Appendix 1

Syntax of FOLDS

classes: KIND

CONTEXT

SORT

SPEC (for SPECIALIZATION)

VAR (for VARIABLE)

such that

$$\mathcal{X} \in \mathrm{CONTEXT} \Rightarrow \mathcal{X} \subset \mathrm{VAR}, \ \mathcal{X} \text{ finite}$$

$$s \in \mathrm{SPEC} \Rightarrow s \text{ is a function; } \mathrm{dom}(s), \mathrm{ran}(s) \in \mathrm{CONTEXT}$$

predicates:

$$\overset{\circ}{\circ} \ \subset \mathrm{KIND} \times \mathrm{CONTEXT}$$

(read $K \overset{\circ}{\circ} \mathcal{Y}$: " K is of arity $\mathcal{Y}$ ")

$$: \ \subset \mathrm{VAR} \times \mathrm{SORT}$$

(read $x : X$: " x is of sort X ")

function:

$$\mathrm{var} : \mathrm{SORT} \longrightarrow \mathrm{CONTEXT}$$

$\boxed{\text{inductive clauses}}$:

1 $\quad \emptyset \in \mathrm{CONTEXT}$.

2 $\quad \mathfrak{X} \in \mathrm{CONTEXT} \Rightarrow \langle 0, \mathfrak{X}, a\rangle \in \mathrm{KIND}$,

$\langle 0, \mathfrak{X}, a\rangle \circ\!\!\circ\, \mathfrak{X}$.

[comment: a is anything]

3 $\quad \mathfrak{X} \in \mathrm{CONTEXT},\ X \in \mathrm{SORT},\ x \in \mathrm{VAR}$,

$x : X,\ \mathrm{var}(X) \subset \mathfrak{X}$

$\Longrightarrow$

$\mathfrak{X} \cup \{x\} \in \mathrm{CONTEXT}$.

4 $\quad s \in \mathrm{SPEC},\ K \in \mathrm{KIND},\ K \circ\!\!\circ\, \mathrm{dom}(s)$

$\Longrightarrow$

$K(s)$ notation $\underset{\text{def}}{=}$ $\langle 1, K, s\rangle \in \mathrm{SORT},\ \mathrm{var}(\langle 1, K, s\rangle) = \mathrm{ran}(s)$.

5 $\quad \emptyset \in \mathrm{SPEC}$.

6 $\quad s \in \mathrm{SPEC},\ X \in \mathrm{SORT},\ x \in \mathrm{VAR},\ x : X$,

$x \notin \mathrm{dom}(s),\ y : X|s$

$\Longrightarrow s \cup \{(x, y)\} \in \mathrm{SPEC}$.

[$X|s \underset{\text{def}}{=} X$ specialized to s; $\langle 1, K, t\rangle|s \underset{\text{def}}{=} \langle 1, K, s\circ t\rangle$]

7 $\quad X \in \mathrm{SORT} \Rightarrow \langle 2, X, a\rangle \in \mathrm{VAR},\ \langle 2, X, a\rangle : X$.

clauses applied — Examples:

1 $\emptyset \in \mathrm{CONTEXT}$

2 $0 \underset{def}{=} \langle 0, \emptyset, \underline{0} \rangle \in \mathrm{KIND}$, $\boxed{0 \mathrel{\substack{\circ\\\circ}} \emptyset}$

5 $\emptyset \in \mathrm{SPEC}$

4 $0 \underset{def}{=} \langle 1, 0, \emptyset \rangle \in \mathrm{SORT}$, $var(0) = \emptyset$

7 $u \underset{def}{=} \langle 2, 0, \underline{u} \rangle \in \mathrm{VAR}$, $\boxed{u : 0}$

$\boxed{v : 0}$, $\boxed{w : 0}$

3 $\{u, v\} \in \mathrm{CONTEXT}$

6 $(u \mapsto u, v \mapsto v) \in \mathrm{SPEC}$, $(u \mapsto u, v \mapsto u), (u \mapsto v, v \mapsto w), \dots \in \mathrm{SPEC}$

2 $A \underset{def}{=} \langle 0, \{u, v\}, \underline{A} \rangle \in \mathrm{KIND}$, $\boxed{A \mathrel{\substack{\circ\\\circ}} \{u, v\}}$

4 $A(u, v) \underset{def}{=} \langle 1, A, (u \mapsto u, v \mapsto v) \rangle \in \mathrm{SORT}$

$A(u, u) \underset{def}{=} \langle 1, A, (u \mapsto u, v \mapsto u) \rangle \in \mathrm{SORT}$

$A(u, w), A(v, w), \dots \in \mathrm{SORT}$

7 $f \underset{def}{=} \langle 2, A(u, v), \underline{f} \rangle \in \mathrm{VAR}$, $\boxed{f : A(u, v)}$

$\boxed{g : A(v, w)}$, $\boxed{h : A(u, w)}$, $\boxed{a : A(u, v)}$, $\boxed{b : A(v, u)}$

$\boxed{i : A(u, u)}$, $\boxed{j : A(v, v)}$

3 $\{u, v, w, f, g, h\} \in \mathrm{CONTEXT}$

$\boxed{T \mathrel{\substack{\circ\\\circ}} \{u, v, w, f, g, h\}}$

6 $(u \mapsto u, v \mapsto v, w \mapsto u, f \mapsto a, g \mapsto b, h \mapsto i) \in \mathrm{SPEC}$

4 $T(u, v, u, a, b, i)$, $T(v, u, v, b, a, j) \in \mathrm{SORT}$

3 $\boxed{\tau : T(u, v, u, a, b, i)}$, $\boxed{\sigma : T(v, u, v, b, a, j)}$

pictorially:

$$\begin{tikzcd} U \arrow[loop left, "i"] \arrow[r, shift left, "a"] & V \arrow[l, shift left, "b"] \arrow[loop right, "j"] \end{tikzcd}$$

$$\begin{tikzcd} & V \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow \tau"] & \\ U \arrow[ur, "a"] \arrow[rr, "i"'] & {} & U \end{tikzcd} \qquad \begin{tikzcd} & U \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow \sigma"] & \\ V \arrow[ur, "b"] \arrow[rr, "j"'] & {} & V \end{tikzcd}$$

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A <u>signature</u> is a set L of kinds such that

$$K \in L, \quad K \mathrel{\substack{\circ \\ \circ}} \mathfrak{X}, \quad \underbrace{x \in \mathfrak{X}}, \quad x : K'(s) \; (= \langle 1, K', s \rangle)$$
$$\Rightarrow K' \in L.$$

L (is) becomes a <u>category</u> whose <u>proper</u> (non-identity) arrows are:

$$K \xrightarrow[p_x^K]{} K' \quad : \quad K \in L, \; K \mathrel{\substack{\circ \\ \circ}} \mathfrak{X}, \; x \in \mathfrak{X}, \; \underbrace{x : K'(s)}$$

Composition:

$$\begin{tikzcd} K \arrow[r, "p_x^K"] \arrow[rr, bend right, "p^K_{\boxed{z}}"'] & \underset{\circ}{K'} \arrow[r, "p_y^{K'}"] & K'' \end{tikzcd}$$

$$K' \mathrel{\substack{\circ \\ \circ}} \mathfrak{X}', \quad y \in \mathfrak{X}', \quad s : \mathfrak{X}' \to \mathrm{var}(K'(s)) \subset \mathfrak{X} \; ; \quad \boxed{z \underset{\text{def}}{=} s(y)}$$

We obtain a category L which

1) <u>has finite fan-out</u>:

for any $K \in L$,

$$\mathrm{af}(K) \underset{\mathrm{def}}{=} \{p \in \mathrm{Arr}(L) : \mathrm{dom}(p) = K,\ p \neq 1_K\}$$

is a finite set;

2) is <u>reverse</u> - <u>well-founded</u> : there is no infinite ('ascending') sequence:

$$K_1 \xrightarrow[p_1]{} K_2 \xrightarrow[p_2]{} K_3 \to \cdots \to K_n \xrightarrow[p_n]{} K_{n+1} \to \cdots$$

Conversely, every such category corresponds to, or *is*, a signature.