Assignment 5/MATH 318/Fall 2007
Due: Friday, November 9

Supply Lambek-style formal deductions (in natural deduction) to prove the listed entailments. Give an informal proof first in the more difficult cases. Make sure that your deductions are fully justified; provide annotation (comments) to steps where the justification is not obvious. A statement of the form $\Phi \equiv \Psi$ abbreviates *two* entailments: $\Phi \vdash \Psi$ and $\Psi \vdash \Phi$.

Only two of the deductions require more than 20 lines: [13] and [14]. Most deductions are about 10 lines or less; there are some between 10 and 20.

[1] $\forall x(Pxy \wedge Rxy) \equiv \forall xPxy \wedge \forall xRxy$ **[2]** $\forall x(Py \vee Rxy) \equiv Py \vee \forall xRxy$

Hints for the left-to-right entailment in [2]: This is an interesting example when an "obvious" kind of approach will not work. There are many other such "difficult" cases (this is the inherent complexity of predicate logic), but they are mostly longer than the present example.

Follow the following informal proof: " $A \vee B$ is logically equivalent to $\bar{A} \to B$; thus, it is sufficient to show $\forall x(Py \vee Rxy) \vdash (\neg Py) \to \forall xRxy$. Assume the premiss and $\neg Py$, and let x be arbitrary, to show Rxy . From the premiss, we obtain $Py \vee Rxy$; since $\neg Py$, Rxy follows."

[3] $\exists x(Pxy \vee Rxy) \equiv \exists xPxy \vee \exists xRxy$ **[4]** $\exists x(Py \wedge Rxy) \equiv Py \wedge \exists xRxy$

[5] $\forall xPxy \vee \forall xRxy \vdash \forall x(Pxy \vee Rxy)$ **[6]** $\exists x(Pxy \wedge Rxy) \vdash \exists xPxy \wedge \exists xRxy$

[7] $\forall x\exists y(x=fy) \vdash \forall x\exists y(x=ffy)$

[8] $\forall x\exists y(x=fy \vee x=gy)$, $\forall xR(fx, x)$, $\forall xR(gx, x)$ $\vdash$ $\forall x\exists yRxy$

[9] $\exists yRxy \equiv \neg\forall y\neg Rxy$ **[10]** $\forall yRxy \equiv \neg\exists y\neg Rxy$

Remarks A statement of the form $\Phi \equiv \Psi$ means two entailments: $\Phi\vdash\Psi$ and $\Psi\vdash\Phi$.

[9] is done in the Notes; see p. 172. However, here a Lambek style deduction is asked for; the proof in the Notes is not directly in that form.

[11] $\vdash$ $\forall x\exists yRxy \longrightarrow \forall x\exists y\exists z(Rxy \wedge Ryz)$ ($\vdash \Phi$ means $\emptyset \vdash \Phi$)

[12] $\vdash$ $(\forall x\forall y(Rxy \to Ryx) \wedge \forall x\forall y((Rxy \wedge Ryx) \to x=y))$
$\longrightarrow \forall x\forall y\forall z((Rxy \wedge Ryz) \to Rxz))$

Remark [11] and [12] are Ψ_4 resp. Ψ_6 in problem [5] in assnmt 4. Informal proofs have been provided by yourself and/or the answer sheet for assnmt 4.

The following entailment addresses the situation involving a binary operation (denoted as addition here), its *graph*, and a way of expressing the associative law for the operation in terms of the graph. Here we *formally prove* that the proposed way is correct. $\forall xyz$ abbreviates $\forall x \forall y \forall z$, and similarly in other cases.

[13] $\vdash \forall xyz(x+y=z \longleftrightarrow ADDxyz) \vdash \forall xyz((x+y)+z=x+(y+z)) \longleftrightarrow$
$\forall xyz \exists uvw(ADDxyu \wedge ADDyzv \wedge ADDuzw \wedge ADDxvw)$

Remarks This entailment is the formal justification of rewriting the associative law (for addition) in the form of a statement using only the relation " $x+y=z$ ". This is the longest of the deductions in this assignment: I have used 45 lines to write it.

[14] **(i)** Let $A=\mathbb{N}-\{0\}$, the set of positive integers, and define the relation R on the set A by

$$R(x, y) \iff \text{there are } u \text{ and } v \text{ in } A \text{ such that } x^u = y^v .$$

Show, by an ordinary mathematical proof, that R is an equivalence relation.

(ii) Formalize the proof of the *transitivity* of R in (i), in the following manner. Write x^u as $e(x, u)$, and $u \cdot w$ as $m(u, w)$, to make sure that you are not using any hidden properties of these two familiar operations. Then write down properties of exponentiation and multiplication, in terms of the binary operations e and m , that were used in your proof in (i). Use the relation symbol R , and make the definition of R into a premiss; it will look like

$\forall x \forall y(Rxy \longleftrightarrow \exists u \ ...)$.

When you think you have all premisses you need, write down what is to be proved in the form of an the entailment in predicate logic, and give a formal deduction for it.

Remarks. In the last problem, part (i) involves a certain amount of ordinary algebraic manipulation. When you formalize this in natural deduction, you will have to use a corresponding number of applications of the E-rule.