"generic" case, from instances of the associativity isomorphisms and other canonical isomorphisms should commute. It is an important *coherence theorem* (due to S. Mac Lane, formulated for the one-object case, that is, for monoidal categories; see [ML]) that this stronger condition is a consequence of the official definition, which is thus nothing but a finite (equational) axiomatization of the totality of all coherence conditions.

This suggests that, possibly, the right approach to the definition of weak n-category is to aim at formulating all coherence conditions at once, regardless the fact that this might give a very "theoretical" definition. It would then be a separate, and still very important, project to find a (hopefully) finite and concise set of coherence conditions that would be enough to imply all coherence conditions.

J. Baez and J. Dolan have produced a very interesting definition of "weak n-category" [BD2]. The similarity type of the Baez/Dolan n-category is radically larger than the usual one. In the usual n-categories, the domain and codomain of a $k+1$-cell are both k- cells; in a Baez/Dolan n-category, there are $k+1$-cells whose domains are arbitrary "pasting diagrams" of k-cells, not the composites of the latter. In addition, the coherence conditions are replaced, and the combinatorial complexity eliminated, by the requirement of the existence of cells satisfying certain universal properties (this phenomenon is well-known from Grothendieck's concept of fibration, relative to the concept of pseudo-functor; see [SGA1]).

The Baez/Dolan definition for $n = 2$ can be shown to be closely related to the concept of "saturated anabicategory" mentioned above. However, Baez and Dolan do not use the ana-versions of morphisms, in particular, of "functor".

As I mentioned in the introduction, the Baez/Dolan definition is only a beginning, albeit a very promising one. Among others, the definition of the Baez/Dolan $n+1$-category of all n-categories is still missing. I believe that thinking in the spirit of anafunctors will help give the latter definition.

8. Dependent types

Dependent types are familiar from [M-L] and [C]; there is a further extensive literature of them. Their use for first-order logic seems to be new, however. First Order Logic with Dependent Sorts (FOLDS; I use "sort" in place of "type") is a variant of ordinary First Order Logic (FOL); the basic metatheory of FOLDS is an extension (generalization) of that of FOL.

A similarity type, or vocabulary, for FOLDS is a *one-way category*; a small category $\mathbf{L}$ is one-way if it has the following two properties:

(i) $\mathbf{L}$ *has finite fan-out*: for any $K \in \mathbf{L}$, there are only finitely many arrows with domain equal to K.

(ii) **L** *is reverse well-founded*: there is no infinite sequence $<K_n \xrightarrow{f_n} K_{n+1}>_{n\in N}$ of composable proper (non-identity) arrows $(f_n \neq 1_{K_n})$.

Immediate consequences of the definition are the following:

(iii) **L** is skeletal: any two isomorphic objects are identical;

(iv) the only arrow from an object to itself is the identity;

and in fact, as a consequence of (iii) and (iv),

(v) **L** has no circuit of positive finite length consisting of proper arrows; there is no $<K_n \xrightarrow{f_n} K_{n+1}>_{n<N}$ with $N \in N$, $K_N = K_0$, and $f_n \neq 1_{K_n} (n < N)$.

It is easy to see that a small category is one-way iff it satisfies (i) and (v); and if the category is finite, then it is one-way iff it satisfies (iii) and (iv).

If **L** is a one-way category, the set $Ob(\mathbf{L})$ of objects is partitioned as in

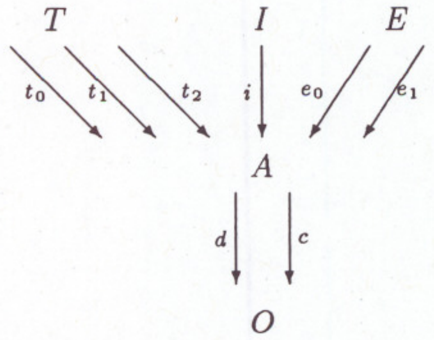
$$Ob(\mathbf{L}) = \bigcup_{i<\ell} \mathbf{L}_i$$

into non-empty *levels* $\mathbf{L}_i$, for $i < \ell$, ℓ the *height* of **L**, $\ell \leq \omega$, such that $\mathbf{L}_0$ consist of the objects A for which there is no proper arrow with domain A, and such that, for $i > 0$, $\mathbf{L}_i$ consists of those objects A for which all proper arrows $A \to B$ have $B \in \mathbf{L}_{<i} = \bigcup_{j<i} \mathbf{L}_j$, and there is at least one proper arrow $A \to B$ with $B \in \mathbf{L}_{i-1}$. All proper arrows go from a level to a lower level. Of course, the height of a finite one-way category is finite.

For **L** a one-way category, an **L**-*structure* is a functor $\mathbf{L} \to Set$.

There is a way of introducing FOLDS that follows Martin-Löf's idea of dependent types closely, without any preconceptions about similarity types; this is then seen to lead precisely to the notion of similarity type we just introduced without prior motivation (see [M3]).

The recognition of role of finite one-way categories in syntax is due to F. W. Lawvere, who pointed out their role in connection with the sketch-syntax of [M1]. Their role in FOLDS is related to their role in the sketch-syntax. I should note that the use of infinite one-way categories, in the sense used here, is essential (see below). Here is an example for a FOLDS similarity type, called $\mathbf{L}_{cat}$:



$$dt_0 = dt_2,\ ct_0 = dt_1,\ ct_1 = ct_2$$
$$di = ci$$
$$de_0 = de_1,\ ce_0 = ce_1$$

The proper arrows of $\mathbf{L}_{cat}$ are the arrows shown, and their composites; the equalities shown identify some of these composites.

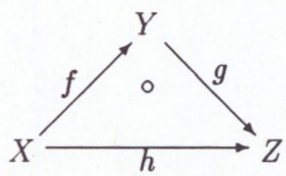
Note that any small category $\mathbf{C}$ gives rise to an $\mathbf{L}_{cat}$-structure

$$M[\mathbf{C}] = M : \mathbf{L}_{cat} \to Set$$

in a natural way. $M(O)$ is the set of objects of $\mathbf{C}$; $M(A)$ is the set of arrows; $M(d)$ and $M(c)$ map an arrow to its domain and codomain, respectively. $M(T)$ is the set of commutative triangles, i.e., tuples

$$(X, Y, Z, f : X \to Y, g : Y \to Z, h : X \to Z)$$

such that $h = gf$. In other words, the elements of $M(T)$ are commutative diagrams of the form



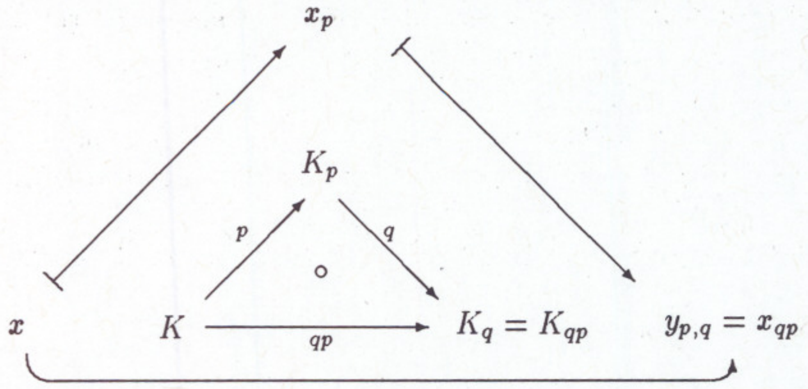
(1)

$M(I)$ is the set of identity arrows, $M(E)$ is the set of pairs (f, f) of equal arrows. With τ standing for (1), $M(t_1)(\tau) = f, M(t_2)(\tau) = g, M(t_3)(\tau) = f$. The rest of the definition of M should be clear. Note that M is indeed a functor.

Of course, not every $\mathbf{L}_{cat}$-structure is a category. On the other hand, for two categories construed as $\mathbf{L}_{cat}$-structures, a natural transformation from one to the other is precisely the same as a functor from one category to the other.

Let $\mathbf{L}$ be a one-way category; we fix $\mathbf{L}$ for a while. The objects of $\mathbf{L}$ are called *kinds*. Let us write K_p for $dom(p)(p \in Arr(\mathbf{L}))$. We use the notation $K|\mathbf{L}$ for the set of all proper arrows $p : K \to K_p$ with domain K. The set $K|\mathbf{L}$ will figure as the *arity* of the symbol K. In particular, the ones with empty arity are exactly the level-0 kinds.

We are going to define what *sorts* are, and what *variables of* a given sort are; we will write $x : X$ to denote that the variable x is of sort X. Every sort will be of the form $K(<x_p>_{p \in K|\mathbf{L}})$, with K a kind (the displayed sort is then said to be *of* the kind K), with variables x_p indexed by the elements p of the arity $K|\mathbf{L}$ of K; additional conditions will have to be satisfied. Let $k \in N$, and suppose we have defined sorts of kinds on levels less than k, and variables of such sorts. Then, for a kind K on level k, $K(<x_p>_{p\in K|\mathbf{L}})$ is a sort (of the kind K) iff for each $p \in K|\mathbf{L}$, we have $x_p : K_p(<y_{p,q}>_{q\in K_p|\mathbf{L}})$ with $p : K \to K_p$ (note that K_p is on a lower level than K), and for every $q \in K_p|\mathbf{L}$, $y_{p,q} = x_{qp}$:



For every sort $X = K(<x_p>_{p\in K|\mathbf{L}})$ thus specified, we declare certain symbols as variables of sort X; the only important things about this declaration are that (i) variables of sort X have to be new, so that every variable uniquely determines its own sort, and that (ii) there are enough (infinitely many) variables of each sort.

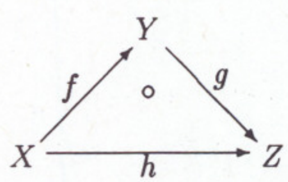
Note that every variable "carries" its own sort with it. This is in contrast with the practice of most of the relevant literature (see e.g. [C]), where variables are "locally" declared to be of certain definite sorts, but by themselves, they do not carry sort information. For a sort

$$X = K(<x_p>_{p\in K|\mathbf{K}}),\ Var(X) \underset{def}{=} \{x_p : p \in K|\mathbf{K}\};$$

and if $x : X$, $Dep(x) \underset{def}{=} Var(X)$; x *depends on* the variables in $Dep(x)$.

Let $\mathbf{L} = \mathbf{L}_{cat}$. Since $O|\mathbf{L}$ is empty, $O(\emptyset)$, with $\emptyset$ the empty sequence of variables, is a sort; we write simply O. We have $A|\mathbf{L} = \{d, c\}$; the sorts of the kind A are of the form $A(X, Y)$, with $X, Y : O$; here, X is indexed by d, Y by c.

$T|\mathbf{L} = \{f_1, f_2, f_3, t_1, t_2, t_3\}$, with $f_1 = dt_1 = dt_3$, $f_2 = ct_1 = dt_2$, $f_3 = ct_2 = ct_3$. A sort of the kind T will have the form $T(X, Y, Z, f, g, h)$; applying the condition in the definition with $p = t_1$, we get that $f : A(X, Y)$ must be the case. In summary, the variables in $T(X, Y, Z, f, g, h)$ have to line up as in



We define *formulas* φ and the set $Var(\varphi)$ of the *free variables of* φ by a simultaneous induction.

The symbols $\mathbf{t}$ ("true"), $\mathbf{f}$ ("false") are formulas; $Var(\mathbf{t}) = Var(\mathbf{f}) = \emptyset$.

The sentential connectives $\wedge, \vee, \rightarrow, \neg, \leftrightarrow$ can be applied in an unlimited manner; $Var(\ )$ for the compound formulas formed using connectives is defined in the expected way; e.g., $Var(\varphi \wedge \psi) = Var(\varphi) \cup Var(\psi)$.

Suppose φ is a formula, x is a variable *such that there is no* $y \in Var(\varphi)$ *with* $x \in Dep(y)$. Suppose $x : X$. Then $\forall x : X.\varphi$, $\exists x : X.\varphi$ are formulas;

$$Var(\forall x : X.\varphi) \underset{def}{=} Var(\exists x : X.\varphi) \underset{def}{=} (Var(\varphi) - \{x\}) \cup Dep(x).$$

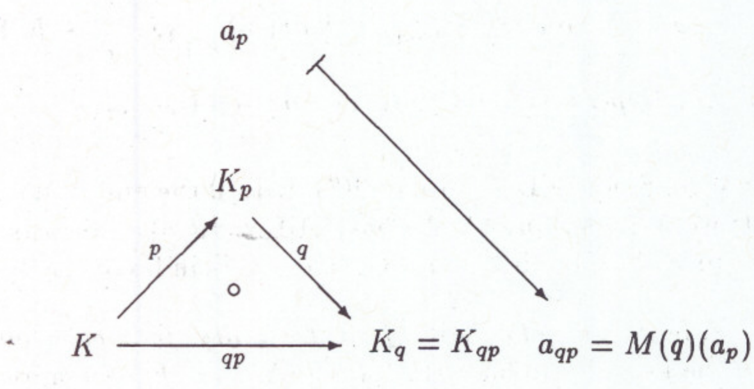
All formulas are obtained as described.

Here is an example of a sentence (formula without free variables) over $\mathbf{L}_{cat}$:

$\forall X : O.\forall Y : O.\forall Z : O.\forall f : A(X,Y).\forall g : A(Y,Z).\exists h : A(X,Z).\exists t : T(X,Y,Z,f,g,h).\mathbf{t}$

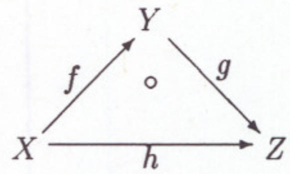
The sentence, referred to below as (2), expresses the existence of the composite $h : X \rightarrow Z$ of composable arrows $f : X \rightarrow Y$, $g, Y \rightarrow Z$.

The sorts are interpreted in structures as certain sets. Let K be a kind, M an $\mathbf{L}$-structure. By $M[K]$ we mean the set of all tuples $<a_p>_{p \in K|\mathbf{L}}$ with $a_p \in M(K_p)$ (where $p : K \rightarrow K_p$), and such that for every $p \in K|\mathbf{L}$ and $q \in K_p|\mathbf{L}$, $M(q)(a_p) = a_{qp}$:



The elements of the set $M[K]$ are called *contexts* for K in M.

For instance, for $\mathbf{L} = \mathbf{L}_{cat}$, when $K = A$ and M is $M[\mathbf{C}]$ for a category $\mathbf{C}$, then $M[A]$ is the set of pairs (X,Y) of objects; when $K = T$, $M[T]$ is the set of all not necessarily commutative triangles in $\mathbf{C}$.



The set $M(K)$ is "fibered over" $M[K]$; $M(K)$ is the disjoint union of sets $MK(\mathbf{a})$, one for each $\mathbf{a} = <a_p>_{p \in K|\mathbf{L}} \in M[K]$;

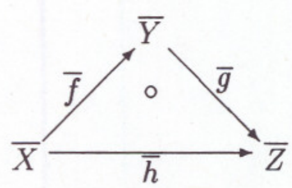
$$MK(\mathbf{a}) \underset{def}{=} \{a \in M(K) : (Mp)(a) = a_p \text{ for all } p \in K|\mathbf{L}\},$$

the *fiber over* $\mathbf{a}$ (it is clear that for any $a \in M(K)$,

$$\mathbf{a} \underset{def}{=} <(Mp)(a)>_{p \in K|\mathbf{L}} \in M[K]).$$

Let $X = K(<x_p>_{p \in K|\mathbf{L}})$ be a sort. An interpretation of X in M is given by a context $\mathbf{a} = <a_p>_{p \in K|\mathbf{L}} \in M[K]$, with a_p *assigned to* x_p, which means the additional condition $a_p = a_{p'}$, every time $x_p = x_{p'}$, $(p, p' \in K|\mathbf{L})$; the interpretation itself is the fiber $MK(\mathbf{a})$ over $\mathbf{a}$.

Returning to the example started above, the interpretation of $A(X, Y)$ is $\hom_{\mathbf{C}}$ (A, B) when A, B are objects assigned to the variables X, Y respectively. The interpretation of the sort $T(X, Y, Z, f, g, h)$ must be given by a (not necessarily commutative) diagram of the shape



in $\mathbf{C}$, and of course, if, say, X is the same variable as Y, then the object $\overline{X}$ must be the same as $\overline{Y}$. The interpretation itself is a singleton set if the displayed diagram commutes, and the empty set otherwise.

For the full definition of the semantics of FOLDS, we need the concept of context (of variables). A *context* is a finite set $\mathcal{Y}$ of variables such that, for all $y \in \mathcal{Y}$ we have that $Dep(y) \subset \mathcal{Y}$.

For instance, in the case $\mathbf{L} = \mathbf{L}_{cat}$, the set $\{X, Y, Z, f, g, h, t\}$, under the conditions

$$X : O, Y : O, Z : O, f : A(X, Y), g : A(Y, Z), z : A(X, Z), t : T(X, Y, Z, f, g, h),$$

is a context.

Note that for any formula φ, $Var(\varphi)$ is a context.

Let $\mathcal{Y}$ be a context, and M an $\mathbf{L}$-structure. We define the set $M[\mathcal{Y}]$, the set of (legitimate) valuations of $\mathcal{Y}$ in M.

For a variable $y \in \mathcal{Y}$, let us display the sort of y in the notation

$$y : K_y(<x_{y,p}>_{p \in K_y|\mathbf{L}}).$$

We define $M[\mathcal{Y}] \underset{def}{=}$

$$\left\{ <a_y>_{y\in\mathcal{Y}} \in \prod_{y\in\mathcal{Y}} M(K_y) : (Mp)(a_y) = a_z \text{ whenever } y \in \mathcal{Y},\ p \in K_y|\mathbf{L} \text{ and } z = x_{y,p} \right\}.$$

The definition says that the elements of $M[\mathcal{Y}]$ are compatible valuations of the variables in $\mathcal{Y}$, where "compatibility" refers to the fact that if two variables y and z from $\mathcal{Y}$ are in a relation of dependence, $z \in Dep(y)$, in a particular way given by the "place" p of the kind K_y (that is, $z = x_{y,p}$), then the corresponding elements of the family have to be related by $M(p)$.

By recursion on the complexity of the formula φ, we define $M[\mathcal{Y} : \varphi]$, *the interpretation of* φ *in* M *in the context* $\mathcal{Y}$, whenever $\mathcal{Y}$ is a context such that $Var(\varphi) \subset \mathcal{Y}$; we will have that $M[\mathcal{Y} : \varphi] \subset M[\mathcal{Y}]$.

$$M[\mathcal{Y} : \mathbf{t}] \underset{def}{=} M[\mathcal{Y}]\ ;$$

$$M[\mathcal{Y} : \mathbf{f}] \underset{def}{=} \emptyset\ .$$

For the propositional connectives, the clauses are the expected ones; *e.g.*,

$$<a_y>_{y\in\mathcal{Y}} \in M[\mathcal{Y} : \psi \wedge \theta] \underset{def}{\Longleftrightarrow}$$

$$<a_y>_{y\in\mathcal{Y}} \in M[\mathcal{Y} : \psi] \text{ and } <a_y>_{y\in\mathcal{Y}} \in M[\mathcal{Y} : \theta].$$

The interpretation of formulas $\forall x : X.\psi$, $\exists x : X.\psi$ will be according to the readings "for all x in X, ψ", and "there is x in X such that ψ". Thus, quantification in FOLDS is a relativized quantification. Here, the sort X is interpreted as a set according to what was said above. For the precise clause, we need a bit more notation.

Let $M, \mathcal{Y}$ and $\mathbf{a} = <a_y>_{y\in\mathcal{Y}} \in M[\mathcal{Y}]$ be as above; assume

$$Var(\forall x : X.\psi) = Var(\exists x : X.\psi) = (Var(\varphi) - \{x\}) \cup Dep(x) \subset \mathcal{Y}.$$

Part of $\mathcal{Y}$ may be discarded. Let $\mathcal{Y}' = \mathcal{Y} - \{x\} - \{y \in \mathcal{Y} : x \in Dep(y)\}$. Then, still, $\mathcal{Y}'$ is a context, and $Var(\forall x\psi) \subset \mathcal{Y}'$ (the reason is that, since $\forall x : X.\psi$ is well-formed, if $y \in Var(\psi)$, then $x \notin Dep(y)$). Also, $\mathcal{Y}' \cup \{x\}$ is a context, and $Var(\psi) \subset \mathcal{Y}' \cup \{x\}$.

Let $X = K(<x_p>_{p \in K|\mathbf{L}})$. Each $x_p \in \mathcal{Y}'$, so $\tilde{a} = <a_{x_p}>_{p \in K|\mathbf{L}}$ is defined, and as easily seen, $\tilde{a} \in M[K]$. Therefore, we have the fiber $MK(\tilde{a})$ of $M(K)$.

Let $\mathbf{a} = <a_y>_{y \in \mathcal{Y}} \in M[\mathcal{Y}]$. Define $\mathbf{a}' = <a_y>_{y \in \mathcal{Y}'}$. For any $a \in MK(\tilde{a})$, let $\mathbf{a}'[a/x]$ be $<b_y>_{y \in \mathcal{Y}' \cup \{x\}}$ for which $b_y = a_y$ when $y \in \mathcal{Y}'$, and $b_x = a$. It is immediately seen that $\mathbf{a}'[a/x] \in M[\mathcal{Y}' \cup \{x\}]$. We define

$$\mathbf{a} \in M[\mathcal{Y} : \forall x : X.\psi] \Leftrightarrow \mathbf{a}'[a/x] \in M[\mathcal{Y}' \cup \{x\} : \psi] \text{ for all } a \in MK(\tilde{a}) ,$$

and

$$\mathbf{a} \in M[\mathcal{Y} : \exists x : X.\psi] \Leftrightarrow \mathbf{a}'[a/x] \in M[\mathcal{Y}' \cup \{x\} : \psi] \text{ for some } a \in MK(\tilde{a}) .$$

This completes the definition of the standard, *Set*-valued semantics of FOLDS.

As usual, we also write $M \models \varphi[\mathbf{a}]$ for $\mathbf{a} \in M[\mathcal{Y} : \varphi]$.

Returning to the example of $\mathbf{L}_{cat}$, the reader will easily write down all axioms for "category" in the form of sentences over $\mathbf{L}_{cat}$; the sentence (2) above is an example. Let us call the resulting finite set of sentences Σ_{cat}. Since equality is not treated as "logical", Σ_{cat} includes axioms concerning equality, the kind E. It will *almost*, but not quite, be the case that an $\mathbf{L}_{cat}$-structure M satisfies Σ_{cat} iff it is of the form $M = M[\mathbf{C}]$ for a category $\mathbf{C}$; for explanation, see below.

9. Formal systems

FOLDS may be regarded as a restricted form of ordinary first order logic. Given a one-way category $\mathbf{L}$, we may consider the multi-sorted first order language with sorts the objects of $\mathbf{L}$, with unary sorted operation symbols the arrows of $\mathbf{L}$, and with equality predicates, one for each sort; let us refer to this language as *First Order Logic* (FOL) *over* $\mathbf{L}$. The formulas of FOLDS over $\mathbf{L}$ can obviously be translated into FOL over $\mathbf{L}$. This consideration will immediately imply the fact that the compactness theorem, a kind of abstract completeness, holds for FOLDS.

We have completeness theorems of appropriate formal systems for FOLDS. Completeness is stronger than compactness; it is also essential for the purpose of adopting FOLDS as a language for an axiomatic system.

We have the classical and the intuitionistic versions of FOLDS. The third version, coherent logic, is a proper part of both classical and intuitionistic logic. Categorical logic, mainly in the pioneering work of Andre Joyal, has shown the fundamental theoretical role of coherent logic in relation to both classical and intuitionistic logic, for example, in connection with completeness theorems (see [MR1]). The coherent fragment of FOLDS continues to play a basic role.