(4)

These are copies of the slides for my talk

Higher dimensional diagrams via computads

at CT 2006, White Point, June 25 – July 1;

The last four slides (pp 19 – 22) were not shown at the talk (for lack of time ...).

More importantly: on bottom p. 7 and on p 9, now you find corrected pictures of "atoms";

Very regrettably, these pictures were shown in wrong variants in the talk.

July 5 / 2006

Higher Dimensional Diagrams (HDD's) via Computads

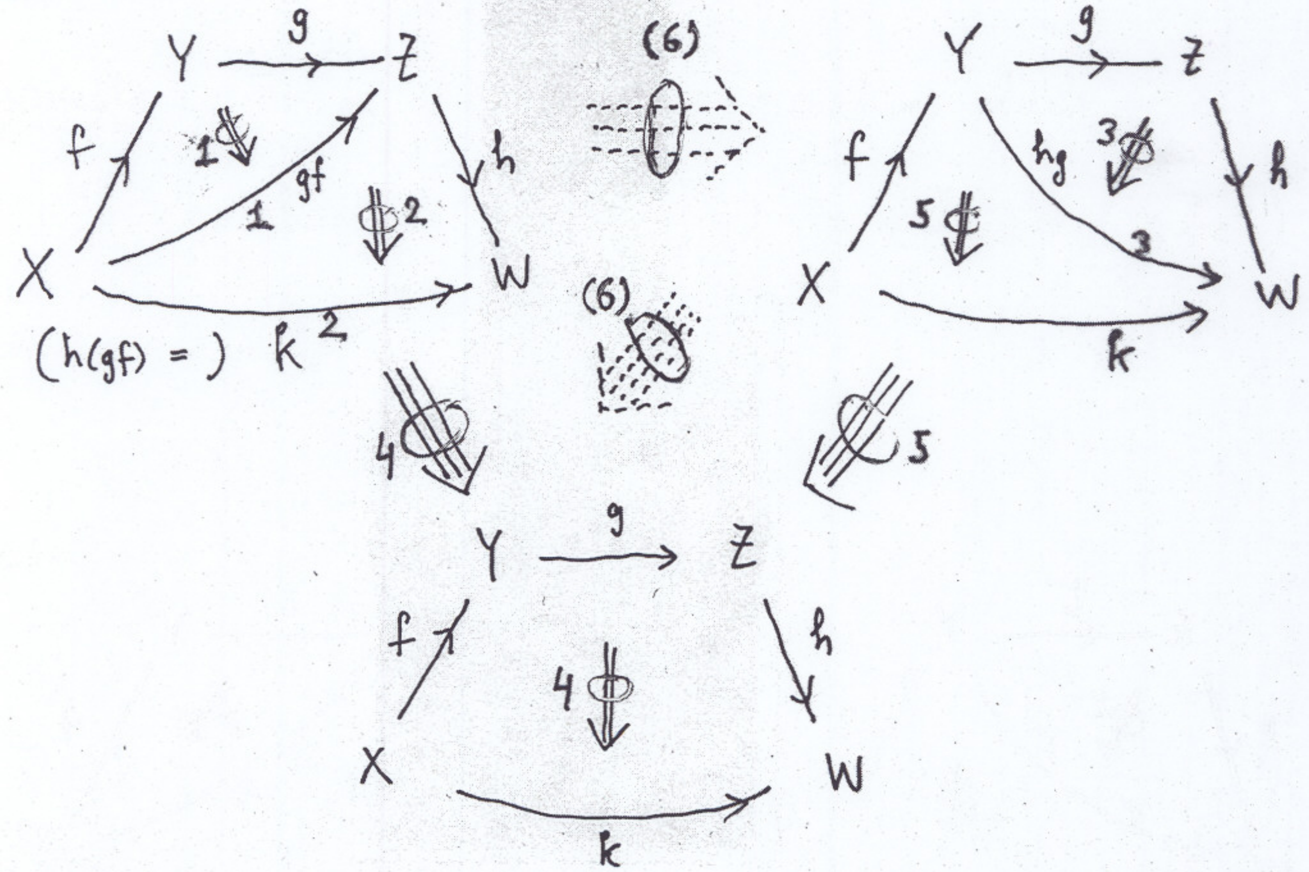
(M. Makkai)

Example: 'Proof' of the associative law for composition: $f(gh) \cong (fg)h \ (\cong fgh)$

Given:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

Derive:



Higher Dimensional Diagrams
(HDD's)
via Computads

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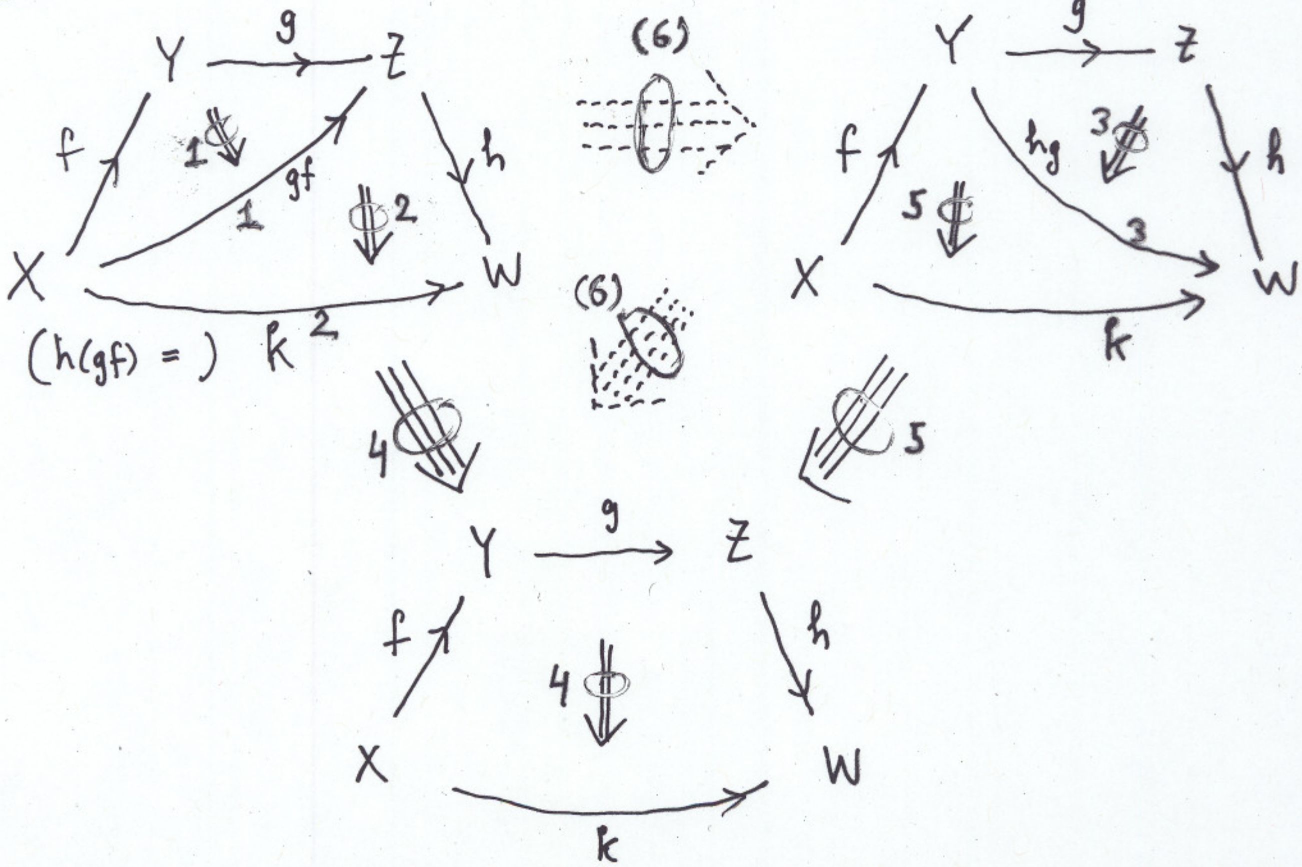
for composition: $f(gh) \cong (fg)h \;(\cong fgh)$

Given:

$$\begin{tikzcd} & Y \arrow[r, "g"] & Z \arrow[dr, "h"] & \\ X \arrow[ur, "f"] & & & W \end{tikzcd}$$

Derive:



SLOGANS:

- $\left.\begin{array}{l}\text{Some}\\ \text{Many}\\ \text{All}\end{array}\right\}$ HDD's are needed for:

  weak n-categories

- HDD = computad (see below)

Sample theorem:

The category $\mathbf{Comp}_3^+$ of positive 3-computads is a presheaf category

N.B.
- The category $\mathbf{Comp}_3$ of all 3-computads is not a presheaf category (M. Zawadowski – M.M.)

- The category $\mathbf{Comp}_2$ of all 2-computads is a presheaf category (Steve Schanuel)

Adjoining indeterminate cells and computads

$\mathbb{X}$: ω-category

(Notation : $\mathbb{X}_n$: set of n-cells ($n = 0, 1, 2, \ldots$)

Convention : $\mathbb{X}_{-1} = \{ * \}$

$$a \in \mathbb{X}_0 \Rightarrow da = ca = *$$

(d : domain ; c : codomain

convention : $* \parallel *$

$$a \parallel b \iff da = db \ \& \ ca = cb$$

$$\|\mathbb{X}\| \overset{\text{def}}{=} \coprod_{n \in \mathbb{N} \cup \{-1\}} \mathbb{X}_n \qquad)$$

Definition:

A set of indeterminates (attached to $\mathbb{X}$)

$$\underline{U} = (U, d, c) \ : \ U \underset{c}{\overset{d}{\rightrightarrows}} \|\mathbb{X}\|$$

set

functions

such that:

$$u \in U \Rightarrow du \parallel cu \quad (\text{in } \mathbb{X})$$

To define: $\mathbb{X}[\underline{U}]$: the result of freely adjoining all $u \in \underline{U}$ to $\mathbb{X}$.

Let: $\mathcal{F}$: category: (of "frames")

objects: $(\mathbb{X}; \underline{U})$

arrows: $(\mathbb{X}; \underline{U}) \xrightarrow{(\Gamma, \Lambda)} (\mathbb{Y}, \underline{V})$

$$\mathbb{X} \xrightarrow{\Gamma} \mathbb{Y}$$

$$\underline{U} \xrightarrow{\Lambda} \underline{V}$$

subject to:

$$\begin{tikzcd}
U \arrow[rr, "\Lambda"] \arrow[d, "{\langle d, c\rangle}"'] \arrow[drr, phantom, "\circ"] & & V \arrow[d, "{\langle d, c\rangle}"] \\
\|\mathbb{X}\| \times \|\mathbb{X}\| \arrow[rr] & & \|\mathbb{Y}\| \times \|\mathbb{Y}\|
\end{tikzcd}$$

'tautological frame':

$$\tau(\underset{\omega\text{-cat}}{\mathbb{Y}}) \overset{\mathrm{DEF}}{=} (\mathbb{Y}; (\overset{U}{\|\mathbb{Y}\|}, \overset{\text{those of } \mathbb{Y}}{d, c}))$$

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, shift left=2, "T"] & & \underline{\omega\mathrm{Cat}} \arrow[ll, shift left=2, "\tau"'] \arrow[ll, dashed, shift left=6, "E"]
\end{tikzcd}$$

$\omega\mathrm{Cat}$ (ordinary) category of (strict) ω-cat's & (strict) ω-functors

$\mathcal{F}$: $E \dashv \tau$

$$(\mathbb{X}; \underline{U}) \longmapsto \mathbb{X}[\underline{U}]$$

0-computad: a set

strict!

<u>Recursively</u>:

(n+1)-computad: an (n+1)-category of the form $\mathbb{X}[\underline{U}]$, with

$\mathbb{X}$: n-computad, $\underline{U} = (U, d, c)$ a set of (n+1)-indeterminates ($u \in U \Rightarrow du, cu \in \mathbb{X}_n$).

Computad (= ω-computad): ω-category $\mathbb{X}$ such that $\forall n \in \mathbb{N}$, $\underbrace{\mathbb{X} \upharpoonright n}_{n\text{-truncation}}$ is an n-computad.

<u>N.B.</u> no need for 'tracking' indet's: they are exactly the indecomposables:

y is indecomposable iff either $\dim(y) = 0$

or: $y \neq 1_b \;\; \forall b$

& $\forall b, e \in \mathbb{Y}_n$, $k < n$, $y = e \circ_k b$

dim(y)

$\Rightarrow \; b = 1_a^{(n)}$ <u>or</u> $e = 1_a^{(n)}$ for some $a \in \mathbb{Y}_k$

($m = \dim(a)$: $1_a^{(m)} = a$;

$1_a^{(p+1)} = 1_{1_a^{(p)}}$.)

$\underline{\text{Comp}}$: the category of computads;

objects: (small) computads

arrows: ω-cat morphisms

taking indet's (indecomposables)

to indet's.

$\therefore$ Non-full inclusion

$$\underline{\text{Comp}} \longrightarrow \underline{\omega\,\text{Cat}}$$

full on isomorphisms.

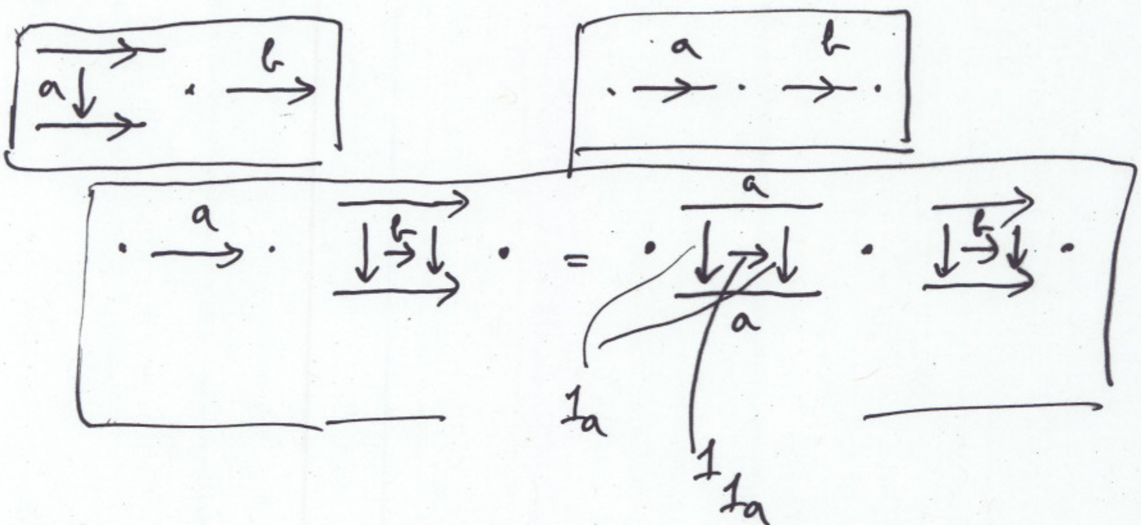
$\boxed{\text{Dot-compositions:}}$

$$a\cdot b \overset{\text{def}}{=} b \circ_k a = 1_b^{(p)} \circ_k 1_a^{(p)}$$

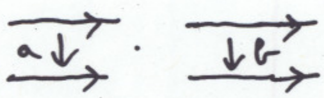
for: $k = k(a,b) \overset{\text{def}}{=} \min(\underbrace{m}_{\dim(a)}, \underbrace{n}_{\dim(b)}) - 1$

$$p \overset{\text{def}}{=} \max(m,n)$$

Ex's:

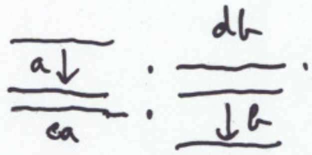


Non example



||

(but:



$(a\cdot db)\cdot(ca\cdot b)$)

The 'commutative' law

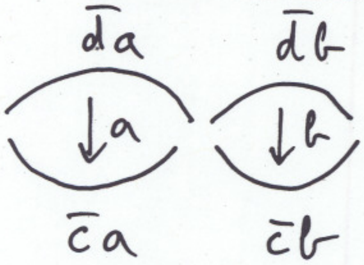
For: $m, n \geq 2 \quad (m = \dim(a),\ n = \dim(b))$

$$k = k(a,b), \quad \bar d = d^{(k)}, \quad \bar c = c^{(k)}$$

assume: $c\bar c a = c \bar d a \Leftrightarrow d \bar d b = d \bar c b$

Then:

$$\boxed{(a \cdot \bar d b)\cdot(\bar c a \cdot b) = ((\bar d a \cdot b)\cdot(a \cdot \bar c b))}$$



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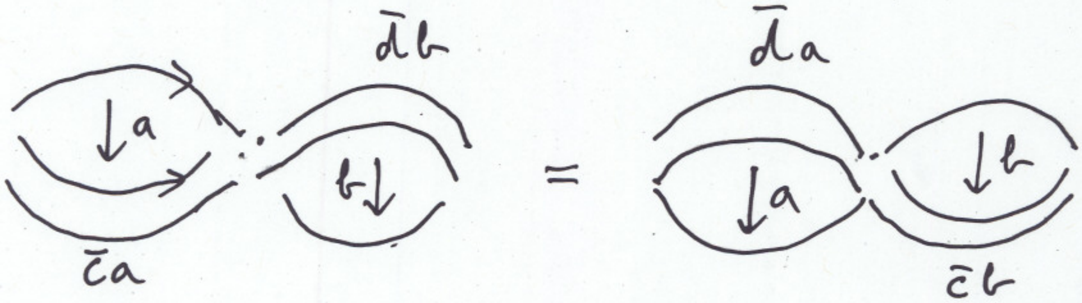


Constructing $\mathbb{X}[\underline{\mathcal{U}}]$

$\mathbb{X}$: n-category

$\underline{\mathcal{U}} = (\mathcal{U}, d, c)$: set of $(n+1)$-indets (attached to $\mathbb{X}$)

$\boxed{\text{Fact 1}}$ Every $(n+1)$-cell of (the $(n+1)$-cat) $\mathbb{X}[\underline{\mathcal{U}}]$ is of the form

$$\varphi_1 \cdot \varphi_2 \cdots \varphi_N \qquad (N = \underset{(!\,\cdots)}{0}, 1, 2, \ldots)$$

where each φ_i is an atom

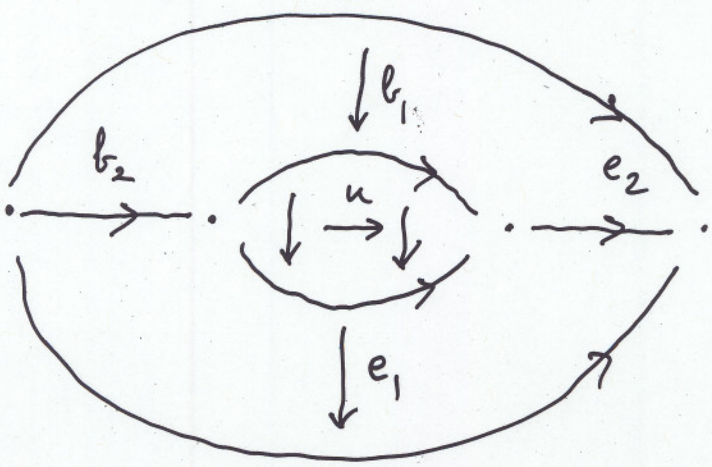
i.e. an $(n+1)$-cell of the form

$$b_n \cdot (b_{n-1} \cdot (\cdots (b_1 \cdot u \cdot e_1) \cdots) \cdot e_{n-1}) \cdot e_n$$

where $u \in \mathcal{U}$; $b_i \in \mathbb{X}_{n-i+1}$, $e_i \in \mathbb{X}_{n-i+1}$

$(i = 1, \ldots, n)$

For $n=2$: $\quad \varphi = b_2 \cdot (b_1 \cdot u \cdot e_1) \cdot e_2$:



Let: $\quad \mathcal{U} \longrightarrow \bar{\mathcal{U}}$ ... new set of n-indet's

$$u \longmapsto \bar{u}$$

$$d\bar{u} \stackrel{\text{def}}{=} ddu = cdu$$

$$c\bar{u} \stackrel{\text{def}}{=} cdu = ccu$$

$$\underline{\mathcal{U}} = (\mathcal{U}, d, c) \rightsquigarrow \underline{\bar{\mathcal{U}}} = (\bar{\mathcal{U}}, d, c)$$

attached to $\mathbb{X}$

$\boxed{\text{Consider}}$: $\quad \mathbb{X}[\underline{\bar{\mathcal{U}}}]$ ← n-category

n-cat   n-indet's   : $\boxed{\text{Known}}$

[N.B. When $\mathbb{X} = \mathbb{Y}[\underline{V}]$

(n-1)-cat   n-indets

then $\mathbb{X}[\underline{\bar{\mathcal{U}}}] = \mathbb{Y}[\underbrace{\underline{V} \sqcup \underline{\bar{\mathcal{U}}}}_{n\text{-indets}}]$

(n-1)-cat

for computads : induction]

The **barred atoms** of $\mathbb{X}[\underline{\bar{u}}]$:

n-cells of the form

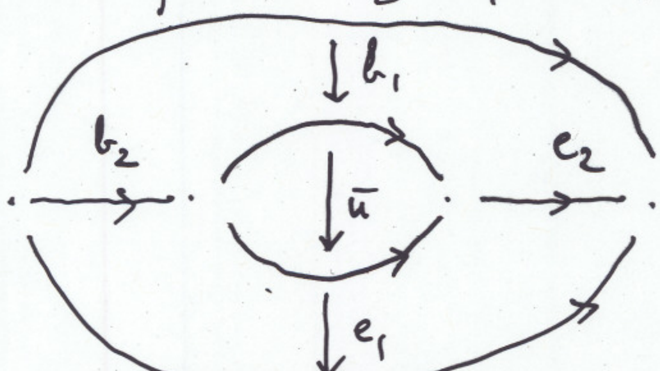
$$b_n\cdot(b_{n-1}\cdot(\cdots(b_1\cdot\bar{u}\cdot e_1)\cdots)\cdot e_{n-1})\cdot b_n$$

$u\in\mathcal{U}$

$$b_i,\ e_i\in\mathbb{X}_{n-i+1}$$

[N.B. A barred atom is not necessarily an atom of $(\mathbb{X}[\underline{\bar{u}}]=)\,\mathbb{Y}[\underline{V}\cup\underline{\bar{U}}]$]

Picture: $n=2$: $\varphi[\bar{u}] = b_2\cdot(b_1\cdot u\cdot e_1)\cdot e_2$:



$\boxed{\text{Fact 2}}$ The map Atoms of $\mathbb{X}[\underline{u}] \longrightarrow$ Barred A's of $\mathbb{X}[\underline{\bar{u}}]$

$$\varphi[u]\longmapsto\varphi[\bar{u}]$$

$$b_n(b_{n-1}(\cdots(b_1 u e_1)\cdots)e_{n-1})e_n$$
$$\longmapsto b_n(b_{n-1}(\cdots(b_1\bar{u}e_1)\cdots)e_{n-1})e_n$$

is a bijection (well def'd ...)

$\therefore$ An atom of $\mathbb{X}[\underline{U}]$ is 'the same' as a barred atom of $\mathbb{X}[\underline{\bar{U}}]$, hence, "inductively understood"

$\boxed{\text{Fact 3}}$

$$d(\varphi[u]) = \varphi[du]$$
$$c(\varphi[u]) = \varphi[cu]$$

—·—

A **molecule**: by definition:

a tuple $\langle \varphi_1, \dots, \varphi_N \rangle$ of $(n+1)$-atoms (of $\mathbb{X}[U]$) such that $\varphi_1 \cdots \varphi_N$ is well-defined

Define: $\sim$

$$(\vec{\varphi} =) \langle \varphi_1, \dots, \varphi_N \rangle \sim \langle \psi_1, \dots, \psi_M \rangle (= \vec{\psi})$$

$$\underset{\text{def}}{\Longleftrightarrow}$$

$$\varphi_1 \cdots \varphi_N = \psi_1 \cdots \psi_M \quad \text{in } \mathbb{X}[U]$$

Fact: $\vec{\varphi} \sim \vec{\psi} \implies N = M$

$\boxed{\text{Explaining } \sim :}$

Define: $\sim_0$:

molecules

$$\langle \varphi_1, \dots, \varphi_N \rangle \sim_0 \langle \psi_1, \dots, \psi_N \rangle$$

$$\underset{\text{def}}{\Longleftrightarrow}$$

there is $i \in \{1, \dots, N-1\}$ such that

$$\varphi_k = \psi_k \quad \text{for } k \in \{1, \dots, N\} - \{i, i+1\}$$

and

$$\left.\begin{array}{l} \text{"} \varphi_i \cdot \varphi_{i+1} = \psi_i \cdot \psi_{i+1} \text{ because of} \\ \qquad \text{the commutative law"} \end{array}\right\} (*)$$

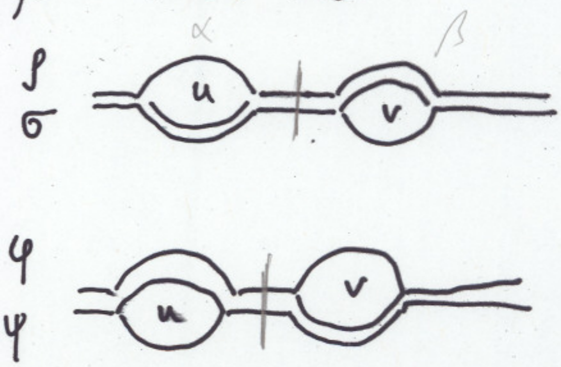
More precisely:

$L(\rho, \sigma, \varphi, \psi) \overset{\text{def}}{\Longleftrightarrow}$ there exist atoms

atoms

α, β such that

$$\begin{aligned} \rho &= \alpha \cdot d\beta \\ \sigma &= c\alpha \cdot \beta \\ \varphi &= d\alpha \cdot \beta \\ \psi &= \alpha \cdot c\beta \end{aligned}$$



(*) above is, precisely,

"$L(\varphi_i, \varphi_{i+1}, \psi_i, \psi_{i+1})$

or

$L(\psi_i, \psi_{i+1}, \varphi_i, \varphi_{i+1})$"

$\boxed{\text{Fact 4}}$ $\sim$ is the transitive/reflexive closure of $\sim_0$.

$\boxed{\text{Fact 5}}$ Assume: $\mathbb{X}$ **is** an n-computad. The $\sim$-equivalence class of any molecule $\langle \varphi_1, \dots, \varphi_N \rangle$ is finite.

$\boxed{\text{Theorem}}$ The word-problem for computads is recursively solvable

The case $n=1$, $n+1=2$: 2-pasting diagrams

(also the subject of:

A. J. Power, A 2-categorical pasting theorem.
J. Algebra 129 (1990), 439–445)

Let : $\mathbb{X}$: 1-computad

arrows of $\mathbb{X}$: composable chains

1-pd's: $\cdot \xrightarrow[f_1]{} \cdot \xrightarrow[f_2]{} \cdot \quad \cdots \quad \cdot \xrightarrow[f_\ell]{} \cdot$

of 1-indeterminates ($\ell = 0$ is allowed ...)

Fix: $N \in \mathbb{N} - \{0\}$

N distinct 2-indet's. $u_1, \dots, u_N$

with $u_i \longmapsto (du_i \parallel cu_i)$

1-pd's : non-identity cells

'Molecule' (for now):

$A = \langle \varphi_1, \dots, \varphi_N \rangle$ such that each u_i ("positivity")

occurs in exactly one φ_k.

Write $\varphi[i]$ for this φ_k.

Define: $i \, <_A \, j \underset{\text{def}}{\iff} k < l$

for $\varphi[i] = \varphi_k$

$\varphi[j] = \varphi_l$

a total order

of $[N] = \{1, \dots, N\}$.

For atoms ρ, σ, write:

$\rho \Rightarrow \sigma$ ("ρ & σ are left switchable")

$\underset{\text{def}}{\iff} \exists : \varphi, \psi$ s.t. $L(\rho, \sigma, \varphi, \psi)$

and:

$\varphi \Leftarrow \psi$

$\underset{\text{def}}{\iff} \exists : \rho, \sigma$ s.t. $L(\rho, \sigma, \varphi, \psi)$

$\boxed{\text{Fact 6}}$ ($n=1$; the <u>positive</u> case only)

$\rho \Rightarrow \sigma$, $\rho \Leftarrow \sigma$ cannot both hold (<u>false</u> for the non-positive context)

& if $L(\rho, \sigma, \varphi, \psi)$,

φ & ψ are uniquely determined (and vice versa)

$\boxed{\text{The tree of variants of a molecule}}$

<u>Define</u> recursively the labelled tree

$$T[A]$$

given molecule
(in terms of $u_1, \dots, u_N$)

At the root r, the label is : A

Suppose we've arrived at node : t ---- level : p
with label A_t, a molecule.

Define the level-$(p+1)$ successors $\hat{t}$ of t,
<u>if</u> <u>any</u>, with their labels. $A_{\hat{t}}$.
(there may be none).

First, define:

$$i \xrightarrow{A} j \overset{\text{def}}{\Longleftrightarrow} \underbrace{j}_{\text{j is next larger to i in $<_{A_t}$}}$$

$$i < j \ \& \ i <_t ! j \ \& \ \varphi[i] \Rightarrow \varphi[j]$$

$$i \xleftarrow{A} j \overset{\text{det}}{\Longleftrightarrow}$$

$$i < j \ \& \ i <_t ! j \ \& \ \varphi[i] \Leftarrow \varphi[j]$$

The successors $\hat{t}$ of t in $T[A]$ are, <u>by definition</u>, in a bijective correspondence with the pairs (i, j) such that

$$i \xrightarrow{A} j \quad \text{or} \quad i \xleftarrow{A} j.$$

The label $A_{\hat{t}}$, for $\hat{t}$ corr. to $i \xrightarrow{A} j$:

$$\hat{\varphi}[h] = \varphi[h] \quad \text{for } h \neq i, j$$

and $\hat{\varphi}[i]$, $\hat{\varphi}[j]$ are determined by:

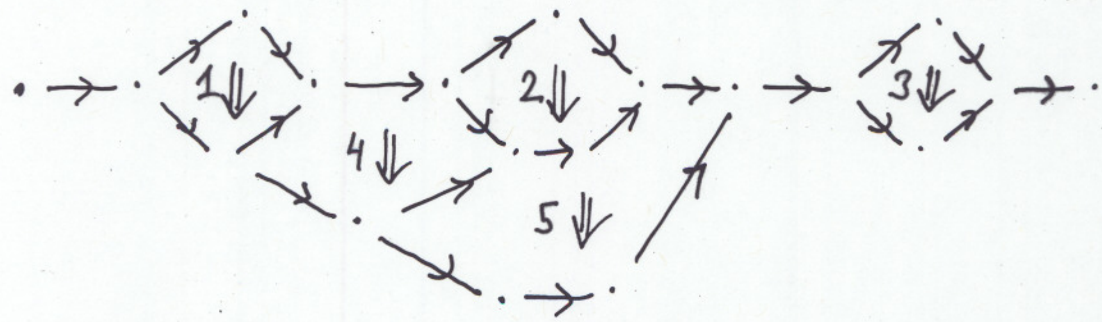
$$L(\varphi[i], \varphi[j], \hat{\varphi}[j], \hat{\varphi}[i])$$

" $\varphi[i]$, $\varphi[j]$ in A_t have been left-switched, to get $A_{\hat{t}}$ "

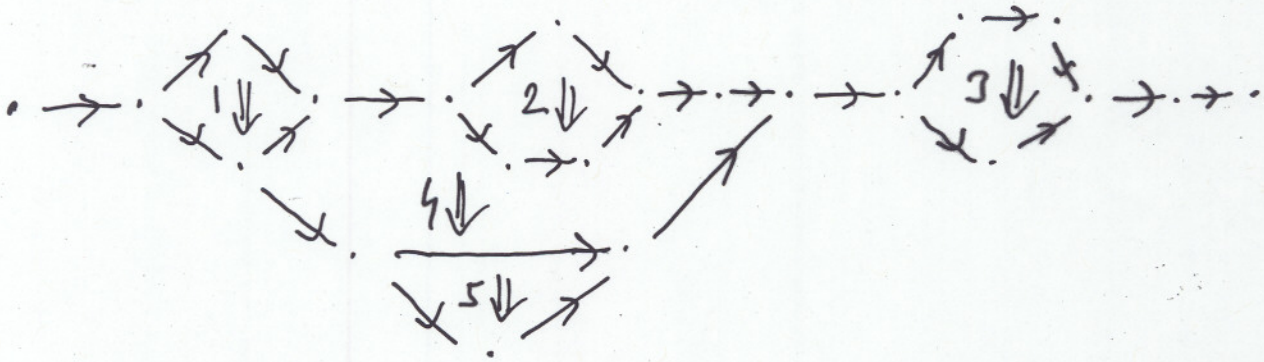
The case $i \xleftarrow{A} j$ is analogous.

(def'n of $T[A]$ is complete)

Ex's:



1
2
3
4
5



1
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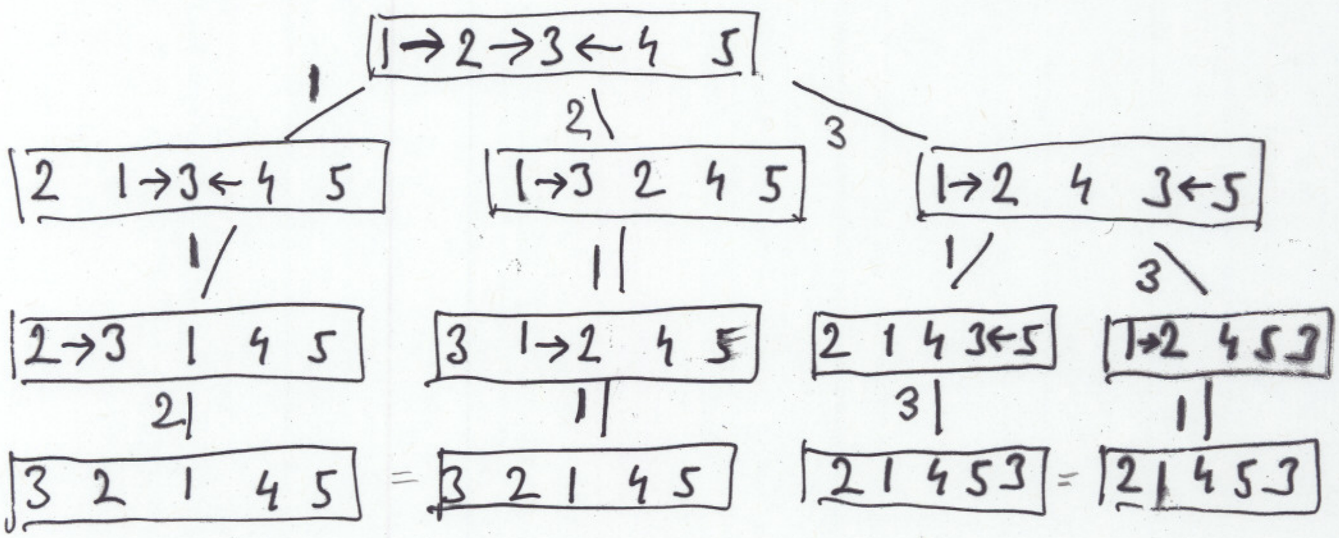

1→2→3←4 5
1
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2 1→3←4 5
1→3 2 4 5
1→2 4 3←5
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2→3 1 4 5
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2 1 4 3←5
1→2 4 5 3
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3 2 1 4 5
2 1 4 5 3
=
2 1 4 5 3

$\boxed{\text{Theorem}}$ ($n=1$; $n+1=2$; positive 2-pd's)

(i) Define: $$\lhd \overset{\text{def}}{=} \bigcap_{t\in T[A]} <_t$$

$$\rightarrow \overset{\text{def}}{=} \bigcup_{t\in T[A]} \xrightarrow{t} \cup \bigcup_{t\in T} (\xleftarrow{t})^{\text{op}}$$

Then: $\lhd$, $\rightarrow$ are irreflexive partial orders of $[N]$ which are complementary:

for every $i, j \in [N]$, $i \neq j$:
exactly one of

$i \lhd j$, $j \lhd i$, $i \rightarrow j$, $j \rightarrow i$

(ii) $<_s = <_t \implies A_s = A_t$

nodes of $T[A]$

molecules agree atom-by-atom

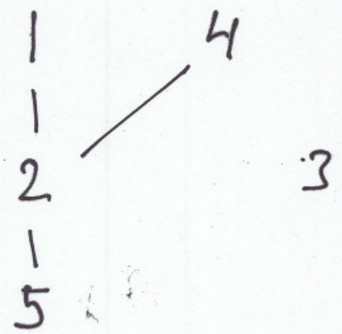
(iii) Every total order of $[N]$ extending $\lhd$ (see (i)) appears as $<_t$ for at least one t.

(iv) Every molecule B for which $B \sim A$ is $B = A_t$ for at least one t.

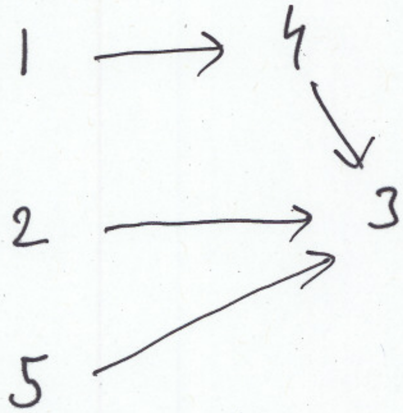
these two are consequences!

To example(s) on p. 18:

Hasse diagram of partial order $\lhd$:



Hasse diagram of partial order $\to$:



Concrete presheaf categories

Concrete category :

$$\mathbb{A} = (\mathbb{A}, |-|_{\mathbb{A}} : \mathbb{A} \to Set)$$

Suppose: $\mathbb{A}, \mathbb{B}$ concrete cat's.

$\mathbb{A} \simeq \mathbb{B} \overset{\text{def}}{\iff}$

equivalent

$$\exists : \begin{tikzcd} \mathbb{A} \arrow[rr, "\simeq"] \arrow[dr, "|-|_{\mathbb{A}}"'] & \arrow[d, phantom, "\cong"] & \mathbb{B} \arrow[dl, "|-|_{\mathbb{B}}"] \\ & Set & \end{tikzcd}$$

$\mathbb{A}$ is a concrete presheaf category **if** it is equivalent to the concrete cat

$$\hat{\mathbb{C}} = (Set^{\mathbb{C}^{op}}, |-|_{\hat{\mathbb{C}}}) \quad \text{defined by}$$

$$|X|_{\hat{\mathbb{C}}} = \coprod_{U \in Ob(\mathbb{C})} X(U)$$

Let X be a finite computad (the set $|X|$ of all indet's in X is finite); and assume there is a unique top-dimensional indet in X; denoted by m_X.

X is **principal** if there is no proper subcomputad of X containing m_X

X is a **computope** if X is principal; and, whenever Y is principal, and $Y \xrightarrow{f} X$ **such** that $f(m_Y) = m_X$, then f is an isomorphism

$\boxed{\text{Theorem}}$ There are enough computopes: for every principal $\mathbf{X}$, there is at least one computope Y, together with a map $Y \xrightarrow{f} X$ such that
$$f(m_Y) = m_X.$$

$\boxed{\text{Proposition}}$ (elementary consequence of the non-elementary $\boxed{\text{Theorem}}$, p 21)

Suppose $\mathbb{A}$ is a full subcategory of $\underline{\text{Comp}}$, which is a sieve in $\underline{\text{Comp}}$: whenever $B \to A$ and $A \in Ob(\mathbb{A})$, then $B \in Ob(\mathbb{A})$.

Then $\mathbb{A}$ is a concrete presheaf category

$= (\mathbb{A}, |-|_{\mathbb{A}}) : \ |X|_{\mathbb{A}} =$ set of all indets in X

$\boxed{\text{if and only if}}$ (a) & (b):

(a) $\mathbb{A}$ is closed under small colimits in Comp

(b) For every Z in $\mathbb{A}$:

(b1) X computope, $X \underset{g}{\overset{f}{\rightrightarrows}} Z$

$f(m_X) = g(m_X) \implies f = g$

(b2) X, Y computopes, $X \xrightarrow{f} Z \xleftarrow{g} Y$

$f(m_X) = g(m_X) \implies f = g$