing but $\bar{A}$, $\mathcal{B} = \bar{A}$, the closure of A in the given system of operations.

Clearly, $\lambda = \aleph_0$. We know that $|A| = 2^{\aleph_0}$; thus, $\mu = 2^{\aleph_0}$. $\kappa = \aleph_1$, the least regular cardinal above $|J_0| = \aleph_0$. Therefore,

$$|\mathcal{B}| \le \max(\aleph_0, 2^{\aleph_0})^{<\aleph_1} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.$$

The most obvious feature of the theorem is that in the estimate for the cardinality of the

closure $\bar{A}$, the cardinality of the set B does not play any role. And indeed, B can be of any size, in fact, even a proper class. The version of the theorem when B is a (possibly proper) class, rather than just a set, is almost the same as the original; the definition of the class B^J remains the same; now, each f_i is possibly a proper class; the only change is that now we cannot talk about the collection of the operations f_i $(i\in I)$ in the same sense as before; but we take a class $\mathcal{F}$ whose elements are of the form (i, a) for $i\in I$, with I a set, and stipulate that $f_i \overline{\text{def}} \{a: (i, a)\in\mathcal{F}\}$ is a Function whose domain is a subclass of B^{J_i}. The proof of the theorem remains esssentially the same.

Finally, we refine the distinction between regular and singular cardinals by introducing the *cofinality* of a cardinal κ; as it will turn out, a cardinal is regular if and only if its cofinality is itself.

We formulate the basic notion in terms of ordinals. For every ordinal α, we have that $\alpha = \operatorname{lsub}_{\beta<\alpha} \beta$; α is the least strict upper bound of all ordinals less than α. It may happen that α can be written in the form

$$\alpha = \operatorname{lsub}_{\xi<\gamma} \beta_\xi \qquad (7)$$

for a system of ordinals β_ξ indexed by an ordinal γ *smaller* than α; the least ordinal γ for which there is a representation of the form (1) of α is called the *cofinality* of α, and it is denoted $\operatorname{cf}(\alpha)$; we have just seen that $\operatorname{cf}(\alpha)\le\alpha$.

When $\alpha=\beta+1$ is a successor ordinal, then we can take $\gamma=1$ and $\beta_0=\beta$; this shows that $\operatorname{cf}(\alpha)=1$; the cofinality of all successor ordinals is 1.

Let us note that if (7) holds, then we can change the β_ξ to some β'_ξ such that the sequence $\langle\beta'_\xi\rangle_{\xi<\gamma}$ is (not necessarily strictly) increasing: $\xi<\xi'<\gamma$ implies that $\beta'_\xi\le\beta'_{\xi'}$; and (7) still holds, with some $\gamma'\le\gamma$:

$$\alpha = \operatorname{lsub}_{\xi<\gamma'} \beta'_\xi\,. \qquad (8)$$

To this end, we define $\beta'_\xi=\operatorname{lub}_{\zeta\le\xi} \beta_\zeta$ for all $\xi<\gamma$. It is clear that the β'_ξ are increasing, and

also that $\beta'_\xi \le \alpha$ for all $\xi<\gamma$. If in fact $\beta'_\xi<\alpha$ for all $\xi<\gamma$, then

$$\alpha = \operatorname{lsub}_{\xi<\gamma} \beta_\xi \le \operatorname{lsub}_{\xi<\gamma} \beta'_\xi \le \alpha$$

and so (8) holds with $\gamma'=\gamma$. If, on the other hand, there is $\xi<\gamma$ such that $\beta'_\xi=\alpha$, then we let γ' be the least such ξ; then $\beta'_\xi<\alpha$ for each $\xi<\gamma'$, and thus

$$\alpha = \operatorname{lub}_{\xi<\gamma'} \beta_\xi \le \operatorname{lsub}_{\xi<\gamma'} \beta'_\xi \le \alpha,$$

and (8) again holds. Note that if, in addition to the above, $\gamma=\mathrm{cf}(\alpha)$, then γ' is necessarily equal to γ, since $\gamma'<\gamma$ and (8) would contradict the "least-ness" of γ. Thus, if $\gamma=\mathrm{cf}(\alpha)$, the we have a representation (7) of α in which the β_ξ are increasing.

The characterization (1') of regular cardinals shows that

(9) *for a limit ordinal* α, $\mathrm{cf}(\alpha)=\alpha$ *implies that* α *is a cardinal, in fact a regular cardinal.*

In fact, more generally,

(10) *for a limit ordinal* α, $\mathrm{cf}(\alpha)$ *is always a regular cardinal.*

This follows from the identity

$$\mathrm{cf}(\mathrm{cf}(\alpha))=\mathrm{cf}(\alpha) \qquad (11)$$

that we prove as follows. Let $\gamma=\mathrm{cf}(\alpha)$ and $\delta=\mathrm{cf}(\gamma)$. We have equalities

$$\alpha = \operatorname{lsub}_{\xi<\gamma}(\beta_\xi) \quad \text{and} \quad \gamma = \operatorname{lsub}_{\zeta<\delta}(\gamma_\zeta);$$

and as we said, we may assume that $\beta_\xi \le \beta_{\xi'}$ whenever $\xi<\xi'<\gamma$. But then

$$\alpha = \operatorname{lsub}_{\zeta<\delta}(\beta_{\gamma_\zeta}):$$

indeed, clearly, α is a strict upper bound of all the β_{γ_ζ} ($\zeta<\delta$); and if $\alpha'<\alpha$, then there is $\xi<\gamma$ such that $\alpha'\leq\beta_\xi$ (why?); and then there is $\zeta<\delta$ such that $\xi\leq\gamma_\zeta$, thus $\beta_\xi\leq\beta_{\gamma_\zeta}$, and so $\alpha'\leq\beta_{\gamma_\zeta}$ -- which shows that α is a the *least* strict upper bound of all the β_{γ_ζ} ($\zeta<\delta$). But then, by the definition of $\gamma=\mathrm{cf}(\alpha)$, it follows that $\gamma\leq\delta$; that is, (9) holds.

Now, (10) follows from (11) and (9).

Exercise: Let $\gamma=\mathrm{cf}(\alpha)$. Then there exists a representation of α in the form of (7) in which the β_ξ are *strictly* increasing: $\beta_\xi<\beta_{\xi'}$ whenever $\xi<\xi'<\gamma$.

We have defined the cofinality of any ordinal, and in particular, of any cardinal; and we saw that it is always a regular cardinal. For instance, if α is a countable limit ordinal, then $\mathrm{cf}(\alpha)=\omega$; this is because $\mathrm{cf}(\alpha)\leq\alpha$, and $\mathrm{cf}(\alpha)$ is an infinite cardinal, and there is only one such, namely $\omega=\aleph_0$. What we have thus seen is that every countable limit ordinal α can be written in the form $\alpha=\mathrm{lsub}_{n<\omega}\,\alpha_n$, and we may also assume that the α_n here are strictly increasing.

For an ordinal α with $|\alpha|=\aleph_1$, the possibilities for $\mathrm{cf}(\alpha)$ are three: it can be 1 (when α is a successor), it can be $\aleph_0$, and it can be $\aleph_1$. Obviously, $\mathrm{cf}(\omega_1+1)=1$, $\mathrm{cf}(\omega_1+\omega)=\aleph_0$ and $\mathrm{cf}(\omega_1+\omega_1)=\aleph_1$ (we may write ω_1 for $\aleph_1$ when "we use it as an ordinal"; similarly for ω_α).

Exercise: show that if α is a limit ordinal, then $\mathrm{cf}(\aleph_\alpha)=\mathrm{cf}(\alpha)$.

Thus, we have $\mathrm{cf}(\aleph_\omega)=\mathrm{cf}(\omega)=\aleph_0$, $\mathrm{cf}(\aleph_{\omega_1})=\mathrm{cf}(\omega_1)=\aleph_1$, and $\mathrm{cf}(\aleph_{\aleph_{\omega_1}})=\mathrm{cf}(\aleph_{\omega_1})=\aleph_1$.