Section 4.2 Generating Boolean subalgebras

A *Boolean subalgebra*, or more simply, a *subalgebra*, of a Boolean algebra $(A;\leq)$ is a subset that contains the top and bottom elements of A, and with any elements x and y, it contains $x\wedge y$, $x\vee y$ and $-x$ as well.

A *sublattice* was defined similarly in section 3.2, but with reference to complements removed. Thus, a subalgebra of a Boolean algebra is a sublattice of the ambient Boolean algebra, which is also closed under taking complements. A subalgebra of a Boolean algebra is again a Boolean algebra, with operations inherited from the ambient algebra.

We *identify* a subalgebra with its underlying set. This is all right, since the order and the operations of the subalgebra are given by those of the ambient (containing) algebra, thus to specify a subalgebra of a Boolean algebra it suffices to specify its underlying set.

Consider the figures on pages 62 and 63 in Section 3.1, Chapter 3. The one on page 62 is the Hasse diagram of $\mathcal{P}(\{0, 1, 2\})$, the Boolean algebra of all subsets of the set $\{0, 1, 2\}$. The one on page 63 is an isomorphic copy of the previous one; in particular, a Boolean algebra. Let us call this A, or more completely, $(A, \leq)$. Its underlying set is $A=\{0, 1, 2, 3, 4, 5, 6, 7\}$.

Now, the set $X = \{0, 1, 6, 7\}$ is a subset of A; and in fact it is a subalgebra of $(A, \leq)$. Indeed, the top and bottom elements, $\top=7$ and $\bot=0$, of A, are in X. The meet and join of any two elements of X are again in X: there is only one pair of incomparable elements, 1 and 6, in X, and $1\wedge 6=0\in X$, $1\vee 6=7\in X$. Finally, $\neg 1 = 6 \in X$, $\neg 6 = 1 \in X$. X is a Boolean algebra on its own right; the top and bottom elements of it are the same as those of A; meets, joins and complements are computed in it as in A. For instance, the equations $1\wedge 6=0\in X$, $1\vee 6=7$, $\neg 1=6$, $\neg 6=1$ are right when understood either in $(A, \leq)$ or in $(X, \leq)$.

On the other hand, the set $Y=\{0, 1, 2, 4\}$ is not a subalgebra of A since the top element of A, 7, is not in Y. On the other hand, note that on its own right, Y, with the ordering inherited from A, *is* a Boolean algebra: it is isomorphic to $\mathcal{P}(\{1, 2\}), \subseteq)$. For instance, Y has a top element, but it is 4, not 7, the top element of A.

The subset $Z=\{0, 2, 6, 7\}$ is a sub*lattice* of A, but not a subalgebra, since $2 \in Z$, but $\neg 2 = 5$ is not in Z.

For any Boolean algebra $(A, \leq)$, the subset $\{\bot, \top\}$ consisting of the top and bottom elements alone is always a subalgebra. The reason is that the operations $\wedge$, $\vee$ and $-$, when applied to $\bot$ or $\top$, result in $\bot$ or $\top$ again. In particular,

$$\top \wedge \bot = \bot , \qquad \top \vee \bot = \top , \qquad \neg\top = \bot , \qquad \neg\bot = \top .$$

The last two relations are particularly important. They directly follow from the first two, by the very meaning of what "complement" ($\neg$) means.

The whole of a Boolean algebra itself is a subalgebra. In fact, the latter is the maximal subalgebra; the set $\{\bot, \top\}$ is the minimal subalgebra.

Given any Boolean algebra $(A, \leq)$ and an arbitrary subset X of A, we have the subalgebra of $(A, \leq)$ *generated* by X. This may be defined as the least (smallest) subalgebra of $(A, \leq)$ containing X. Let us consider this carefully.

Consider the set $\mathrm{Subalg}(A, \leq)$ of all subalgebras of $(A, \leq)$. E.g., the set A is always in it; and so is the set $\{\bot, \top\}$. When the algebra is A on page 63 of Chapter 3, the set $\mathrm{Subalg}(A, \leq)$ consists of five elements:

$$\begin{aligned}\mathrm{Subalg}(A, \leq) = \{&\{0, 7\} = \{\bot, \top\} ,\\ &\{0, 1, 6, 7\},\\ &\{0, 2, 5, 7\},\\ &\{0, 3, 4, 7\},\\ &A\ \}\end{aligned}$$

To see this, note that a subalgebra must be of size 2, 4 or 8, by what we learned in the last section. The one of size 2 is $\{\bot, \top\}$, the one of size 8 is A; the ones of size 4 are given by a pair of elements (x, y) which are different from $\bot$ and $\top$, and which are complements of each other: $\neg x = y$. There are three such pairs: 1 and 6; 2 and 5; 3 and 4.

$\mathrm{Subalg}(A, \leq)$ is a subset of the power-set $\mathcal{P}(A)$; we claim that

$\mathtt{Subalg}(A, \leq)$ *is closed under intersection in* $\mathcal{P}(B)$:

when X_1 , X_2 , $\ldots X_k$ are subalgebras of $(A, \leq)$, then the intersection $\bigcap_{i=1}^{k} X_i$ is again a subalgebra of $(A, \leq)$. (In fact, even an infinite intersection may be taken.)

The proof of this is quite easy once one decides to do it. Moreover, the essence of the matter has nothing to do with Boolean algebras. What is essential is only that we have a set A , some distinguished elements of it, in this case $\perp$ and $\top$, and a couple of operations on the set, in this case $-$ (unary) , $\wedge$, $\vee$ (both binary). The set $\mathtt{Subalg}(A, \leq)$ is, in the general case, replaced by the set S of all subsets that contain the distinguished elements, and which are closed under the operations. S is a subset of $\mathcal{P}(A)$, and, we assert, S is closed under intersection; the intersection of any number of elements of S is again a member of S .

For the sake of concreteness, let us return to the Boolean case. Consider the fact that the intersection of the second and third subalgebra of A in the example above is the first subalgebra; the assertion at hand says in this case that the intersection of any two members of that five-element set of sets is again a member of that five-element set.

Let us prove the assertion for $\mathtt{Subalg}(A, \leq)$ as stated above. To say that $\bigcap_{i=1}^{k} X_i$ is a subalgebra is to say, among others, that $\perp$ and $\top$ belong to $\bigcap_{i=1}^{k} X_i$. But this is clear since $\perp$ and $\top$ belong to each X_i . Further, we should see that $\bigcap_{i=1}^{k} X_i$ is closed under the operations $\wedge$, $\vee$, $-$; e.g., if x and y are in $\bigcap_{i=1}^{k} X_i$, then so is $x \wedge y$. But if x and y are in $\bigcap_{i=1}^{k} X_i$, then they are in X_i for each i ; since each X_i is a subalgebra, $x \wedge y \;\varepsilon\; X_i$; since this holds true for all i , $x \wedge y \;\varepsilon \bigcap_{i=1}^{k} X_i$ as required. The argument for $\vee$ and $-$ is identical. We have proved our claim.

Now, let X be an arbitrary subset of A , $(A, \leq)$ a Boolean algebra. Then we may look at the intersection of *all* subalgebras of $(A, \leq)$ containing X :

$$\langle X\rangle = \bigcap \{Y \mid X\subseteq Y \ \& \ Y \text{ is a subalgebra of } (A, \le)\} .$$

As an intersection of *some* subalgebras, $\langle X\rangle$ is again a subalgebra. Since X is a subset of each member of the set whose intersection is $\langle X\rangle$, it is clear that $X\subseteq\langle X\rangle$; $\langle X\rangle$ is a subalgebra containing X. But also, $\langle X\rangle$ is the *least* subalgebra containing X : whenever Y is a subalgebra of $(A, \le)$, and $X\subseteq Y$, then necessarily, $\langle X\rangle\subseteq Y$: this is clear, since Y is *a particular member of the set whose intersection is* $\langle X\rangle$. We conclude that

$\langle X\rangle$, *the intersection of all subalgebras containing* X , *is the lea*st *subalgebra containing* X .

We call $\langle X\rangle$ the *subalgebra generated by* X. It may happen that $\langle X\rangle=A$, the generated subalgebra is the whole algebra; in this case, we say that X *generates the algebra* $(A, \le)$.

Let us look at the example of $(A, \le)$ on page 63 of Chapter 3 again. Let $X = \{1, 6\}$. Looking at the list of all subalgebras of A given above, we see that the ones containing $\{1, 6\}$ are A itself, and $\{0, 1, 6, 7\}$; the intersection of these two is the smaller one, $\{0, 1, 6, 7\}$;

$$\langle\{1, 6\}\rangle = \{0, 1, 6, 7\} .$$

Let us note a curious aspect of the definition of $\langle X\rangle$. $\langle X\rangle$ itself is a subalgebra containing X , hence, $\langle X\rangle$ itself belongs to the set whose intersection is $\langle X\rangle$! This seems to make the definition through intersection pointless; the definition of $\langle X\rangle$ seems to refer to $\langle X\rangle$ itself, among all subalgebras containing X . However, before we considered the intersection, we did not know that the least subalgebra containing X existed; the definition through the intersection is necessary to have a hold on the thing theoretically.

In general, it would very difficult to find the subalgebra generated by a subset by following the definition. The definition requires the consideration of all subalgebras containing the given set X . Of course, once we have one, say Y , then there is no need to consider any but the ones that are contained in Y ; $\langle X\rangle$ will be contained in Y . If Y is large, it is of little help. We, of course, have A itself as one of the subalgebras containing X ; taking this as Y does not cut down on the subalgebras we have to consider for getting $\langle X\rangle$.

There is another approach to the subalgebra $\langle X\rangle$ generated by X that is better from the point of view of calculation, but messier from a theoretical point of view. Let X be an arbitrary subset of the Boolean algebra $(A, \leq)$. Then, besides the elements of X, all elements of the form

$$\bot\,,\ \top\,,\ \ x \wedge y\,,\ \ x \vee y\,,\ \ -x\,, \tag{8}$$

with x and y in X, are in $\langle X\rangle$. Moreover, all elements of the form

$$u \wedge v\,,\ \ u \vee v\,,\ \ -u\,,$$

where u and v come from the elements (8) just considered, will also belong to $\langle X\rangle$. E.g., this includes all elements of the form

$$(x \wedge y) \vee (x' \wedge y')$$

with $x,\ y,\ x',\ y'$ from X. Clearly, we can continue in this way, and we may say that

all elements, including the elements of X *itself, that can be expressed in terms of* $\bot$, $\top$ *and the elements of* X, *using, possibly repeatedly, the operations* $\wedge$, $\vee$ *and* $-$, *belong to* $\langle X\rangle$.

Now, we claim, the elements mentioned in the last displayed paragraph are *precisely* the elements of $\langle X\rangle$. Since we already know that they all belong to $\langle X\rangle$, we only have to convince ourselves that the set S of these elements *is a subalgebra*: since $\langle X\rangle$ is the least subalgebra, and $S \subseteq \langle X\rangle$, it must then be the case that $S = \langle X\rangle$! But, once we said this, the assertion is clear: the set S contains, by its definition, the elements $\bot$, $\top$; and if u, v belong to S, then they are given as expressions in terms of the elements of X, and $\bot$, $\top$, $\wedge$, $\vee$, $-$, and so, the elements $u \wedge v$, $u \vee v$ are also given as expressions, one level more complicated though, in terms of the elements of X, and $\bot$, $\top$, $\wedge$, $\vee$, $-$, and thus, $u \wedge v$, $u \vee v$ belong to S again. We conclude that

the subalgebra $\langle X\rangle$ *consists of all elements, including the elements of* X *itself, that can be expressed in terms of* $\bot$, $\top$ *and the elements of* X, *using, possibly repeatedly, the operations* $\wedge$, $\vee$ *and* $-$.

It is good to realize that this last description of the subalgebra generated by a set is again quite independent of the concrete kind of "algebra" we are considering (in this case, "Boolean algebra"). E.g., clearly, in the same way, we may talk about the sublattice of a lattice generated by a given set. Besides the definition as an intersection, we get that the sublattice generated by X is given as the Boolean subalgebra in the last description, except that we ignore the references to $-$. There are many other kinds of "algebras" considered in mathematics; the considerations just given apply to all of them.

Consider again this last description of the Boolean subalgebra generated by X . Constructing $\langle X\rangle$ involves collecting all *Boolean expressions* involving elements of X (this is just a short-hand for the elements described). Suppose we collect *some* of them, and we notice that the set of elements we have collected is already closed under the Boolean operations. Then, of course, we may stop, and have the given set equal to $\langle X\rangle$.

Let us reconsider the example of the Boolean algebra A on p. 63, Chapter 3, and its subset $X=\{0, 6\}$. $1=-6$ and $7=\top$; 1 and 7 are given as Boolean expressions in terms of X , and so, they belong to $\langle\{0, 6\}\rangle$. But, then we see that the set $X\cup\{1, 7\}=\{0, 1, 6, 7\}$ is already closed under the operations $\wedge$, $\vee$ and $-$ and it contains $\perp$ and $\top$ as well (in principle, by looking at all possible $u\wedge v$, $u\vee v$, with u and v from this set; in fact, now we know that $\{0, 1, 6, 7\}$ is a subalgebra of A since it appears in $\mathrm{Subalg}(A, \leq)$ above.); therefore, we conclude again that

$$\langle\{0, 6\}\rangle=\{0, 1, 6, 7\} .$$

Next, we give a theorem that describes the Boolean subalgebra $\langle X\rangle$ of any Boolean algebra generated by a set X in more explicit terms. This description will be quite specific to Boolean algebras; one does not have a similar description e.g. for the sublattice generated by a subset.

Let $(A, \leq)$ be a Boolean algebra, $X\subseteq A$. We consider the specific elements of A which are of the form

$$y_1\wedge y_2\wedge\ldots\wedge y_n$$

where each y_i is either x , or $-x$, with $x\in X$. These elements are called the *meet-expressions based on* X , or more simply, the *meets based on* X . The top element, $\top$, as the empty meet, is always, by definition, a meet based on X . Likewise, as the meet of the one-element set $\{x\}$, any element x of X is a meet based on X , and so is the complement of any element of X .

If x_1, x_2, x_3, x_4 are elements of X (X may have further elements), then

$$-x_1 \wedge -x_2 \wedge x_3$$

$$x_1 \wedge -x_2 \wedge -x_3 \wedge x_4$$

$$-x_1 \wedge x_2 \wedge x_3 \wedge x_4$$

are examples of meets based on X .

We use an abbreviation to denote meets. We write

$$\overline{x_1}\,\overline{x_2}x_3\ ,\quad x_1\overline{x_2}\,\overline{x_3}x_4\ ,\quad \overline{x_1}x_2x_3x_4$$

for the above three examples, by ignoring the symbol $\wedge$, and putting the minus sign on top of the letters.

Next, still with the given X , we consider *arbitrary joins of X-based meets*. By definition, $\perp$ is always such a join: $\perp$ is the join of the empty set. In the example when
$x_1, x_2, x_3, x_4 \,\varepsilon\, X$,

$$\overline{x_1}x_2x_3 \vee \overline{x_1}x_2x_3x_4 \vee \overline{x_1}\,\overline{x_2}x_3x_4$$

is another example of such an element. We call these elements the *join-meet expressions* based on X . The theorem promised above is as follows:

The set of the values of all the join-meet expressions based on the subset X of any Boolean algebra is identical to the subalgebra generated by X .

Somewhat sloppily, we will omit the phrase "the value of" from "the value of the join-meet expression E " and just say "the join-meet expression E ". The *expression* E is not the same thing as its *value*: an expression is a linguistic entity, its value, on the other hand, is an element of the Boolean algebra under consideration. But this kind of sloppiness is practiced in ordinary algebra all the time.

To be able to speak more briefly, let us denote the set of join-meet expressions based on X by $\hat{X}$. We are claiming that

$$\langle X\rangle = \hat{X} .$$

Clearly, every element of $\hat{X}$ is an element of $\langle X\rangle$, $\hat{X} \subseteq \langle X\rangle$ (why?). To see that $\langle X\rangle \subseteq \hat{X}$, we have to see that $\hat{X}$ is a subalgebra containing X , that is, the following facts:

(1) Every element of X is equal to a join-meet expression based on X ;

(2) $\bot$ and $\top \,\varepsilon\, \hat{X}$;

(3) If u and $v \,\varepsilon\, \hat{X}$, then $u \wedge v \,\varepsilon\, \hat{X}$, $u \vee v \,\varepsilon\, \hat{X}$, $-u \,\varepsilon\, \hat{X}$.

(1) and (2) are clear; likewise, the case of join in (3) (why?). Let $u, v \,\varepsilon\, \hat{X}$; then $u = \bigvee_{i=1}^{k} u_i$, $v = \bigvee_{j=1}^{\ell} v_j$, with each u_i, v_j a meet based on X . But then, by using the distributive law in its generalized form,

$$u \wedge v = \bigvee_{i=1}^{k} u_i \wedge \bigvee_{j=1}^{\ell} v_j = \bigvee_{\substack{i=1,\ldots,k \\ j=1,\ldots,\ell}} u_i \wedge v_j .$$

Now, notice that the meet of two meets based on X is again a meet based on X . This shows that $u \wedge v$ is a join-meet expression based on X . For use in the next argument, let us note that, as a consequence of the fact that the meet of any two elements of $\hat{X}$ is in $\hat{X}$, the meet of any finitely many elements of $\hat{X}$ belongs to $\hat{X}$ (we already know that $\top \,\varepsilon\, \hat{X}$).

Finally, let us turn to complements; assume $u = \bigvee_{i=1}^{k} u_i \,\varepsilon\, \hat{X}$, with each u_i a meet based on X; we want to show that $-u \,\varepsilon\, \hat{X}$. By the De Morgan law, we have

$$-u = -\bigvee_{i=1}^{k} u_i = \bigwedge_{i=1}^{k} -u_i \,. \tag{9}$$

But, each u_i is of the form $u_i = \bigwedge_{p=1}^{q_i} y_p$, with y_p either an element, or a negated (complemented) element of X. By De Morgan's law, now applied to negating a meet rather than a join, we get that $-u_i = \bigvee_{p=1}^{q_i} -y_p$. Here, each $-y_p$ is still either x, or $-x$ for some $x \,\varepsilon\, X$. Namely, if y_p is x, then $-y_p$ is $-x$, and if y_p is $-x$, then $-y_p$ is $--x = x$. So in particular, each of the $-y_p$'s is a meet expression (of a very simple kind) based on X, hence, their join is a join-meet expression, and so, $-u_i \,\varepsilon\, \hat{X}$. Now, above we saw that the meet of any finitely many elements of $\hat{X}$ is again in $\hat{X}$. Therefore, by (9), $-u$ belongs to $\hat{X}$ as desired.

This completes the proof of the theorem asserting that $\langle X \rangle$ coincides with the set of all join-meet expressions based on X.

Now, let us consider the special case when X (but not necessarily A) is a finite set; $X = \{x_1, x_2, \ldots, x_n\}$. The *complete meets* based on X are those meet-expressions that use each x_i, straight or negated, exactly once. That is, the complete meets based on X are the expressions of the form $(\overline{x_1})(\overline{x_2})\ldots(\overline{x_n})$, with bars present or not present at will. If $n = 4$, then $x_1\overline{x_2}\,\overline{x_3}x_4$, $\overline{x_1}x_2x_3x_4$ are complete meets based on X, but $\overline{x_1}\,\overline{x_2}x_3$ is not necessarily one, unless it happens to be equal to one for an individual reason (this last happens, e.g., if $x_4 = \top$; then $\overline{x_1}\,\overline{x_2}x_3 = \overline{x_1}\,\overline{x_2}x_3x_4$).

In the case when X has 4 elements, we may form $2^4 = 16$ of these complete meets; this is the number of ways we can assign positive or negative signs to four elements independently of each other. Of course, there is no guarantee that all the elementary meets are distinct; the actual number of distinct complete meets may be less than 16.

We claim that, in case X is finite,

every meet based on X is the join of some complete meets based on X .

Indeed, note that

$$u = (u \wedge v) \vee (u \wedge -v) \ ,$$

or in an abbreviated form,

$$u = uv \vee u\overline{v} \ .$$

Namely, the distributive law, read backwards, says that

$$(u \wedge v) \vee (u \wedge -v) \ = \ u \wedge (v \vee -v) = \ u \wedge \top \ = \ u \ .$$

Using this equality, any meet-expression can be "completed" to read as a join of complete meet-expressions. E.g., suppose $X = \{x_1, x_2, x_3, x_4\}$, and let $u = x_1\overline{x_2}$. Then

$$\begin{aligned} x_1\overline{x_2} &= x_1\overline{x_2}x_3 \vee x_1\overline{x_2}\,\overline{x_3} \\ &= x_1\overline{x_2}x_3x_4 \vee x_1\overline{x_2}x_3x_4 \vee x_1\overline{x_2}\,\overline{x_3}x_4 \vee x_1\overline{x_2}\,\overline{x_3}\,\overline{x_4} \ . \end{aligned}$$

Thus, if X is a finite set, then every meet-expression, and hence, every join-meet expression, based on X , is the join of complete meets based on X , and we get that

for a finite subset X of a Boolean algebra, the subalgebra $\langle X \rangle$ generated by X is the set of all joins of complete meets based on X .

We obtain an estimate of the cardinality of the subalgebra $\langle X \rangle$.

The cardinality of the subalgebra generated by a set of cardinality n is at most

$2^{(2^n)}$.

The reason is that the number of complete meets based on X is at most 2^n ; we have at most $2^{(2^n)}$ sets of complete meets of which to form joins.

In particular,

in any Boolean algebra, a finite set generates a finite subalgebra.

This is not obvious on the basis of the definitions alone, and in fact, it does not hold true for many other kinds of "algebras" in mathematics; the sublattice generated by a finite subset of a lattice is not always finite.

The last result, together with the one from the last section that says that every finite Boolean algebra is isomorphic to a power-set algebra, allows us to make an inference of a general "logical" nature. We ask ourselves about *all possible identities* holding for the Boolean operations on sets. Certainly, since the power-set algebras are particular Boolean algebras, any identity that can be derived from the Boolean identities will hold for sets. But, is the converse true? That is, could there be an identity, say $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$, made up of the variables $x_1, \ldots, x_n$, and the Boolean operations $\top$, $\bot$, $\wedge$, $\vee$, $-$, which *does* hold whenever the x_i mean subsets of a fixed set A , and $\top$, $\bot$, $\wedge$, $\vee$, $-$ mean A , $\emptyset$, $\cap$, $\cup$ and $A-(\ )$, respectively, but which *does not* follow from the axioms for Boolean algebras? If such an identity existed, one could argue that the concept of Boolean algebra is "incomplete".

The answer to this question is "no"; such an identity cannot exist. In other words, if an identity holds in power-set algebras, then it holds in all Boolean algebras;

the concept of Boolean algebra is "complete" as far as identities for the Boolean operations on sets are concerned.

To see this, suppose that the identity $p(x_1, ..., x_n) = q(x_1, ..., x_n)$ holds in power-set algebras; let A be an arbitrary Boolean algebra, and $x_1, ..., x_n$ particular elements in A; we want to see that

$$p(x_1, ..., x_n) = q(x_1, ..., x_n) \tag{9'}$$

holds in A. Take $B \overset{=}{\text{def}} \langle\{x_1, ..., x_n\}\rangle$, the subalgebra of A generated by the given elements. It is enough to show that (9') holds in B, since the operations in B are the same as in A, except that they are restricted to a smaller set. But by what we proved, B is a finite Boolean algebra, and thus, it is isomorphic to $\mathcal{P}(C)$ for a finite set C. But, we assumed that (9') was true for any elements $x_1, ..., x_n$ in any algebra of the form $\mathcal{P}(C)$. Since B is isomorphic to $\mathcal{P}(C)$, (9') must hold in B for any elements $x_1, ..., x_n$ in B, in particular, also for the originally selected elements $x_1, ..., x_n$. And this is what we wanted to prove.

Next, let us point out that, for X a finite set,

the atoms of the Boolean algebra $\langle X\rangle$ *are exactly those complete meet-expressions based on* X *which are not equal to* $\perp$.

To see this, let Y be the set of all the complete meet-expressions based on X *that are not equal to* $\perp$. Still, every element of $\langle X\rangle$ is a join of elements of Y; the bottom can always be omitted from every join. Now, assume a is an atom in $\langle X\rangle$; it is a non-empty join of elements of Y; but, then for at least one $y \in Y$, $y \leq a$; since $y \neq \perp$, we must have that $a = y$. We have proved that every atom of $\langle X\rangle$ must be an element of Y. Conversely, let $y \in Y$, and

$$y = y_1 y_2 ... y_n$$

with $y_i = x_i$, or $-x_i$, where x_i $(i = 1, ..., n)$ are all the elements of X. We want to see that y is an atom in $\langle X\rangle$. Then, since $y \neq \perp$, as we noted above, there is an atom a of $\langle X\rangle$ below y, $a \leq y$. We just saw that a must belong to Y,

$$a = a_1 a_2 \ldots a_n ,$$

where each $a_i = x_i$, or $-x_i$. We must have $a_i = y_i$; otherwise, $y_i = x_i$, and $a_i = -x_i$, or vice versa; thus, $a_i \wedge y_i = x_i \wedge -x_i = \perp$. But then $a \wedge y \leq a_i \wedge y_i = \perp$, and so, $a = a \wedge y = \perp$, contradicting that a is an atom. We thus have that, for each i, $a_i = y_i$; which of course implies that $a = a_1 a_2 \ldots a_n = y_1 y_2 \ldots y_n = y$. Since a is an atom, and $y = a$, y is an atom as asserted.

Knowing the atoms of the Boolean algebra $\langle X \rangle$ gives us all the elements of that algebra, by the proposition proved in the previous section, according to which, in a finite Boolean algebra, each element is the join of the atoms below it. This now tells us again that every element of $\langle X \rangle$ is the join of some complete meets based on X. Note, however, that to reach the conclusion about the atoms of $\langle X \rangle$, we first had to prove the statement in the previous sentence -- thus, we did not do anything superfluous!

Let $X = \{x_1, x_2, \ldots, x_n\}$, with the x_i's (distinct) elements of the Boolean algebra $(A, \leq)$. Let us look at the subalgebra $\langle X \rangle$ generated by X more closely, with the exact determination of its cardinality in mind. Among the 2^n formal expressions

$$(\bar{x}_1)(\bar{x}_2)\ldots(\bar{x}_n) ,$$

where each letter x_i is either negated or non-negated, there are some which are equal to $\perp$; let us say that their number is k. The remaining $2^n - k$ expressions represent atoms. Moreover, it is clear that any two formally different expressions that are not equal to $\perp$ are *different* atoms: here, "formally different" means that at least one x_i appears in one unnegated, in the other negated. The reason is that the two expressions are disjoint, as a consequence of x_i and $-x_i$ being disjoint. Therefore, the algebra $\langle X \rangle$ has exactly $2^n - k$ atoms. But then, by what we learned in the previous section, we know that the cardinality of $\langle X \rangle$ is: $|\langle X \rangle| = 2^{(2^n - k)}$. In fact, $\langle X \rangle$ is isomorphic to a power-set algebra $\mathcal{P}(Y)$, where Y is a set of cardinality $2^n - k$; Y can be taken to be what it denoted above: the non-bottom complete meets based on X.

When $k=0$, we say that the values $x_1, x_2, ..., x_n$ are *independent*. A set of elements of a Boolean algebra are independent if all the complete meet expressions based on them are different from the bottom element. This is equivalent to saying that the subalgebra generated by $x_1, x_2, ..., x_n$ is of cardinality $2^{(2^n)}$.

We can give a very neat description of *all* the finite Boolean subalgebras of a given Boolean algebra by looking at the above things a bit longer. Looking at any *finite* Boolean algebra, we know that every element, hence in particular the top element $\top$, is the join of the atoms below it, that is, all the atoms in the algebra. To repeat:

in any finite Boolean algebra, the top element $\top$ *is the disjoint join of the set of all atoms in the algebra*.

(Of course, a "disjoint join" means a join of pairwise *disjoint* elements, elements such that for any two distinct ones of them, say x and y , we have $x \wedge y = \bot$.)

If $x_1,\ x_2,\ ...,\ x_n$ are non-bottom elements of a Boolean algebra, which are pairwise disjoint and whose join is $\top$, we say that $\{x_1, x_2, ...x_n\}$ *is a partition of* $\top$. Of course, the expression comes from the fact that a partition of $\top$ in $\mathcal{P}(B)$ is the same thing as a partition of the set B in the usual sense.

Now, if C is a finite subalgebra of the Boolean algebra $(A, \leq)$, the atoms of C are disjoint in C , hence, *they are disjoint in the sense of* $(A, \leq)$ *as well*, since meet and bottom element are the same in $(C, \leq)$ and $(A, \leq)$. Also, for a similar reason, the join of the atoms of C is the top $\top$ of $(A, \leq)$ (which is the same as the top of $(C, \leq)$). We conclude that

the atoms of a finite subalgebra of a Boolean algebra form a partition of the top element of the Boolean algebra.

Note however that an atom of a subalgebra is far from necessarily being an atom of the big algebra; an atom of the subalgebra has no non-bottom element *of the subalgebra* under it, but it may have plenty of elements of the big algebra under it.

In particular, we conclude that

the atoms of any finite subalgebra of $\mathcal{P}(B)$ *form a finite partition of the set* B *in the usual sense*.

By the *Venn diagram* generated by a system $\{X_1, X_2, \ldots, X_n\}$ of subsets of a fixed set B we mean the partition of the set B whose cells are the *non-empty* complete intersection(=meet)-expressions based on $\{X_1, X_2, \ldots, X_n\}$, that is, all *non-empty* sets of the form

$$(-)X_1 \cap (-)X_2 \cap \ldots \cap (-)X_n ,$$

with the minus signs present or not in an arbitrary manner. The cells of the Venn-diagram generated by a system of sets are the atoms of the Boolean subalgebra generated by the given sets. The elements of that generated subalgebra are precisely the unions (joins) of the cells.

We have just seen that any finite subalgebra of a Boolean algebra gives rise to a partition of $\top$. Moreover, the subalgebra is completely given by this partition; it consists of the joins that can be formed using the elements of the partition.

Now, let us consider an *arbitrary* partition $\{x_1, x_2, \ldots, x_n\}$ of $\top$ in a Boolean algebra $(A, \leq)$. What are the atoms of the subalgebra generated by $\{x_1, x_2, \ldots, x_n\}$? Not surprisingly, the elements x_1 , x_2 , $\ldots$ x_n themselves! Just consider: an expression of the form

$$(-)x_1 \cap (-)x_2 \cap (-)x_3 \cap \ldots \cap (-)x_n$$

(which is the form any atom of the generated subalgebra will take) must be $\bot$ if there are two or more terms without $-$ in the expression; namely, any two distinct atoms are disjoint, and thus, any intersection in which two distinct atoms and possibly other things are terms is bottom. If all terms have minus signs, we just have

$$\begin{aligned} &-x_1 \wedge -x_2 \wedge -x_3 \wedge \ldots \wedge -x_n \\ &\quad = -(x_1 \vee x_2 \vee x_3 \vee \ldots \vee x_n) = -\top = \bot ; \end{aligned}$$

again, we got $\bot$. Therefore, the only way we get a non-bottom element, that is, an atom of the subalgebra, is when we have precisely one term without minus sign in the expression. Now, suppose this term without minus sign is the first one:

$$x_1 \wedge -x_2 \wedge -x_3 \wedge \ldots -x_n \ . \tag{10}$$

But, since $x_1 \wedge x_2 = \bot$, we have

$$x_1 = x_1 \wedge \top = x_1 \wedge (x_2 \vee -x_2) = (x_1 \wedge x_2) \vee (x_1 \wedge -x_2) = \bot \vee (x_1 \wedge -x_2) = x_1 \wedge -x_2 \ .$$

We got that $x_1 \wedge -x_2 = x_1$. So, the element under (10) is

$$x_1 \wedge -x_3 \wedge \ldots -x_n \ .$$

Of course, by the same argument, $-x_3$, $\ldots -x_n$ can also be taken away, and we get that the element under (10) is just x_1 .

Similar conclusion can be drawn if, instead of x_1 , another x_i is the one term without a negative sign. We obtain, as promised, that

the atoms of the subalgebra generated by a partition of $\top$ *are precisely the elements of the partition.*

We conclude

the finite subalgebras of a Boolean algebra are in a one-to-one correspondence with the finite partitions of $\top$.

If we take $\mathcal{P}(B)$ to be the Boolean algebra, and if we consider that $\top$ is the set B itself, and a finite partition of B is the same as an equivalence relation with finitely many equivalence classes, we get that *the finite subalgebras of* $\mathcal{P}(B)$ *are in a one-to-one correspondence with those equivalence relations on* B *which have finitely many classes.*

Of course, if B itself is finite, we do not have to stipulate that the equivalence relation have finitely many classes. Also, it is easy to see that if X and Y are two Boolean subalgebras of $\mathcal{P}(B)$, and E and F are the corresponding equivalence relation on B , then $X \subseteq Y$ iff $F \subseteq E$. We conclude:

For a finite set B *, the lattice of Boolean subalgebras of* $\mathcal{P}(B)$ *is isomorphic to the converse of the lattice of equivalence relations on* B *.*