Chapter 1 Sets and functions

Section 1.1 Sets

The concept of set is a very basic one. It is simple; yet, it suffices as the basis on which all abstract notions in mathematics can be built.

A set is *determined by its elements*.

If A is a set, we write $x \in A$ to say that x is an *element of* A. Other readings for "$x \in A$" are: "x *belongs to* the set A", "x is *in* A".

Anything may be an element of a set; any two, possibly unrelated, things may be elements of the same set. In fact, any way of collecting things into a whole results in a set; the things collected are the *elements* of the set.

To say that a set is *determined* by its elements is to say that any set is completely given by specifying what its elements are. We may express this in the following mathematical style:

Principle of extensionality. Two sets A and B are equal if for all things x, x is an element of A if and only if x is an element of B.

We will frequently use the following logical abbreviations:

$\forall x$: "for all x ...".
$\exists x$: "there is x such that ..."
$\longleftrightarrow$: "if and only if"
$\Longrightarrow$: "if ..., then ..."
& : "... and..."

Principle of Extensionality, in abbreviated form: For sets A and B,

$$\forall x(x \in A \longleftrightarrow x \in B) \Longrightarrow A = B .$$

A set may be given by *listing* its elements. A list enclosed in curly brackets denotes the set whose elements are the things in the list. E.g.,

$$\{0,\ 2,\ 142,\ 96,\ 3\} \tag{1}$$

denotes the set whose elements are the five integers listed. The order in which the elements are listed is immaterial. Thus,

$$\{2,\ 142,\ 0,\ 3,\ 96\}$$

is a notation for the same set as (1). Also, if a list contains repetitions, when enclosed in curly brackets, it will denote the same set as the list with the repetitions removed. E.g.,

$$\{2,\ 142,\ 0,\ 0,\ 3,\ 96,\ 3\}$$

still denotes the same set as the previous two notations.

There are sets that cannot be listed; they are *infinite*. E.g., the set of all non-negative integers, or natural numbers, denoted by $\mathbb{N}$, is such an infinite set. We may write

$$\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ \ldots\} ,$$

but this is a very incomplete notation! Other important sets of numbers are as follows:

$$\mathbb{Z} = \{0,\ 1,\ -1,\ 2,\ -2,\ \ldots\} ,$$

the set of all (positive, negative and zero) integers;

$$\mathbb{Q} = \text{the set of all rational numbers}$$

(a number is *rational* if it is of the form $\frac{p}{q}$ for some integers p and q ($q \neq 0$));

$$\mathbb{R} = \text{the set of all real numbers}$$

and

$$\mathbb{C} = \text{the set of all complex numbers}$$

($\sqrt{2}$, π and e are real numbers, but they are not rational; $\mathtt{i} = \sqrt{-1}$ is a complex number which is not real).

A set may be specified by giving a property or condition that its elements, and only its elements, have or satisfy; the elements of the set are exactly the things that have the property (satisfy the condition) in question. In fact, the five number-sets just introduced are given in this way. E.g., any thing x is a member of $\mathbb{Q}$ if and only if it has the property that there are integers p and q , $q \neq 0$, such that $x = \frac{p}{q}$. By using arbitrary conditions, we may define an endless variety of sets.

The curly brackets are also used for specifying a set by a condition. E.g.,

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} ,\ q \neq 0\} ,$$

$$\mathbb{C} = \{x+y\sqrt{-1} : x, y \in \mathbb{R}\} .$$

In other sources, the colon is replaced by a vertical line:

$$\mathbb{C} = \{x+y\sqrt{-1} \mid x, y \in \mathbb{R}\} .$$

In each of these formulas, in front of the colon (:), one finds an expression denoting a quantity depending on certain variables; in the first case, these variables are p and q , in the second x and y . The complete symbol denotes the set of all values of the expression when the variables range over all values satisfying the condition stated after the colon. Thus, one should read the first formula as "the set of all $\frac{p}{q}$ such that p and q are integers and $q \neq 0$ ".

A set given in the form of a list may be specifiable more conveniently by a condition. E.g., the set

$$\{0,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24,\ 26,\ 28,\ 30,\ 32\}$$

is the same as the set

$$\{n\in\mathbb{N} \mid n \text{ is even and } n < 34\}\ .$$

This latter symbolic expression should be read as:

"the set of all n *in* $\mathbb{N}$ such that n is even and $n<34$ ".

Note the use of the symbol " $\in$ " in this set-notation.

Differently worded conditions may give rise to the same set. E.g., the following two brackets define the same set:

$$\{n\in\mathbb{N} \mid n \text{ is a prime number and } 11 \le n \le 120\} \tag{2}$$

$$\{n\in\mathbb{N} \mid n \text{ is not divisible by } 2,\ 3,\ 5 \text{ and } 7\text{ , and } 1<n\le 120\} \tag{3}$$

(can you prove that these last sets are the same?).

It is a fundamental point that sets themselves may be elements of other sets. E.g., the set

$$A = \{\{1, 2\},\ \{4, 7\}\} \tag{4}$$

has two elements, $\{1, 2\}$ and $\{4, 7\}$, both of which are sets themselves. In axiomatic set theory, mathematical objects are constructed as sets of sets of sets ... with "arbitrary" complexity.

Note that the set

$$B = \{1,\ 2,\ 4,\ 7\}$$

is something very different from A in (4); whereas the elements of A are sets, the elements of B are numbers, not sets. (Unlike in the usual versions of axiomatic set theory, we do not identify numbers with sets. Foundationally, our set theory has "urelements", non-sets with no internal structure; natural numbers are such "ur-elements"). Also note that the set $\{\{7, 4\}, \{2, 1\}\}$ is equal to A, but the set $\{\{1, 4\}, \{2, 7\}\}$ is not (why?).

One specific set, the *empty set*, has to be pointed out. The empty set is the set which has no elements; it is given by any condition that is contradictory, that is, a condition that has nothing to satisfy it. E.g., the set

$$\{n\in\mathbb{N} \mid n^2+1 \text{ is divisible by } 4\}$$

is empty, since there is no natural number n for which n^2+1 is divisible by 4. (If n is even, $n = 2k$, then $n^2+1 = 4k^2+1$, which is not divisible by 4 (gives the remainder 1 when divided by 4); if n is odd, $n = 2k+1$, then $n^2+1 = 4(k^2+k)+2$, which is not divisible by 4 either.)

There is just one empty set; if both A and B are empty, then for any x, $x \in A$ just in case $x \in B$, namely never; hence, $A = B$ by the principle of extensionality. The symbol for the empty set is $\emptyset$.

Let us state the general principle behind the curly-bracket notation.

Principle of Comprehension The set

$$\{x\colon P(x)\}$$

that is, the set whose elements are those, and only those, x that have property $P(x)$, *exists*.

Sadly, there are some exceptions to the validity of the principle of comprehension. Notably, the *vacuous* condition $P(x)$ that is identically true (which can be represented by the expression $x=x$, since everything is equal to itself) cannot be used in the principle. It would give rise to the set $\{x\colon x=x\}$, that is, the set of all things; and the set of all things does not exist; it gives rise to the famous paradoxes (contradictions) of set theory.

Unfortunately, it is quite difficult to describe precisely the extent of the validity of the principle of comprehension; this is done in the discipline called *axiomatic set theory*. For us, it should suffice to say that *normally*, the principle of comprehension is valid; the exceptions are rare.

Note the following consequences of the meaning of the term $\{x\colon P(x)\}$:

If $P(x)$, then $x \in \{x\colon P(x)\}$;
If not $P(x)$, then $x \notin \{x\colon P(x)\}$;
$x \in \{x\colon P(x)\}$ if and only if $P(x)$.

We use the logical abbreviation $\neg$ for "not"; we read $\neg P(x)$ as: "it is not the case that $P(x)$ "; briefly: $\mathtt{not}(P(x))$.

Using logical abbreviations, we have:

$$P(x) \implies x \in \{x\colon P(x)\}$$

$$\neg P(x) \implies x \notin \{x\colon P(x)\}$$

$$x \in \{x\colon P(x)\} \iff P(x) .$$

Variants of the comprehension notation can be explained thus:

$$\{x{\in}X\colon P(x)\} \underset{\mathrm{def}}{=} \{x\colon x{\in}X \;\&\; P(x)\}$$

$$\{f(x)\colon P(x)\} \underset{\mathrm{def}}{=} \{y\colon \exists x.y{=}f(x) \;\&\; P(x).\} .$$