Section 3.2 Lattices

Let $(A, \leq)$ be a (reflexive) order, considered fixed throughout this section. Thus, the symbol $\leq$ now means an arbitrary order on the arbitrary set A, rather than the usual less-than-or-equal relation on numbers.

As usual, $x \geq y$ means the same as $y \leq x$.

When, as sometimes happens, we also want to refer to the ordinary meaning of $\leq$ in the same context with an "arbitrary" order, we have to use a letter like R for the "arbitrary" order. Thus, you should be able to see what follows also with R replacing $\leq$.

Let $X \subseteq A$, a subset of A, and $y \in A$, an element of A.

Definition y is an *upper bound* of X if for all $x \in X$, we have $x \leq y$. In symbols

y is an upper bound of X $\iff$ $\forall x . x \in X \longrightarrow x \leq y$.

Note the following obvious facts:

If y is an upper bound of X, and $y \leq z$, then z is also an upper bound of X.
x is an upper bound of $\{x\}$.
Any $y \in A$ is an upper bound of $\emptyset$. (Note: the empty set $\emptyset$ is a subset of A; thus, we can take X to be $\emptyset$.)

Exercise 1. Prove the assertions just made.

The set of all upper bounds of X is denoted by $X\uparrow$. Thus, to say that y is an upper bound of X is the same as to write $y \in X\uparrow$.

The concept of *lower bound* is similar; it is obtained by replacing $\leq$ with $\geq$:

$$y \text{ is an lower bound of } X \iff \forall x.\, x\in X \longrightarrow x \geq y \,.$$

The set of all lower bounds of X is denoted by $X\downarrow$.

It is important to keep in mind that the expressions $X\uparrow$, $X\downarrow$ make an implicit reference to the order in which they are evaluated.

Example 1. Let $A=\mathbb{R}$, and $\leq$ the usual less-than-or-equal relation on $\mathbb{R}$. Let $X=\{\frac{n}{n+1} : n\in\mathbb{N}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ ...\}$. Then $y=1$ is an upper bound of X since $\frac{n}{n+1}\leq 1$ for all $n\in\mathbb{N}$ (in fact, $\frac{n}{n+1}<1$). In fact, $y\in\mathbb{R}$ is an upper bound of X if and only if $y\geq 1$ (why?) . In other words,

$$\{\frac{n}{n+1} : n\in\mathbb{N}\}\uparrow = \{y\in\mathbb{R} : y\geq 1\} \,.$$

Also,

$$\{\frac{n}{n+1} : n\in\mathbb{N}\}\downarrow = \{y\in\mathbb{R} : y\leq 0\}$$

(why?).

Example 2. Let $(A, \leq)$ be represented by the Hasse diagram

[Figure 22]

Now, we have

$$\{3, 4\}\uparrow = \{6, 7, 8\} , \quad \{2, 3\}\uparrow = \{5, 6, 7, 8\} , \quad \emptyset\uparrow = A , \quad \{5, 8\}\uparrow = \emptyset ,$$
$$\{5, 6\}\downarrow = \{1, 2, 3\} , \quad \{7, 8\}\downarrow = \{1, 2, 3, 4, 6\} , \quad \emptyset\downarrow = A \,.$$

Definition For any subset Y of A, we say that $b \in Y$ is *the least element of* Y if for all $y \in Y$, we have $b \leq y$. Similarly, $b \in Y$ is the *greatest element* of Y if for all $y \in Y$, we have $b \geq y$.

A subset Y of A may or may not have a least element; but if it has one, the least element is uniquely determined (why is that?). Similar statements can be made for the "greatest element".

Example 1 (continued) The set $\{y \in \mathbb{R} \colon y \geq 1\}$ has a least element; it is 1. The set $\{y \in \mathbb{R} \colon y > 1\}$ has no least element (why?). The subset $\mathbb{R}$ of $\mathbb{R}$ has no least element, and no greatest element (why?). The set $\{y \in \mathbb{R} \colon y \leq 0\}$ has a greatest element, 0; but $\{y \in \mathbb{R} \colon y < 0\}$ has no greatest element.

Example 2 (continued) The set $\{6, 7, 8\}$ has a least element, 6: $6 \leq 6$, $6 \leq 7$, and $6 \leq 8$. The set $\{5, 6, 7, 8\}$ has no least element. $\emptyset$ has no least, or greatest, element, since it has no element at all.

The concept of "least element" has to be compared to that of "minimal element" carefully.

By a *minimal element* of a subset X of A we mean any element $a \in X$ such that for all $x \in X$, $x \quad a$: it is *not* the case that $x < a$. The least element of X is certainly a minimal element of X, but not the other way around. For instance, in Example 2, $\{5, 6, 7, 8\}$ has two minimal elements, 5 and 6, but no least element. On the other hand, if a subset X of A does have a least element, that element is necessarily the *unique* minimal element of X.

The concept of *unique minimal* element agrees with the least element in case the set A is finite; but not in general. Consider the following example. Let $A = \mathbb{Z} \times \mathbb{Z}$, and let R be the relation on A defined by

$$(a, b) R (c, d) \iff a \leq c \text{ and } b \leq d$$

(here, we used $\leq$ in its usual sense as the less-than-or-equal relation on numbers). Let $X = \{(0, b) \colon b \in \mathbb{Z}\} \cup \{(1, c) \colon c \in \mathbb{N}\}$. The element $(1, 0)$ is a minimal element of X in the present order (A, R). In fact, $(1, 0)$ is the *only* minimal element in X: $(1, 0)$ is the *unique minimal* element. However, X has *no least* element in the given order (A, R).

On the other hand, for a *finite* order $(A, \le)$, a unique minimal element of A is the same as a least element of A (can you prove this?). For any *total* order $(A, \le)$, a minimal (maximal) element is necessarily unique if it exists, *and* it is the least (the greatest) element of A (can you prove this?).

Definition Let X be a subset of A.

The *join of* X, written as $\bigvee X$, is the least element of $X\uparrow$, if it exists.

The *meet of* X, written as $\bigwedge X$, is the greatest element of $X\downarrow$, if it exists.

The join of X is also called the *supremum*, or more briefly the *sup* of X; for "meet" we also say "*infimum*" or "*inf*".

Example 1 (continued) We have $\bigvee \{\frac{n}{n+1}: n\in\mathbb{N}\} = 1$. This is because $\{\frac{n}{n+1}: n\in\mathbb{N}\}\uparrow = \{y\in\mathbb{R}: y\ge 1\}$, and the least element of $\{y\in\mathbb{R}: y\ge 1\}$ is 1. On the other hand, $\bigvee \mathbb{R}$ does not exist. This is because $\mathbb{R}\uparrow = \emptyset$, and $\emptyset$ has no least element (no element at all).

We have $\bigwedge \{\frac{n}{n+1}: n\in\mathbb{N}\} = 0$, since $\{\frac{n}{n+1}: n\in\mathbb{N}\}\downarrow = \{y\in\mathbb{R}: y\le 0\}$, and the greatest element of $\{y\in\mathbb{R}: y\le 0\}$ is 0.

Example 2 (continued) We have $\bigvee \{3, 4\} = 6$, since $\{3, 4\}\uparrow = \{6, 7, 8\}$, and the least element of $\{6, 7, 8\}$ is 6. On the other hand, $\bigvee \{2, 3\}$ does not exist, since $\{2, 3\}\uparrow = \{5, 6, 7, 8\}$, and $\{5, 6, 7, 8\}$ has no least element. $\bigvee \emptyset$ does not exist, since $\emptyset\uparrow = A$, the total set, and A has no greatest element.

We have $\bigwedge \{7, 8\}=6$, since $\{7, 8\}\downarrow = \{1, 2, 3, 4, 6\}$, and the greatest element of $\{1, 2, 3, 4, 6\}$ is 6. On the other hand, $\bigwedge \{5, 6\}$ does not exist, since $\{5, 6\}\downarrow = \{1, 2, 3\}$, and $\{1, 2, 3\}$ has no greatest element. $\bigwedge \emptyset = 1$, since $\emptyset\downarrow=A$, and A has a least element, 1.

Two reformulations of the concepts of "join" (and "meet"):

$$\bigvee X = a \quad \text{if and only if} \quad X\uparrow = \{a\}\uparrow .$$

(*) $\qquad \bigvee X = a$ if and only if the following holds:

$$\text{for all } u \in A, \quad u \geq a \iff \forall x . x \in X \longrightarrow u \geq x$$

(and similarly for $\bigwedge$).

Exercise 2. Prove the two assertions above.

We always have the following facts:

$$\bigvee \{a\} = a , \qquad\qquad \bigwedge \{a\} = a .$$

$\bigvee \emptyset$, if it exists, is the *bottom element* of the order; it is denoted by $\perp$. Note that $\emptyset\uparrow$ is the whole of A (every element of A is an upper bound of $\emptyset$); thus, $\perp = \bigvee \emptyset$, being the least element of $\emptyset\uparrow$, it is the least element of A (if it exists). $\perp$ is characterized by the fact that for all $a \in A$, we have $\perp \leq a$.

Similarly, $\bigwedge \emptyset$, if it exists, is the *top element*, or *greatest* element of the order; it is denoted by $\top$; we have $a \leq \top$ for all $a \in A$.

$$\bigwedge A = \bigvee \emptyset = \perp , \qquad\qquad \bigvee A = \bigwedge \emptyset = \top .$$

If the set X has a least element a , then $\bigwedge X = a$;
if the set X has a greatest element a , then $\bigvee X = a$.

Of course, the converses of the last statements are, in general, false. For instance, in Example 2, we have $\bigwedge \{7, 8\} = 6$, but 6 is not the least element of $\{7, 8\}$; 6 is not an element of $\{7, 8\}$ at all.

Exercise 3. Prove that

$$\bigvee X = \bigwedge (X\uparrow)$$
$$\bigwedge X = \bigvee (X\downarrow) \ ;$$

more precisely, in each case, the (value of the) left-hand-side expression exists if and only if the right-hand-side does, and when they exist, they are equal. (Hint: first prove that, for any subset $X \subseteq A$, if $a = \bigwedge (X\uparrow)$ exists, then a must be the least element of $X\uparrow$; and a similar statement for $\bigvee (X\downarrow)$.)

When the set X is given in the form $X = \{x_i : i \in I\}$, where x_i is some expression of the variable i ranging over some set I, then we may write $\bigvee_{i\in I} x_i$ for $\bigvee X$, and $\bigwedge_{i\in I} x_i$ for $\bigwedge X$. For instance, if $\mathcal{X}$ is a set of some subsets of A, in other words, every X in $\mathcal{X}$ is a subset of A, then the expression $\bigvee_{X\in\mathcal{X}} (\bigvee X)$ means the same as $\bigvee \{\bigvee X : X \in \mathcal{X}\}$. In turn, this means taking the join $\bigvee X$ of each set X in the collection $\mathcal{X}$, and then taking the join of the set of all the joins so obtained.

Let us recall that $\bigcup \mathcal{X}$ stands for the union of all the sets that are elements of $\mathcal{X}$. We can say the same by the formula

$$x \in \bigcup \mathcal{X} \iff \exists X \in \mathcal{X} . x \in X \ .$$

We may also write

$$x \in \bigcup_{i\in I} X_i \iff \exists i \in I . x \in X_i \ .$$

Exercise 4. Prove that

$$\bigvee (\bigcup \mathcal{X}) = \bigvee_{X\in\mathcal{X}} (\bigvee X) \ ,$$

meaning that if one side exists, so does the other, and they are equal.

Examples for the last equality:

$$\bigvee\{x_1, x_2, x_3\} = \bigvee\{\bigvee\{x_1, x_2\}, x_3\} ,$$

$$\bigvee\{x_1, x_2, x_3, x_4, x_5\} = \bigvee\{\bigvee\{x_1, x_2\}, \bigvee\{x_3, x_4, x_5\}\}$$

Special notation:

$$x\vee y \underset{\overline{\overline{\mathrm{def}}}}{} \bigvee\{x, y\} .$$
$$x\wedge y \underset{\overline{\overline{\mathrm{def}}}}{} \bigwedge\{x, y\} .$$

Note the obvious facts:

if $x\leq y$, then $x\vee y = y$ and $x\wedge y = x$.

Item (*) above becomes, for $X=\{x, y\}$, the following characterizations of $x\vee y$ and $x\wedge y$:

$$\frac{u \geq x\vee y}{u \geq x \quad \text{and} \quad u \geq y} \qquad \frac{u \leq x\wedge y}{u \leq x \quad \text{and} \quad u \leq y}$$

These abbreviated statements use the horizontal lines to mean "if and only if". It is understood that the statements hold true for all u in A .

Definition A *lattice* is an order $(A, \leq)$ in which $\bigvee X$, $\bigwedge X$ exist for all *finite* subsets X of A .

A *complete lattice* is an order $(A, \leq)$ in which $\bigvee X$, $\bigwedge X$ exist for all subsets X of A .

Facts:

1a) $(A, \leq)$ is a complete lattice if and only if $\bigwedge X$ exists for every subset X of A .

1b) Similarly, $(A, \leq)$ is a complete lattice if and only if $\bigvee X$ exists for every subset X of A.

2) If A is a *finite* set, then $(A, \leq)$ is a lattice if and only if it is a complete lattice.

3) An order $(A, \leq)$ is a lattice if and only if $\top$ (= $\bigwedge \emptyset$) , $\bot$ (= $\bigvee \emptyset$) , $x \wedge y$, $x \vee y$ all exist in it, the latter two for all $x, y \in A$.

4a) If A is a *finite* set, then the order $(A, \leq)$ is a lattice if and only if $\top$ (= $\bigwedge \emptyset$) and $x \wedge y$ exist in it, the latter for all $x, y \in A$.

4b) The previous statement, with $\bot$ (= $\bigvee \emptyset$) replacing $\top$, and $x \vee y$ replacing $x \wedge y$.

Exercise 5. Prove the facts (**Hints**: For 1a) and 1b): use exercise 3. For 2): every subset of a finite set is finite. For 3): use exercise 4 and the examples after it, to obtain the join or meet of a set of finitely many elements by the successive evaluation of joins of two elements. For 4a) and 4b): use several previous things.

Example 3. We modify the Hasse diagram (A, H) given in Example 2 by removing the element $(2, 6)$ from H , adding the element 9 to A , and adding the pairs $(7, 9)$, $(8, 9)$ to H :

[Figure 23]

This represents a lattice (that is, its transitive closure is the irreflexive version of a lattice) which we call $(A, \leq)$; now, $A=\{i \in \mathbb{N} : 1 \leq i \leq 9\}$.

We can display this fact as follows. First of all, the Hasse diagram shows that we have a unique maximal element and a unique minimal element:

$$\top = 9 \qquad \bot = 1 .$$

Secondly, for any x and y in A that are *comparable* in the order, that is, either $x \le y$ or $y \le x$, we know that $x \wedge y$, $x \vee y$ both exist (in this case, these values are all equal to either x or y). Thus, in checking joins $x \vee y$ and meets $x \wedge y$, we may confine our attention to *incomparable* x and y. But also, it is not necessary to consider both $x \vee y$ and $y \vee x$, since one of these exists if the other does, and they are equal: they are equal to $\bigvee \{x, y\} = \bigvee \{y, x\}$. Therefore, in our example, we make a list of all pairs (x, y) of incomparable elements x, y for which, also, $x<y$ in the usual ordering $<$ of the integers; the latter condition is to make sure that we do not list both (x, y) and (y, x).

Here is the list of incomparable pairs:

$(2, 3)$, $(2, 4)$, $(2, 6)$, $(2, 8)$,
$(3, 4)$,
$(4, 5)$,
$(5, 6)$, $(5, 8)$,
$(7, 8)$.

We have

$(2, 3)\uparrow = \{5, 7, 9\}$,	$2 \vee 3 = 5$;	$(2, 3)\downarrow = \{1\}$,	$2 \wedge 3 = 1$;
$(2, 4)\uparrow = \{7, 9\}$,	$2 \vee 4 = 7$;	$(2, 4)\downarrow = \{1\}$,	$2 \wedge 4 = 1$;
$(2, 6)\uparrow = \{7, 9\}$,	$2 \vee 6 = 7$;	$(2, 6)\downarrow = \{1\}$,	$2 \wedge 6 = 1$;
$(2, 8)\uparrow = \{9\}$,	$2 \vee 8 = 9$;	$(2, 8)\downarrow = \{1\}$,	$2 \wedge 8 = 1$;
$(3, 4)\uparrow = \{6, 7, 8, 9\}$,	$3 \vee 4 = 6$;	$(3, 4)\downarrow = \{1\}$,	$3 \wedge 4 = 1$;
$(4, 5)\uparrow = \{7, 9\}$,	$4 \vee 5 = 7$;	$(4, 5)\downarrow = \{1\}$,	$4 \wedge 5 = 1$;
$(5, 6)\uparrow = \{7, 9\}$,	$5 \vee 6 = 7$;	$(5, 6)\downarrow = \{1, 3\}$,	$5 \wedge 6 = 3$;
$(5, 8)\uparrow = \{9\}$,	$5 \vee 8 = 9$;	$(5, 8)\downarrow = \{1, 3\}$,	$5 \wedge 8 = 3$;
$(7, 8)\uparrow = \{9\}$,	$7 \vee 8 = 9$;	$(7, 8)\downarrow = \{1, 3, 4, 6\}$,	$7 \vee 8 = 6$;

It may be noted that it would have been sufficient to verify the existence of all joins $x \vee y$, or alternatively, of all meets $x \wedge y$, because now the underlying set A is finite (see Exercise 5 above).

Examples for lattices:

L1. For any set B, the order $(\mathcal{P}(B), \subseteq)$ of all subsets of B is a lattice. In fact, in this case, we have

$$\begin{aligned} \top &= B \\ \bot &= \emptyset \\ X \wedge Y &= X \cap Y \\ X \vee Y &= X \cup Y . \end{aligned}$$

$(\mathcal{P}(B), \subseteq)$ is in fact a complete lattice. We call any lattice of the form $(\mathcal{P}(B), \subseteq)$ a *power-set lattice*.

Particularly important is the case when $B = \{0\}$. Now, $A = \mathcal{P}(\{0\}) = \{\emptyset, B\}$. Now, $\emptyset = \bot$ and $B = \{0\} = \top$; thus, $A = \{\top, \bot\}$. This lattice has two elements; it is frequently denoted by **2**.

The meet and join tables for **2** are those for "and" (conjunction) and "inclusive or" (disjunction). This interpretation depends on reading $\top$ as "true", and $\bot$ as "false". Here are the tables:

$\vee$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

$\wedge$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

L2. Here is a particular infinite lattice: $(\mathbb{N}, \mid)$. The relation $\mid$ is "divides":

$$a \mid b \iff \exists c \in \mathbb{N}.\ a \cdot c = b$$

In this case, we have:

$$\begin{aligned} \top &= 0 \\ \bot &= 1 \\ a \wedge b &= \mathrm{gcd}(a, b) \\ a \vee b &= \mathrm{lcm}(a, b) \end{aligned}$$

$\mathtt{gcd}$ means "greatest common divisor"; $\mathtt{lcm}$ means "least common multiple". $(\mathbb{N}, |)$ is not a complete lattice. Can you say why?

L3. There are many lattices formed by certain special subsets, as opposed to all subsets, of a given set. For instance, let B be any set, and let us consider the set of all equivalence relations on the set B; this set is denoted by $\mathcal{E}(B)$. Since every equivalence relation is, in particular, a subset of $B\times B$, we have that $\mathcal{E}(B) \subseteq \mathcal{P}(B\times B)$. We may consider the subset relation $\subseteq$ restricted to $\mathcal{E}(B)$; this is the order induced by $\subseteq$ on $\mathcal{E}(B)$. It turns out that $(\mathcal{E}(B),\subseteq)$ is a lattice; in fact, a complete lattice.

Similarly, the set $\mathcal{T}r(B)$ of all transitive relations on the set B, again with $\subseteq$ as the order, is a complete lattice.

Exercise 6. Prove that L1, L2 and L3 are indeed lattices, with the indicated lattice operations (when such is given).

(**Hint** for L3: show that the intersection of any non-empty collection of equivalence relations is again an equivalence. Conclude that in $(\mathcal{E}(B),\subseteq)$, the meet of any non-empty set of elements exists, and is equal to the intersection of those elements. The meet of the empty set of elements also exists (obviously). Finally, use Exercise 3.)

L4. Let V be a vector space (over any scalar field). Let $\mathrm{Sub}(V)$ be the set of all subspaces of V. $(\mathrm{Sub}(V), \subseteq)$ is a lattice. We have that, in this case,

$$\top = V$$
$$\bot = \{0\}$$
$$X\wedge Y = X\cap Y$$
$$X\vee Y = X+Y = \{x+y : x\in X, y\in Y\}$$

Exercise 6.1 Prove the assertions just made.

L5. Here is an important, but somewhat special, construction of a lattice.

Let us start with two fixed sets, M and N, and a relation T between them: $T\subseteq M\times N$ (thus, T is somewhat more general then our relations so far, since not one, but two "underlying sets" are involved). Let us use the variable U to denote subsets of M, V for subsets of N, a elements of M, x elements of N. Given any $U\subseteq M$, $V\subseteq N$, we define $U^{*}\subseteq N$, $V^{\#}\subseteq M$ by

$$x\in U^{*} \iff \forall a\in U.aTx$$
$$a\in V^{\#} \iff \forall x\in V.aTx$$

We let $A=\{U\subseteq M:U^{*\#}=U\}$. A is a subset of $\mathcal{P}(M)$. The induced order $(A,\subseteq)$ is a lattice, called the *concept lattice* derived from $T\subseteq M\times N$.

Exercise 7[*] Prove the last-stated assertion.

L6 One of the most important examples for a complete lattice is $([a,b],\leq)$, where a and b are fixed real numbers such that $a\leq b$, $[a,b]$ is the closed interval $\{x\in\mathbb{R}:a\leq x\leq b\}$, and $\leq$ is the usual less-than-or-equal relation (we should have written $\leq\restriction[a,b]$ instead of $\leq$, but this kind of abbreviation was already practiced before). The fact that $([a,b],\leq)$ is a complete lattice, for any a and b as described, is responsible for such fact as the existence of $\sqrt{2}$: $\sqrt{2}$ may be defined as $\sqrt{2}=\bigvee\{x\in[a,b]:x^2<2\}$, for any a and b such that $a^2<2$ and $b^2>2$.

Laws holding in all lattices:

$x\wedge y=y\wedge x$	$x\vee y=y\vee x$	(**commutative laws**)
$(x\wedge y)\wedge z=x\wedge(y\wedge z)$	$(x\vee y)\vee z=x\vee(y\vee z)$	(**associative laws**)
$x\vee(x\wedge y)=x$	$x\wedge(x\vee y)=x$	(**absorption laws**)
$x\wedge x=x$	$x\vee x=x$	(**idempotent laws**)
$\top\wedge x=x$ $\quad$ $\top\vee x=\top$	$\bot\wedge x=\bot$ $\quad$ $\bot\vee x=x$	(**identity laws**)

$$x \leq y \iff x \wedge y = x \iff x \vee y = y$$

$x \leq y$ and $u \leq v$ imply that $x \wedge u \leq y \wedge v$ and $x \vee u \leq y \vee v$

$$(x \vee y) \wedge z \geq (x \wedge z) \vee (y \wedge z)$$

$$(x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$$

$x \leq \top$ $\qquad$ $x \geq \bot$

$x \leq \bot \Rightarrow x = \bot$ $\qquad$ $x \geq \top \Rightarrow x = \top$

Exercise 8. Prove the laws.

Sublattice of a lattice

A *sublattice* of lattice (A, R) is a lattice (B, S) for which B is a subset of A, and for which the meanings of $\top$ (top), $\bot$ (bottom), and of $x \wedge y$, $x \vee y$ for elements x and y in the (smaller) set B, are the same in the two lattices. When (B, S) is a sublattice of (A, R), then, for $x, y \in B$, xRy iff xSy; but, this condition is not enough to ensure that (B, S) is a sublattice of (A, R).

At any rate, a sublattice (B, S) of (A, R) is completely determined by its underlying set B. However, if we take a subset B of A, define S by $xSy \iff xRy$ for all $x, y \in B$ (this order S on B is called the *order induced by* R *on* B), then (B, S) is not necessarily a sublattice of (A, R) *even if* (B, S) *is a lattice on its own right*.

Example 3 (continued) Now, (A, R) as in Example 3 above.

Let $B = (1, 2, 4, 7, 9\}$. Then the order induced by R on B is given by the Hasse diagram

[Figure 24]

Notice that in the Hasse diagram of (A, R), there is no arc from 2 to 7; although, of course, we have $2R7$. However, in the induced order S, it is not only the case that $2S7$, but we also have that 7 *covers* 2, since there is no element *in* B between 2 and 7. This is why we do have an arc from 2 to 7 in the Hasse diagram of (B, S).

Next, we verify that (B, S) is a lattice, and in fact, a sublattice of (A, R). There is only one incomparable pair, up to the order of mention: $(2, 4)$; and $2\vee 4=7$, $2\wedge 4=1$ in (B, S). When we check what the values of $2\vee 4$, $2\wedge 4$ were in (A, R), we see that these are 7 and 1, respectively. We can conclude that (B, S) is a sublattice of (A, R).

Next, let B be the subset $B=\{1, 3, 4, 8, 9\}$ of A. The induced order (B, S) now is given by the Hasse diagram

[Figure 25]

This is also a *lattice*. However, it is *not a sublattice of* (A, R), since in (B, S), $3\vee 4=8$, but in (A, S), $3\vee 4=6$.

Let B be a subset of A. When does B determine a sublattice of $(A; \leq)$; when is $(B, \leq\restriction B)$ a sublattice of $(A, \leq)$? For this,

$\top$, the top element of $(A, \leq)$, must belong to B;
$\bot$, the bottom element of $(A, \leq)$, must belong to B;
if x and y in B, then $x\wedge y$, the meet of x and y in the sense of the lattice $(A, \leq)$, must belong to B;
if x and y in B, then $x\vee y$, the join of x and y in the sense of the lattice $(A, \leq)$, must belong to B.

These conditions are also enough: if they hold true, then $(B, \leq\restriction B)$ is a lattice; in fact, $\top$ is the top element of $(B, \leq\restriction B)$; $\bot$ is the bottom element of $(B, \leq\restriction B)$; and if $x, y\in B$, then $x\wedge y$, $x\vee y$ are the meet and join, respectively, of $\{x, y\}$ in $(B, \leq\restriction B)$ as well.

Given any lattice $(A, \leq)$, and *any* subset X of A, we can form the sublattice $\langle X \rangle$ *of* $(A, \leq)$ *generated* by X. The underlying set B of $\langle X \rangle$ is the *least subset* B of A for which

X is a subset of B;
$\top$, the top element of (A, R), belongs to B;
$\perp$, the bottom element of (A, R), belongs to B;
for any x and y in B, both $x \wedge y$, $x \vee y$ computed in the given lattice (A, R) belong to B again.

The way to obtain $\langle X \rangle$ is to build B by (i) throwing in all the elements of X into B, (ii) throwing in $\top$ and $\perp$ into B, and (iii) every time we have x and y already in B for which $x \wedge y$ and/or $x \vee y$ is not yet in B, throwing $x \wedge y$ and/or $x \vee y$ into B -- until B is *closed under* said operations.

Example 3 (continued) We take (A, R) as in Example 3 again. Let $X = \{2, 4, 8\}$. We construct the underlying set B of

$$\langle X \rangle = \langle \{2, 4, 8\} \rangle = \langle 2, 4, 8 \rangle$$

by starting with the elements 2, 4, 8 and $1 = \perp$, $9 = \top$. We have $2 \vee 4 = 7$, therefore $7 \in B$. But then, since 7 and 8 are both in B, we must have that $7 \wedge 8 = 6$ belongs to B. When we take the set of the elements listed so far, $B = \{2, 4, 8, 1, 9, 7, 6\}$, we see that for each of the incomparable pairs of elements of B, which are $(2, 4)$, $(2, 6)$, $(2, 8)$ and $(7, 8)$, the join and the meet of the two elements of the pair is again an element of B; this can be checked by the listing for (A, R) given above. (Of course, when x and y are comparable elements of B, then $x \wedge y$, $x \vee y$, being elements of $\{x, y\}$, do already belong to B.) This ensures that B is closed under the lattice operations in (A, R), and we conclude that the underlying set of $\langle 2, 4, 8 \rangle$ is $\{2, 4, 8, 1, 9, 7, 6\}$.

The Hasse diagram of the lattice $\langle 2, 4, 8 \rangle$ is

[Figure 26]

Let us return to the general situation. When $\langle X \rangle$ equals the whole lattice $(A, \leq)$, that is, $B=A$, we say that X *generates* the lattice $(A, \leq)$, or, that X is a *set of generators for* the lattice $(A, \leq)$.

In the example, the only elements of $(A, \leq)$ missing from $\langle 2, 4, 8 \rangle$ are 3 and 5. Adding either of these to the set $\{2, 4, 8\}$ results in a set of generators for the lattice. For instance, $\{2, 3, 4, 8\}$ is a set of generators for $(A, \leq)$, $\langle 2, 3, 4, 8 \rangle = (A, \leq)$. The reason is that $2 \vee 3 = 5$, and thus all elements of A are in the underlying set of $\langle 2, 3, 4, 8 \rangle$.

Consider the **distributive identity**:

$$(x \vee y) \wedge z \overset{?}{=} (x \wedge z) \vee (y \wedge z)$$

This may or may not hold in a lattice. A lattice is said to be *distributive* if the distributive identity holds for all values of the variables x, y, z.

Example 3 (continued) This lattice is distributive. It is not easy to check this directly, since the number of triples (x, y, z) to check is quite large. However, there is a characterization, given below, of distributivity that makes the verification somewhat easier.

The most important examples for distributive lattices are the power-set lattices. $(\mathcal{P}(B), \subseteq)$ is distributive, since the distributive identity holds for union and intersection of sets:

$$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z) .$$

This was proved in Section 1.2.

Exercise 9. Prove that the inequality $(x \vee y) \wedge z \geq (x \wedge z) \vee (y \wedge z)$ always holds in any lattice.

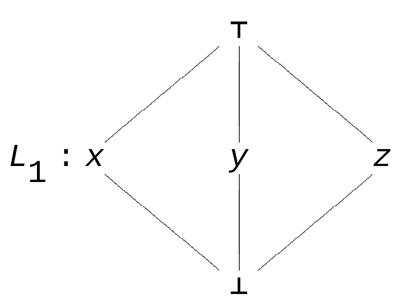
We need two auxiliary concepts.

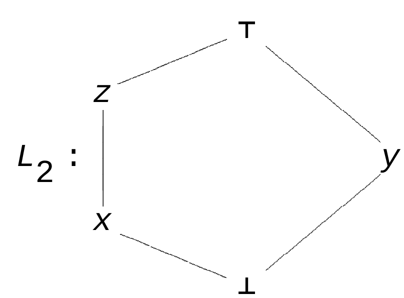
Let us say that the lattice (B, S) is a *weak sublattice* of (A, R) if all the conditions for (B, S) being a sublattice of (A, R) are satisfied except possibly the ones for "top" and "bottom"; that is, it may happen that $\top \notin B$ or $\bot \notin B$.

Example 3 (continued) For instance, $B=\{1, 3, 4, 6\}$ is the underlying set of a weak sublattice (B, S) of the lattice of Example 3, because $3\wedge 4=1$ and $3\vee 4=6$ in both (A, R) and (B, S) , and the only pair of incomparables in B is $(3, 4)$. However, (B, S) is not an (ordinary) sublattice of (A, R) , since the top element of (B, S) is $6\neq 9$.

Recall that isomorphic relations share all "mathematical" properties. For instance, one is a lattice if and only if the other is; one is distributive if and only if the other is.

To return to distributivity, here are two particular non-distributive lattices:





In the first lattice, $(x\vee y)\wedge z = z$ and $(x\wedge z)\vee(y\wedge z) = \bot$; in the second case, $(x\vee y)\wedge z = z$ and $(x\wedge z)\vee(y\wedge z) = x$.

It follows that if either of L_1 , L_2 is *isomorphic* to a weak sublattice of a lattice $(A, \leq)$, the latter cannot be distributive: one or the other of the above counterexamples to distributivity will be present in $(A, \leq)$. It is a **theorem** that conversely, if $(A, \leq)$ is a non-distributive lattice, then either L_1 or L_2 is *isomorphic* to a weak sublattice of $(A, \leq)$.

Example 3 (continued) It is not too difficult to see that in this example, neither L_1 nor L_2 is isomorphic to a weak sublattice of $(A, \leq)$. Therefore, $(A, \leq)$ is distributive. However, when we modify the example by removing the arc $(6, 7)$ from the Hasse diagram, the resulting lattice (A, R') contains the weak sublattice (B, S) with $B=\{3, 5, 6, 7, 9\}$, which is isomorphic to L_2; therefore, the modified lattice is not distributive. In fact, in (A, R'), $(5 \vee 8) \wedge 7 \neq (5 \wedge 7) \vee (8 \wedge 7)$.