

Syntax of FOLDS

classes: KIND

CONTEXT

SORT

SPEC (for SPECIALIZATION)

VAR (for VARIABLE)

such that

$X \in \text{CONTEXT} \Rightarrow X \subset \text{VAR}, X \text{ finite}$

$s \in \text{SPEC} \Rightarrow s \text{ is a function; } \text{dom}(s), \text{ran}(s) \in \text{CONTEXT}$

predicates:

$\circ \subset \text{KIND} \times \text{CONTEXT}$

(read $K \circ y$: "K is of arity y")

$:$ $\subset \text{VAR} \times \text{SORT}$

(read $x:X$: "x is of sort X")

function:

$\text{var}: \text{SORT} \longrightarrow \text{CONTEXT}$

inductive clauses:

- 1 $\emptyset \in \text{CONTEXT.}$
- 2 $X \in \text{CONTEXT} \Rightarrow \langle 0, X, a \rangle \in \text{KIND},$
 $\langle 0, X, a \rangle \circ X.$

[comment: a is anything]

- 3 $X \in \text{CONTEXT}, X \in \text{SORT}, x \in \text{VAR},$
 $x : X, \text{var}(X) \subset X$
 $\Rightarrow X \cup \{x\} \in \text{CONTEXT.}$

- 4 $s \in \text{SPEC}, K \in \text{KIND}, K \circ \text{dom}(s)$

$K(s)$

$\Rightarrow$

notation $\nearrow$ def $\langle 1, K, s \rangle \in \text{SORT}, \text{var}(\langle 1, K, s \rangle) = \text{ran}(s).$

- 5 $\emptyset \in \text{SPEC.}$

- 6 $s \in \text{SPEC}, X \in \text{SORT}, x \in \text{VAR}, x : X,$
 $x \notin \text{dom}(s), y : X|s$

$\Rightarrow s \cup \{(x, y)\} \in \text{SPEC.}$

[$X|s \stackrel{\text{def}}{=} X$ specialized to s ; $\langle 1, K, t \rangle|s \stackrel{\text{def}}{=} \langle 1, K, \text{sort} \rangle$]

- 7 $X \in \text{SORT} \Rightarrow \langle 2, X, a \rangle \in \text{VAR}, \langle 2, X, a \rangle : X.$

clauses applied

Examples:

3

1

$\emptyset \in \text{CONTEXT}$

2

$O \stackrel{\text{def}}{=} \langle 0, \emptyset, \underline{O} \rangle \in \text{KIND}$

$O \circ \emptyset$

5

$\emptyset \in \text{SPEC}$

4

$O \stackrel{\text{def}}{=} \langle 1, O, \emptyset \rangle \in \text{SORT}$

$\text{var}(O) = \emptyset$

7

$U \stackrel{\text{def}}{=} \langle 2, O, \underline{U} \rangle \in \text{VAR}$

$U: O$

$V: O$

$W: O$

3

$\{U, V\} \in \text{CONTEXT}$

6

$(U \mapsto U, V \mapsto V) \in \text{SPEC}, (U \mapsto U, V \mapsto U), (U \mapsto V, V \mapsto W), \dots \in \text{SPEC}$

2

$A \stackrel{\text{def}}{=} \langle 0, \{U, V\}, \underline{A} \rangle \in \text{KIND}$

$A \circ \{U, V\}$

4

$A(U, V) \stackrel{\text{def}}{=} \langle 1, A, (U \mapsto U, V \mapsto V) \rangle \in \text{SORT}$

$A(U, U) \stackrel{\text{def}}{=} \langle 1, A, (U \mapsto U, V \mapsto U) \rangle \in \text{SORT}$

$A(U, W), A(V, W), \dots \in \text{SORT}$

7

$f \stackrel{\text{def}}{=} \langle 2, A(U, V), \underline{f} \rangle \in \text{VAR}$

$f: A(U, V)$

$g: A(V, W)$

$h: A(U, W)$

$a: A(U, V)$

$b: A(V, U)$

$i: A(U, U)$

$j: A(V, V)$

3

$\{U, V, W, f, g, h\} \in \text{CONTEXT}$

$T \circ \{U, V, W, f, g, h\}$

6

$(U \mapsto U, V \mapsto V, W \mapsto U, f \mapsto a, g \mapsto b, h \mapsto i) \in \text{SPEC}$

4

$T(U, V, U, a, b, i), T(V, U, V, b, a, j) \in \text{SORT}$

3

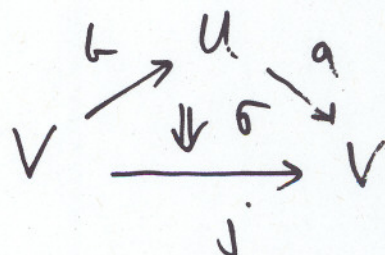
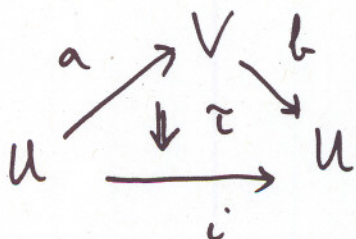
$\tau: T(U, V, U, a, b, i)$

$\sigma: T(V, U, V, b, a, j)$

picturelly:

4

$$i \hookrightarrow U \xrightleftharpoons[b]{a} V \hookrightarrow j$$



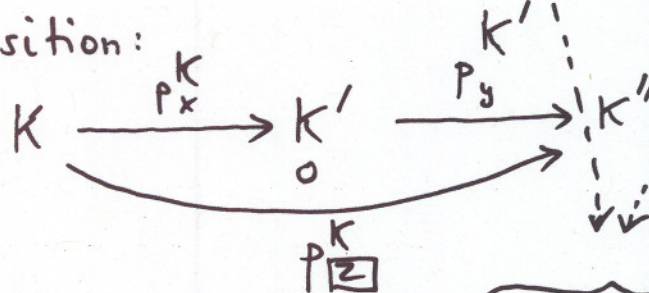
A signature is a set L of kinds such that

$$K \in L, K \circ x, x \in \mathcal{X}, x: K'(s) (= \langle I, K', s \rangle) \Rightarrow K' \in L.$$

L becomes (is) a category whose proper (non-identity) arrows are:

$$K \xrightarrow[p_x^K]{\quad} K' \quad : K \in L, K \circ x, x \in \mathcal{X} \quad \underbrace{x: K'(s)}$$

Composition:



$$K' \circ x', y \in \mathcal{X}', s: \mathcal{X}' \rightarrow \text{var}(K'(s)) \subset \mathcal{X}; \boxed{z =_{\text{def}} s(y)}$$

We obtain a category L which

1) has finite fan-out:

for any $K \in L$,

$$\text{ar}(K) \stackrel{\text{def}}{=} \{p \in \text{Arr}(L) : \text{dom}(p) = K, p \neq 1_K\}$$

is a finite set;

2) is reverse-well-founded : there is no infinite 'ascending' sequence:

$$K_1 \xrightarrow{p_1} K_2 \xrightarrow{p_2} K_3 \rightarrow \dots \rightarrow K_n \xrightarrow{p_n} K_{n+1} \rightarrow \dots$$

Conversely, every such category corresponds to, or is, a signature.