**Assignment 1/MATH 318/Fall 2007**

**Due: Monday, September 17**

**[1]** We consider the following eight properties of relations:

$P_1$:reflexive $P_2$:symmetric $P_3$:transitive $P_4$:irreflexive

$P_5$:antisymmetric $P_6$:strictly antisymmetric

$P_7$:dichotomous $P_8$:trichotomous

We also look at the seven kinds of relation:

$Q_1$:preorder $Q_2$:equivalence relation $Q_3$:reflexive order

$Q_4$:irreflexive order $Q_5$:total reflexive order

$Q_6$:total irreflexive order $Q_7$:graph

**(i)** Draw the digraphs of the following relations $R_1, \ldots, R_8$, all on the same set $A=\{1, 2, 3, 4\}$, :

$R_1=\{(3, 1), (1, 3), (1, 4), (1, 2), (4, 1), (2, 1)\}$

$R_2=\{(3, 3), (3, 4), (3, 2), (1, 3), (1, 1),$
$(1, 4), (1, 2), (4, 3), (4, 4), (4, 2), (2, 2)\}$

$R_3=\{(3, 3), (3, 1), (1, 3), (1, 1), (4, 4), (4, 2), (2, 4), (2, 2)\}$

$R_4=\{(3, 3), (3, 1), (3, 4), (3, 2), (1, 1), (1, 4), (1, 2), (4, 4), (2, 2)\}$

$R_5=\{(3, 4), (3, 2), (1, 3), (1, 4), (1, 2), (2, 4)\}$

$R_6=\{(3, 3), (3, 2), (1, 3), (1, 1), (1, 4), (1, 2), (4, 3), (4, 4), (4, 2), (2, 2)\}$

$R_7=\{(3, 1), (3, 4), (3, 2), (1, 1), (1, 4), (1, 2), (4, 1), (4, 4), (4, 2)\}$

$R_8=\{(3, 1), (3, 4), (3, 2), (1, 4), (1, 2)\}$

**(ii)** Make an $8\times 8$ table whose $(i, j)$-entry is YES or NO according to whether or not $R_i$ has the property $P_j$.

**(iii)** Make a similar $8\times 7$ table to record which of the relations are and which are not of each the seven kinds $Q_1$ , ..., $Q_7$ .

**[2]** Consider the relations listed:

$R_1$ on $\mathbb{N}$ (remember: " $R$ on $A$ " is to say $R \subseteq A \times A$ ):
$aR_1 b \iff a \neq b$ but $a$ and $b$ have a common prime divisor.

$R_2$ on $\mathbb{N} \times \mathbb{N}$ :
$(a, b)R_2(c, d) \iff$ either $a<c$ , or ( $a=c$ and $b<d$ ).

$R_3$ on $\mathbb{Q}^{\neq 0}$ (=the set of all non-zero rational numbers):
$xR_2 y \iff x/y$ is an integer

$R_4$ on $\mathbb{R}^{\geq 0}$ (=the set of all non-negative real numbers) :
$xR_4 y \iff y-3x > 0$ .

$R_5$ on $\mathbb{R}$ :
$xR_5 y \iff (x-y) \in \mathbb{Q}$ .

Do the work of problem [1] for these relations. Draw up a $5 \times 8$ table and a $5 \times 7$ table containing the information on the relations as to whether they do or do not have the properties $P_i$ , $Q_k$ defined in [1].

**[3]** Remember that relations are sets, namely, sets of ordered pairs. Therefore, the intersection and the union of two relations on a set $A$ are relations on $A$ as well.

Let $P$ be a property of relations $R$ . E.g, $P$ could be $P_3$ in [1] above: " $R$ has property $P_3$ " means that $R$ is transitive. But also, each $Q_k$ in [1] above is a possible property of relations: e.g., $R$ has property $Q_1$ means that $R$ is a preorder.

We say that property $P$ is "*preserved by intersections of relations*" if it is true that every time $R$ and $S$ are relations on the same set $A$ , both having property $P$ , we have that $R \cap S$ also has property $P$ . For instance, $P_1$ (reflexivity) is preserved under intersection of relations.

We can talk about a property being preserved by unions of relations in the analogous sense.

The question is:

*which of the properties* $P_1, ...P_8, Q_1, ..., Q_7$ are *preserved by intersections of relations, which are not? Which by unions of relations?*

Give a table with two columns and 8+7=15 rows, the first column for $R \cap S$, the second for $R \cup S$, the rows for the $P_i$ and the $Q_j$, and the entries containing "yes" or "no". Give brief justifications for the less obvious "yes" answers for the $P_i$, and counterexamples for the "no" answers for the $P_i$'s.

**[4]** Let $A=\{1, 2, 3, 4\}$. For each of the combinations (a) to (g) of properties below, give, *if possible*, an example of a relation $R$, by drawing a digraph, on the set $A$ satisfying it -- or else, if that is not possible, explain why that is so.

- **(a)** symmetric, irreflexive and trichotomous;
- **(b)** symmetric, dichotomous and *not* transitive;
- **(c)** symmetric, antisymmetric and irreflexive;
- **(d)** symmetric, antisymmetric and reflexive;
- **(e)** strictly antisymmetric, trichotomous and *not* transitive;
- **(f)** symmetric, antisymmetric and *not* transitive.
- **(g)** symmetric, irreflexive, transitive and trichotomous;

**[5]** Let $n$ be a fixed positive integer, and let $A = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. We list some relations on the set $A$:

$$R_1 = \{(i, i+1) : i \in A \ \& \ i+1 \in A\}$$
$$R_2 = \{(i, i+2) : i \in A \ \& \ i+2 \in A\}$$
$$R_3 = \{(i, i+1) : i \in A \ \& \ i+1 \in A\} \cup \{(n, 1)\}$$
$$R_4 = \{(2i, 2i+1) : 2i \in A \ \& \ 2i+1 \in A\} \cup \{(2i, 2i-1) : 2i \in A \ \& \ 2i-1 \in A\}.$$

**(i)** Draw digraphs for the each of the above when $n=5$.

**(ii)** Describe the transitive closure $R_j^{tr}$ of each the above relations, for general $n$, in a simple way, and determine the number of elements (which are ordered pairs) in each set $R_j^{tr}$ .

**[6]** Define the relation $E$ on $\mathbb{N}$ by the condition

$$aEb \iff \text{there are integers } i \text{ and } j \text{ such that } b \mid a^i \text{ and } a \mid b^j .$$

**(i)** Prove that $E$ is an equivalence relation on $\mathbb{N}$ .

**(ii)** Show that $aEb$ holds if an only if either $a=b=0$ , or $a$ and $b$ have the same prime factors (none if $a=b=1$ ).

**(iii)** Let $A=\{i\in\mathbb{N}:i<20\}$ . Give the partition $A/(E\restriction A)$ corresponding to the equivalence $E\restriction A$ on $A$ (for $E\restriction A$ , see "*restriction*": p.36, line 8).

[**Note:** Recall that $x^0=1$ , even for $x=0$ . For partitions, see Section 2.2, in particular, p. 42].

**[7]** Consider the following relations $R$ and $S$ on the set $A=\{1, 2, 3, 4\}$ :

$$R: \begin{tikzcd} 2 \arrow[r, dash] \arrow[d, dash] & 3 \arrow[d, dash] \\ 1 \arrow[r, dash] & 4 \end{tikzcd} \qquad S: \begin{tikzcd} 2 \arrow[dr, dash] & 3 \\ 1 \arrow[ur, dash] & 4 \end{tikzcd}$$

(An edge without an arrow-head is equivalent to the two arrows in both directions between the two vertices.) Give the network ("digraph" without arrow-heads) as well as the adjacency matrix of each of the relations of the form $R^{\circ m}\circ S^{\circ n}$ , $S^{\circ n}\circ R^{\circ m}$ , for all $m.n\in\mathbb{N}$ (there are just a few distinct ones among of these). Also give $R^{tr}$ , $S^{tr}$ , $R^{r/tr}$ , $S^{r/tr}$ in both ways.

**[8]** Exercise 1 (p.31),

**[9]** Exercise 2 (p.33)

**[10]** Exercise 3 (p.35) .

**[11]** Exercise 4 (p.35).

**Remark:** When doing any one of the above exercises, collected in [9] to [11] , you may use results appearing anywhere earlier in the text; in particular, any of the earlier exercises. For instance, in doing Exercise 4(p.35), you may use the results of Exercise 3(p.35) even if you did not do the latter.

**[12]** We have three graphs $G_i = S_i \cup S_i^*$ , $i = 1, 2, 3$ , all on the set $\{1, 2, 3, 4, 5, 6, 7\}$ of vertices:

$$S_1 = \{(1, 2), (1, 4), (2, 3), (2, 4), (2, 5), (3, 5), (4, 5), (4, 6), (5, 6), (5, 7), (6, 7)\}$$
$$S_2 = \{(1, 2), (1, 3), (1, 4), (2, 4), (2, 5), (2, 6), (3, 4), (4, 6), (4, 7), (5, 6), (6, 7)\}$$
$$S_3 = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6), (5, 7), (6, 7)\} .$$

**(i)** Draw networks (edges without arrow-heads) of the three graphs in such a way that the (straight) edges do not cross each other (in these cases this is possible; of course, this is not always possible.)

**(ii)** Two of the graphs are isomorphic to each other; which ones are they? Give an isomorphism between them.

**(iii)** Show that one of the graphs is not isomorphic to either of the other two.