Chapter 5 Predicate logic

Section 5.1 Quantifiers

Suppose U is a *non-empty* set (I use the letter "U" since the word "universe" is going to be used to signify this set), and R a binary relation on U. The expression

$$Rxy \tag{1}$$

is an indefinite one; when the variables x and y are given values that are from the set U, then Rxy assumes a definite *truth-value*, which is either $\top$ (*true*) or $\bot$ (*false*).

For instance, when $U=\mathbb{N}$, and R denotes the ordinary less-than relation $(<)$, then

$$R(1,3) \sim \top,$$
$$R(4,3) \sim \bot,$$
$$R(2,10) \sim \top.$$

We are using the symbol $\sim$ to indicate equality of truth-values. We do not want to use ordinary equality, since $\sim$ does not mean that the two expressions in question are the *same*; it only means that under the given interpretation, they *evaluate* to the same truth-value. For instance, we may say that

$$R(1,3) \sim R(2,10)$$

under the interpretation of R as the ordinary less-than relation; but

$$R(1,3) = R(2,10)$$

would suggest that the expressions $R(1,3)$, $R(2,10)$ "are the same" -- which we do not want to say.

Note that we did not need to write $R(x, y)$ in (1), in place of the simpler Rxy ; however, we did not want to write $R12$ in place of $R(1, 2)$, or, worse, $R210$ in place of $R(2, 10)$, for obvious reasons. It would *not* be a mistake to write $R(x, y)$ for Rxy .

Now, consider the expression

$$Rxy \wedge Ryz \ . \tag{2}$$

Under the same interpretation as above -- that is, with the universe being $\mathbb{N}$, and R being the ordinary less-than relation --, this expression also takes up a definite truth-value when the variables are given values from $U=\mathbb{N}$. E.g., when $x=1$, $y=2$ and $z=3$, the expression becomes

$$Rxy \wedge Ryz \ \sim \ R(1, 2) \wedge R(2, 3) \ \sim \ \top \wedge \top \ \sim \ \top \ .$$

Here we used the interpretation of $\wedge$ as an operation on truth-values as was specified in Chapter 4, Section 3, p. 122. Also remember that you can read $\wedge$ as "and" ; thus $Rxy \wedge Rxz$ can be read as " Rxy and Rxz ". It was explained in Chapter 4 how we can use the Boolean connectives to build compound sentences (Boolean expressions) out of simpler ones, starting with letters such as A , B , ... Here we use the connectives in the same way to build compound logical expressions (indefinite sentences) out of simpler ones, starting with *atomic* expressions such as Rxy .

Now, to the new element: *quantifiers*. Out of the expression (2), we can form, by prefixing $\exists x$ to it, the new expression

$$\exists y(Rxy \wedge Ryz) \ . \tag{3}$$

Read " $\exists y$ " as "there exists y such that". Suppose we are still using the same interpretation as before: $\mathbb{N}$ for the universe, the ordinary less-than relation for R . The role of the universe now becomes more important than before; it fixes the meaning of the *existential quantifier* $\exists y$. The actual meaning of " $\exists y$ ", under the given interpretation, is: "there exist $y \in \mathbb{N}$ such that". If we changed $\mathbb{N}$ to something else, the meaning of this would change. If the universe is U , for us *always assume non-empty*, the meaning of $\exists y$ is:

$\exists y \quad \sim \quad$ "there exist $y \in U$ such that"

(we again used the symbol $\sim$ to signify "means the same under the given interpretation as").

Now, what does (3) evaluate to? First of all notice that the expression (3) really has only two "free" variables: x and z ; these are the ones that we can freely give values to. Thus, when $x=1$ and $z=3$, (3) becomes

$$\begin{aligned} \exists y(Rxy \wedge Ryz) \ &\sim \ \exists y(R(1, y) \wedge R(y, 3)) \\ &\sim \text{ there exists } y \in \mathbb{N} \text{ such that } R(1, y) \text{ and } R(y, 3) \\ &\sim \text{ there exists } y \in \mathbb{N} \text{ such that } 1 < y \text{ and } y < 3 , \end{aligned}$$

and this is true, since $y=2$ is a suitable value making " $1 < y$ and $y < 3$ " true:

$$\exists y(Rxy \wedge Ryz) \ \sim \ \exists y(R(1, y) \wedge R(y, 3)) \ \sim \ \top .$$

On the other hand, when $x=1$ and $z=2$, then

$$\begin{aligned} \exists y(Rxy \wedge Ryz) \ &\sim \ \exists y(R(1, y) \wedge R(y, 2)) \\ &\sim \text{ there exists } y \in \mathbb{N} \text{ such that } R(1, y) \text{ and } R(y, 2) \\ &\sim \text{ there exists } y \in \mathbb{N} \text{ such that } 1 < y \text{ and } y < 2 , \end{aligned}$$

and this is *false* since there is no natural number (element of $\mathbb{N}$) which would be both greater than 1 and smaller than 2 :

$$\exists y(Rxy \wedge Ryz) \ \sim \ \exists y(R(1, y) \wedge R(y, 2)) \ \sim \ \perp .$$

On the other hand, *it does not make sense to give a value to the variable* y in (3): it does not make sense to write

?? $\qquad \exists 6(R(x, 6) \wedge R(6, z))$

?? $\qquad$ there exists 6 such that $R(x, 6)$ and $R(6, z)$.

If we must, then we would give the same value to this as to

$$R(x,6) \text{ and } R(6,z) ;$$

but for this, we did not need the "quantifier" $\exists 6$.

We say that the variables x and z in the above formal expression are *free variables*; and that y is a *bound variable*.

It is not a mistake to have the same variable both as a free and a bound variable in the same expression; but this can be avoided, by renaming the bound variable. For instance,

$$\exists y(Rxy \wedge Ryz) \rightarrow \exists zRxz$$

is a legitimate logical expression in which z is both free and bound; but it is the same, in essence, as

$$\exists y(Rxy \wedge Ryz) \rightarrow \exists yRxy \ ;$$

and the latter is clearer in its meaning.

Let us emphasize again that in a logical expression,

only the free variables can be freely evaluated; when all the free variables are evaluated, the whole expression assumes a definite truth-value.

Besides the existential quantifier, we also have the *universal quantifier*. With any variable, we can combine the symbol $\forall$, and prefix the result to any logical expression. For instance, we may write

$$\forall y(Rxy \rightarrow \neg Ryx) \ . \tag{4}$$

The universal quantifier $\forall y$ is read "for all y "; or, "for every y ". Thus, the last-displayed expression is read as

"for all y , if Rxy , then *not*-Ryx ".

When we use the same U as before ($U=\mathbb{N}$), and the same R as well, the expression evaluates to

"for all $y \in \mathbb{N}$, if $x<y$, then *not*-$y<x$ " .

In this expression (4), we have one free variable, x ; and one bound variable, y . We see that, under the given interpretation, (4) is true no matter what the value of the free variable x is:

$$\forall y(Rxy \rightarrow \neg Ryx)$$
$$\rightsquigarrow \text{"for all } y\in\mathbb{N}\text{, if } x<y\text{, then } not\text{-}y<x\text{" } \rightsquigarrow \top . \qquad (4')$$

Just like Boolean connectives, quantifiers too can be used iteratively. For instance, we may write the logical expression

$$\forall x \exists y Rxy \qquad (5)$$

and this receives the value $\top$ identically, when our choice of the universe and the interpretation of R are as before:

$$\forall x \exists y Rxy \rightsquigarrow$$
$$\text{for all } x\in\mathbb{N}\text{, there exists } y\in\mathbb{N} \text{ such that } x<y$$
$$\rightsquigarrow \top .$$

Note that in (5), there is no free variable at all. As a consequence, as soon as the universe and the relation R are specified, the sentence (5) receives a definite truth-value.

Let us summarize what we have in the way of *predicate logic*, also called *first order logic*.

First of all, we have a notion of *logical expression*. Logical expressions are also called *well-formed formulas*, or simply, *formulas*, of first order logic. We will simply say "formula" now, although this is clearly a terminology that could be confusing, given the fact that in mathematics, many things are called "formulas".

A *formula* is built up out of *atomic formulas*. An atomic formula is of the form

$$Rx_1x_2...x_k\,,$$

where the x_i's are *variables*, and R is a k-*ary relation symbol*. For instance, R above was used as a binary relation symbol ($k=2$). But we could use, profitably, S as a ternary relation symbol ($k=3$) , for instance, when we want to talk about the relation

$$Sxyz \sim x+y=z$$

on the universe $U=\mathbb{N}$ (or, for that matter, $U=\mathbb{Z}$, $U=\mathbb{Q}$, $U=\mathbb{R}$ or $U=\mathbb{C}$ -- or even many more). The language of first order logic allows k-ary relation symbols for arbitrary natural numbers $k=0$, 1, 2, $\ldots$. The case $k=0$ corresponds to a propositional atom: the "atomic formulas of propositional logic".

Out of atomic formulas, we build compound formulas by using the Boolean connectives

$$\wedge\,,\ \vee\,,\ \neg\,,\ \rightarrow\,,\ \longleftrightarrow,\ \top\,,\ \bot\,,$$

and the quantifiers

$$\exists x\,, \qquad \forall x$$

with arbitrary variables x . The connectives and the quantifiers are jointly called *logical operators.* (Note that it is $\exists x$, rather than $\exists$, what we call a quantifier.) Of course, these operators must be applied "meaningfully". For instance, $\wedge$ is always applied to two formulas: of Φ and Ψ are formulas, then so is $\Phi\wedge\Psi$; when Φ and Ψ are compound formulas, we have to use parentheses: $(\Phi)\wedge(\Psi)$. Note that we listed $\top$ and $\bot$ as connectives; they are 0-ary connectives; they do not apply to given formulas, but they are, by themselves, formulas. Thus, for instance, if Φ is a formula, then $\Phi\rightarrow\bot$ is also a well-formed formula.

All formulas are obtained by the iterated application of the logical operators, starting with the atomic formulas (and $\top$ and $\bot$).

Formulas may have *free* variables and *bound* variables in them.

Consider the following formula and an annotated version of it:

$$\forall x(\exists y(\forall x(Pxy \vee Qzx) \wedge \neg Qxz) \longrightarrow \exists z(Pzy \wedge \neg Qxz))$$

$$\forall x\;\overset{1}{(}\;\exists y\;\overset{2}{(}\;\forall x\;\overset{4}{(}\;Pxy \vee Q\overset{\text{free}}{z}x\;\overset{4}{)} \wedge \neg Qx\overset{\text{free}}{z}\;\overset{2}{)} \longrightarrow \exists z\;\overset{3}{(}\;Pz\overset{\text{free}}{y} \wedge \neg Qxz\;\overset{3}{)}\;\overset{1}{)}$$

There are four quantifiers in the formula: $\forall x$ occurring twice, $\exists y$ and $\exists z$. Each quantifier has a definite *scope*: the part of the formula it has effect over. The scope of the first $\forall x$ is enclosed by the pair of parentheses marked by 1, the scope of $\exists y$ by $\overset{2}{(}\ \overset{2}{)}$; the scope of $\exists z$ by $\overset{3}{(}\ \overset{3}{)}$; the scope of the second $\forall x$ by $\overset{4}{(}\ \overset{4}{)}$.

For a quantifier $\mathrm{Q}u$ (where Q may be $\forall$ or $\exists$),

$\mathrm{Q}u$ binds each occurrence of the variable u in its scope unless the occurrence is already bound by another copy of $\mathrm{Q}u$ inside the scope of the first one;

$\mathrm{Q}u$ does not bind any other occurrence of u, and any variable other than u. When we talk about "occurrences" of a variable, we disregard those copies of the variable that are inside a quantifier. Thus, in the example, the copies of the variables that have the arrows coming out of them (these are the ones that are inside quantifiers) do not count as "occurrences" in what we are saying here.

For instance, in our example, the first $\forall x$ binds two occurrences of x; it does not bind two other occurrences of x. All occurrences of x (there are four of them) are in the scope of the first $\forall x$. However, two of them are bound by the second $\forall x$, which is inside the scope of the first $\forall x$. The binding relations are shown by arrows connecting occurrences of variables with variables inside quantifiers.

A *bound variable occurrence* is one that is bound by a quantifier; a *free variable occurrence* is one that is not bound by any quantifier. A *free variable* of a formula is one that has at least one free occurrence in the formula; similarly, for "bound variable".

Bound variables are "dummy"; they can be replace by others provided one does not change the binding pattern of the formula. In our example, the formula may be changed, without changing the meaning of the formula, to this:

$$\forall x(\exists y(\forall u(Puy \vee Qzu) \wedge \neg Qxz) \longrightarrow \exists v(Pvy \wedge \neg Qxv))$$

$$\forall x \overset{1}{(} \exists w \overset{2}{(} \forall u \overset{4}{(} Puw \vee Q\overset{\overset{\text{free}}{\downarrow}}{z}u \overset{4}{)} \wedge \neg Qx\overset{\overset{\text{free}}{\downarrow}}{z} \overset{2}{)} \longrightarrow \exists v \overset{3}{(} Pv\overset{\overset{\text{free}}{\downarrow}}{y} \wedge \neg Qxv \overset{3}{)} \overset{1}{)}$$

We have changed $\exists y$ to $\exists w$ and the y bound by $\exists y$ to w. We have changed the second $\forall x$ to $\forall u$, and we changed the two x's bound by the second $\forall x$ to u's. We have changed $\exists z$ to $\exists v$, and changed the two z's bound by $\exists z$ to v's. The binding pattern of the formula has not changed.

(Changing *free* variables is not allowed; it changes the meaning of the formula!)

The new version we obtained above of our formula has the advantages that (1) no variable is both free and bound in it (in the original version, all three variables x, y, z were both free and bound variables in the formula); and (2) no quantifier appears twice (in the original formula, $\forall x$ appeared twice). These features make for better readability. However, it should be emphasized that the original formula was perfectly well-formed as it was.

Let us emphasize that changing of bound variables is allowed as long as it does not alter the binding pattern of the formula. In our example, changing $\exists y$ to $\exists x$, with changing the y bound by $\exists y$ to x, would be inadmissible; it would result in

$$\forall x \overset{1}{(} \exists x \overset{2}{(} \forall x \overset{4}{(} Pxx \vee Q\overset{\overset{\text{free}}{\downarrow}}{z}x \overset{4}{)} \wedge \neg Qx\overset{\overset{\text{free}}{\downarrow}}{z} \overset{2}{)} \longrightarrow \exists z \overset{3}{(} Pz\overset{\overset{\text{free}}{\downarrow}}{y} \wedge \neg Qxz \overset{3}{)} \overset{1}{)}$$

which has a different binding pattern. However, we can always make a change of bound variables if we use a *brand new* variable: one that is not used elsewhere in the formula at all. In fact, this is what we did three times in our first series of changes of bound variables. Sometimes, however, one can repeat a variable without harm (that is, without altering the binding pattern). For instance, in the formula that we obtained before, we could change v to either u or w without harm.

Note that it is not necessary that a variable appear anywhere else in the formula, other than the quantifier. For instance, $\exists yRxz$ is a meaningful formula. In the previously considered interpretation, it means that "there is $y\in\mathbb{N}$ such that $x<z$ ". x and z are free variables in this. Although meaningful, $\exists yRxz$ is of the same meaning as Rxz , and thus, the use of $\exists y$ was superfluous in this case. (This is the point where U being non-empty becomes essential; for empty U, what we just said is incorrect.)

As we noted, it may happen that a formula has the same variable both free and bound; on the other hand, this can always be avoided, and in fact, it is advisable to avoid it.

Formulas can be *interpreted.* This takes place upon the specification of a universe of discourse, or simply, *universe*, and the specification of a *relation corresponding to* each relation symbol in the formula. A specification of a universe, together with specific relations corresponding to the relation symbols is called a *structure*, or an *interpretation*.

We stipulate that *the universe be always a non-empty set*. It would not be meaningless to consider an empty universe -- but if we did not exclude the empty universe, certain things would become more complicated (e.g., what we said about $\exists yRxz$ above would become incorrect).

When R is a k-ary relation symbol, its interpretation has to be a k-*ary relation*. For instance, a ternary ($k=3$) relation on the set U is a subset $R\subseteq U\times U\times U=U^3$, that is, R is a set of ordered triples (a, b, c) of elements a, b, c of U . An example was given above: with $U=\mathbb{N}$ (for instance), we could interpret the ternary relation symbol R as the relation $\{(a, b, c)\in\mathbb{N}^3 : a+b=c\}$; this is the same thing as to stipulate that

$$Rxyz \sim x+y=z ,$$

with the understanding that the variables x, y, z *range over* $\mathbb{N}$, which phrase is just another way of saying that the universe is $\mathbb{N}$.

When an *interpretation* is fixed, *and* the free variables are assigned specific values (which have to be elements of the universe), the formula evaluates to a definite truth-value. We saw some examples for this above. We also saw that if the formula has no free variable, then it evaluates directly to a truth-value, without our having to specify any additional value of a (free) variable.

The word "predicate" signifies the entity that is obtained from a formula when an interpretation (structure) is given, but the values of the free variables are left undetermined. Predicates are almost the same as relations, except that their places have been filled by variables, and the identity of these variables is important. For instance, from a relation R (not just a relation-symbol!), we can form the predicates Rxy , Ruv , Ryx , ...; these are all *distinct* predicates.

It should not be thought that the evaluation of a formula is always an easy matter. Let us use relation symbols R (binary), S , P (both ternary), $\mathbf{1}$, $\mathbf{2}$ (both unary, k=1), and $=$ (binary; "equality"), and write down the formulas

$$\Phi(x) \equiv \neg\mathbf{1}(x) \wedge \forall y\forall z(Pyzx\rightarrow(\mathbf{1}(y)\vee\mathbf{1}(z)) ; \qquad (5')$$

and

$$\Psi \equiv \forall u\exists x\exists w(Rux \wedge \Phi(x) \wedge \Phi(w) \wedge \exists v(\mathbf{2}(v) \wedge S(x, v, w))) .$$

Here, the second formula is indicated in an abbreviated manner; the previous formula Φ is used in it twice, once with x , and once with w as the free variable. By the way, writing $\Phi(x)$ is a way of indicating that the formula Φ has x as its sole free variable. The unabbreviated way of writing Ψ is

$$\Psi \equiv \forall u\exists x\exists w(Rux \wedge \neg\mathbf{1}(x) \wedge \forall y\forall z(Pyzx\rightarrow(\mathbf{1}(y)\vee\mathbf{1}(z)))$$
$$\wedge \neg\mathbf{1}(w) \wedge \forall y\forall z(Pyzw\rightarrow(\mathbf{1}(y)\vee\mathbf{1}(z)))$$

$$\wedge \exists v(\mathbf{2}(v) \wedge S(x, v, w))) .$$

Now, note that, when we put $U=\mathbb{N}$,

$$\begin{aligned} Rxy &\sim x<y \,, \\ Sxyz &\sim x+y=z \,, \\ Pxyz &\sim x\cdot y=z \,, \\ \mathbf{1}(x) &\sim x=1 \,, \\ \mathbf{2}(x) &\sim x=2 \,, \end{aligned}$$

and, by $x=y$ we mean ordinary equality (as we always do in first order logic), then the formula Φ becomes

$\Phi(x)$ $\sim$ $x\neq 1$ and for all $y, z\in\mathbb{N}$,
if $y\cdot z=x$, then either $y=x$ or $z=x$

$\sim$ x is a prime number ;

and the sentence Ψ becomes

Ψ $\sim$ for all $u\in\mathbb{N}$, there are $x, w\in\mathbb{N}$ such that $u<x$,
both x and w are prime numbers, and $x+2=w$

$\sim$ there are infinitely many pairs of prime numbers x and w
such that $x+2=w$

$\sim$ there are infinitely many twin primes

(x and w are "twin primes" if they are both prime numbers and $x+2=w$. Note that 3 and 5 , 17 and 19 , are twin primes. Also note that

$\exists v(\mathbf{2}(v) \wedge S(x, v, w))$ $\sim$ there is $v\in\mathbb{N}$ such that $v=2$ and $x+v=w$

$\sim$ $x+2=w$.)

Nobody knows whether Ψ evaluates to $\top$, or to $\bot$ under the given interpretation, because nobody knows whether there are infinitely many twin primes.

Many mathematical matters can be expressed in first order logic. In fact, if we use the universe of all sets, *everything* mathematical can be expressed in first order logic -- or at least, experience seems to show that this is the case. (Note that such a statement is not subject to proof; its truth depends on what we deem to belong to mathematics; and this cannot be done by any means other than agreement.)

Let us use another piece of notation. Given a *sentence* Φ, meaning a formula without any free variable, and a structure $(U; R, S, \ldots)$ where the universe is U, and the relation R is meant to interpret the relation-symbol R (here, we employ a double meaning of the same letter R; but this kind of thing is done a lot in mathematics; one tries to be careful not to get confused ...), the relation S interprets the relation symbol S, etc, we write

$$(U; R, S, \ldots) \models \Phi$$

to mean that Φ is true, $\Phi \sim \top$ under the given interpretation $(U; R, S, \ldots)$. The symbol $\models$ is read as "satisfies": the whole phrase reads "the structure $(U; R, S, \ldots)$ satisfies the sentence Φ".

Let us consider a few examples for expressing properties of a (binary) relation in first order logic. We have in mind a binary relation (A, R); $R \subseteq A \times A$. We also use the *equality relation* $x=y$; a binary relation (symbol). Since $x=y$ is always interpreted as ordinary equality, it does not have to be listed in an interpretation.

Then

$$(A, R) \text{ is reflexive } \iff (A, R) \models \forall x Rxx \;;$$

$$(A, R) \text{ is symmetric } \iff (A, R) \models \forall xy(Rxy \rightarrow Ryx)$$

($\forall xy$ abbreviates $\forall x \forall y$);

(A, R) is transitive $\iff$ $(A, R) \models \forall xyz((Rxy \wedge Ryz) \rightarrow Rxz)$;

(A, R) is irreflexive $\iff$ $(A, R) \models \forall xy(Rxy \rightarrow x \neq y)$;

(A, R) is antisymmetric $\iff$ $(A, R) \models \forall xy((Rxy \wedge Ryx) \rightarrow x = y)$;

(A, R) is strictly antisymmetric $\iff$ $(A, R) \models \forall xy \neg(Rxy \wedge Ryx)$;
(note that, following (4'), another way of expressing strict antisymmetry is
(A, R) is strictly antisymmetric $\iff$ $(A, R) \models \forall xy(Rxy \rightarrow \neg Ryx)$)

(A, R) satisfies dichotomy $\iff$ $(A, R) \models \forall xy(Rxy \vee Ryx)$;

(A, R) satisfies trichotomy $\iff$ $(A, R) \models \forall xy(Rxy \vee x = y \vee Ryx)$.

In applications of predicate logic, one wants to use, besides *relations*, also *operations*. As a matter of fact, the formal use of operations is not necessary; it is something that may be called "syntactic sugar" on predicate logic. We saw above how to express arithmetical matters by relations only.

Each operation symbol comes with a definite *arity*, a natural number; if the arity is n , we talk about an n-*ary*, or n-*place* operation symbol. We use the letters f, g, h for operation symbols -- but also, special symbols such as $+$, $\cdot$, etc.

The syntax of operations is this. We define a grammatical category of entities called *terms* as follows:

every variable is a *term*;
if f is an n-ary operation symbol, and $t_1, t_2, \ldots, t_n$ are *terms*, then the expression $f(t_1, t_2, \ldots, t_n)$ is a *term*;
the only *terms* are obtained by the previous two clauses.

For instance, if f is ternary, g is unary, and c is a zero-ary operation symbol (denoting, in any interpretation, a distinguished element of the universe), and x , y are variables, then

$$f(c, g(f(x, c, y)), g(y))$$

is a term. The special binary symbols $+$, $\cdot$ and some others are used with infix notation: instead of $+(t_1, t_2)$, one writes t_1+t_2.

Terms are used to define *atomic formulas*:

an *atomic formula* is an expression of the form $R(t_1, t_2, ..., t_n)$, where R is an n-ary relation symbol, or an expression of the form $t_1=t_2$, with each t_i being a term.

Finally, the general definition of *formula* is as follows:

every atomic formula is a *formula*;
if Φ and Ψ are formulas, then
$\neg(\Phi)$, $(\Phi)\wedge(\Psi)$, $(\Phi)\vee(\Psi)$, $(\Phi)\longrightarrow(\Psi)$, $(\Phi)\longleftrightarrow(\Psi)$ are *formulas*;
if Φ is a *formula*, and x is a variable, then $\forall x(\Phi)$, $\exists x(\Phi)$ are *formulas*;
the only *formulas* are obtained by the previous three clauses.

Interpretations of operation symbols are operations on the given universe. For instance, if f is a ternary operation symbol, then it may be interpreted by a three-place operation, also denoted by f, with $f:U\times U\times U\longrightarrow U$.

The interpretation of terms, based on that of the operation symbols, is self-explanatory: it follows the usual algebraic practice of using compound algebraic expressions.

The important thing to emphasize is that all operations must be everywhere defined on U. This is necessary for the general rules of logic (see section 5.3) to remain valid in the presence of the operations.

For instance, using the arithmetic division $/$ in formal logic is admissible only if we artificially agree that $a/0$ is something specific, say 0 itself. Then, of course, one still has to worry whether $a/b = 0$ means that $a=0$ (the "genuine" case), or $b=0$ (the "artificial" case). It will *not* be true that

? $\quad \forall a\forall b((a/b)\cdot b=1)$;

instead,

$$\forall a \forall b (b \neq 0 \longrightarrow (a/b) \cdot b = 1) \, .$$