3.1

Notes for:

talk at

"Logic in Hungary"

Aug 2005

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Higher-dimensional diagrams: the basis for the syntax of a new foundations of mathematics.

Sets form a category, rather than a set — or even, a class: the category of sets, Set, is the primary category. Other categories arise naturally from it: think of the category of groups, that of topological spaces, etc. We need totalities of categories, in particular, the totality of all categories. Cat is no longer a category: it is a 2-dimensional category. (analogously to the fact that Set is a 1-category, rather than a 0-category, that is, a set.) This is the beginning of the hierarchy of n(-dimensional) categories, with n ranging over all natural numbers.

I will not speak in any detail about n-categories, especially the so-called weak n-categories, which are the ones we actually need: this is a long story, going back now for over a decade, and I have talked about them several times, at logic meetings too. I want to concentrate on the (a discourse about) syntax of the language that is suitable for n-categories, and in fact, only on the algebraic part of that syntax, the one that comes before the logical operators. Someone familiar with mathematical logic should think of

a kind of type theory here, a type theory that has a fairly elaborate system of types, in fact, dependent types, ones that could themselves depend on variables of "earlier" types.
More specifically, the basic entities of our syntax are the higher-dimensional diagrams of the title.

Very little exposure to category theory is needed to recognize the extensive role of diagrams in that theory. Consider, for instance the following diagram:

$$
\begin{tikzcd}
X_3 \arrow[r, "f_3"] & X_6 \arrow[r, "f_6"] & X_9 \\
X_2 \arrow[u, "g_4"] \arrow[r, "f_2"'] \arrow[ur, phantom, "2"] & X_5 \arrow[u, "g_5"'] \arrow[r, "f_5"] \arrow[ur, phantom, "4"] & X_8 \arrow[u, "g_6"'] \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] \arrow[ur, phantom, "1"] & X_4 \arrow[u, "g_2"'] \arrow[r, "f_4"'] \arrow[ur, phantom, "3"] & X_7 \arrow[u, "g_3"']
\end{tikzcd}
\qquad (1)
$$

One thinks of objects $X_1, \dots, X_9$ and arrows f_i, g_j in a category — in particular, of sets $X_1, \dots, X_9$, and functions f_i, g_j from one of the sets to another, in the definite pattern fixed by the diagram.

Eventually, we will have a firm concept of this diagram, and of many others, as an entity on its own right, just as in logic a formula is an object on its own right before we interpret it, variously, in models. But, at the outset, we have to rely on interpretations to motivate the appearance of higher-dimensional diagrams.

We have the concept of an <u>interpreted</u> diagram being <u>commutative</u>. For instance, the diagram

$$\begin{tikzcd}
X_2 \arrow[r, "f_2"'] & X_3 \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] & X_4 \arrow[u, "g_2"']
\end{tikzcd}$$

now with given objects and arrows in a category (for instance, Set), being commutative means that the <u>composites</u> are equal: $f_2 \circ g_1 = g_2 \circ f_1$.

Returning to (1), imagined interpreted in a fixed category, we have:

(2) <u>if</u> the four small squares 1, 2, 3, 4 all commute <u>then</u>, as a consequence, the total square

$$\begin{tikzcd}
X_3 \arrow[r, "f_3"] & X_6 \arrow[r, "f_6"] & X_9 \\
X_2 \arrow[u, "g_4"] & X_5 & X_8 \arrow[u, "g_6"'] \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] & X_4 \arrow[r, "f_4"'] & X_7 \arrow[u, "g_3"']
\end{tikzcd}$$

also commutes: $f_6 \circ f_3 \circ g_4 \circ g_1 = g_6 \circ g_3 \circ f_4 \circ f_1$

This is a fact, one that will be seen as obvious on the basis of a minimal amount of experience with such things.

Having agreed on this fact, one asks, as a logician should,

what are the general laws behind fact (2) and its countless analogs? The answer is: the laws codified in the notion of ω-category (an ω-category being an umbrella notion of n-categories for all $n = 0, 1, 2, \ldots$)

3.2

Copies of slides for
the Aug 2005
Logic in Hungary
talk

Higher-dimensional diagrams

Aug 9/05
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"Logic in Hungary"

1. ω-categories

Ex:

$$\begin{tikzcd}
X_3 \arrow[r, "f_3"] & X_6 \arrow[r, "f_6"] & X_9 \\
X_2 \arrow[u, "g_2"] \arrow[r, "f_2"'] \arrow[ur, phantom, "②"] & X_5 \arrow[u, "g_4"] \arrow[r, "f_5"'] \arrow[ur, phantom, "④"] & X_8 \arrow[u, "g_6"'] \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] \arrow[ur, phantom, "①"] & X_4 \arrow[u, "g_3"] \arrow[r, "f_4"'] \arrow[ur, phantom, "③"] & X_7 \arrow[u, "g_5"']
\end{tikzcd}$$

① 'commutes' $\underset{def}{\Longleftrightarrow}$ $g_1 f_2 = f_1 g_3$.

(! geometric notation : $f_1 g_3 \overset{def}{=} f_1 \# g_3 \overset{def}{=} g_3 \circ f_1$
usual composition notation)

'boundary':

$$\begin{tikzcd}
X_3 \arrow[r, "f_3"] & X_6 \arrow[r, "f_6"] & X_9 \\
X_2 \arrow[u, "g_2"] & & X_8 \arrow[u, "g_6"'] \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] \arrow[uurr, phantom, "⑤"] & X_4 \arrow[r, "f_4"'] & X_7 \arrow[u, "g_5"']
\end{tikzcd}$$

Fact(*): 1, 2, 3, 4 commute $\Rightarrow$ 5 commutes.

Want: general laws behind (*) and others like it.

Answer: ω-category

Approach: insert (imaginary) process transforming one composite into the other.

"1 commutes 'via' a_1": $g_1 f_2 \xrightarrow{a_1} f_1 g_3$

& similarly for the others:

$$\begin{tikzcd}
X_3 \arrow[r, "f_3"] & X_6 \arrow[r, "f_6"] & X_9 \\
X_2 \arrow[u, "g_2"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\Downarrow a_2"] & X_5 \arrow[u, "g_4"] \arrow[r, "f_5"'] \arrow[ur, phantom, "\Downarrow a_4"] & X_8 \arrow[u, "g_6"'] \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] \arrow[ur, phantom, "\Downarrow a_1"] & X_4 \arrow[u, "g_3"] \arrow[r, "f_4"'] \arrow[ur, phantom, "\Downarrow a_3"] & X_7 \arrow[u, "g_5"']
\end{tikzcd}$$

$$\begin{tikzcd}
X_3 \arrow[r, "f_3"] & X_6 \arrow[r, "f_6"] & X_9 \\
X_2 \arrow[u, "g_2"] \arrow[rr, phantom, "\Downarrow b"] & & X_8 \arrow[u, "g_6"'] \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] & X_4 \arrow[r, "f_4"'] & X_7 \arrow[u, "g_5"']
\end{tikzcd}$$

In an ω-category:

the implication $(*)$ becomes the

composite operation

$$(a_1, a_2, a_3, a_4) \longmapsto b;$$

laws ensure that all ways of composing

a_1, a_2, a_3, a_4 lead to the same result.

ω-category has: n-cells for all $n = 0, 1, 2, 3, \dots$

E.g., the

Fact $(**)$: $\quad \mathrm{comp}(a_1, a_2, a_3, a_4) = b$

is "generalized" to a "3-cell":

$$\begin{tikzcd}
X_3 \arrow[r, "f_3"] \arrow[dr, phantom, "a_2"] & X_6 \arrow[r, "f_6"] \arrow[dr, phantom, "a_4"] & X_9 & & X_3 \arrow[r, "f_3"] & X_6 \arrow[r, "f_6"] & X_9 \\
X_2 \arrow[u, "g_2"] \arrow[r, "f_2"'] \arrow[dr, phantom, "a_1"] & X_5 \arrow[u, "g_4"] \arrow[r, "f_5"'] \arrow[dr, phantom, "a_3"] & X_8 \arrow[u, "g_6"'] & \overset{\Phi}{\Rrightarrow} & X_2 \arrow[u, "g_2"] \arrow[rr, phantom, "b"] & & X_8 \arrow[u, "g_6"'] & (*) \\
X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] & X_4 \arrow[u, "g_3"] \arrow[r, "f_4"'] & X_7 \arrow[u, "g_5"'] & & X_1 \arrow[u, "g_1"] \arrow[r, "f_1"'] & X_4 \arrow[r, "f_4"'] & X_7 \arrow[u, "g_5"']
\end{tikzcd}$$

The picture represents 2-dimensional views of a 3-dimensional geometric structure (engineering drawing)

(note the repetitions of $X_1, \dots, X_9$, $f_1, \dots, g_6$).

And so on, for diagrams of 3-cells, 4-cells, ...

DEFINITION

An ω-category $\mathbb{X}$ consists of:

for each $n = 0, 1, 2, \ldots$:

set $\mathbb{X}_n$ of n-cells

(convenient : add $\mathbb{X}_{-1} = \{*\}$).

These are related by domain (d) and codomain (c)

maps

$$\mathbb{X}_n \underset{c_n}{\overset{d_n}{\rightrightarrows}} \mathbb{X}_{n-1} \qquad (n = 0, 1, \ldots)$$

such that

$$d_n \circ d_{n+1} = d_n \circ c_{n+1}$$
$$c_n \circ d_{n+1} = c_n \circ c_{n+1} \quad .$$

(abbreviated : $d \circ d = d \circ c$
$c \circ d = c \circ c$)

$$\begin{tikzcd} dda = dca \bullet \arrow[rr, bend left, "da"] \arrow[rr, bend right, "ca"'] \arrow[rr, phantom, "\Downarrow a"] & & \bullet \; cda = cca , \end{tikzcd}$$

There are two operations :

$$a \longmapsto 1_a \qquad \text{(identity on a)}$$

and

Composition: for $a, b \in \mathbb{X}_n$, $0 \leq k < n$:

$$\underbrace{c^{(k)}a = d^{(k)}b}_{\text{"}a \#_k b \text{ is well-defined"}} \Rightarrow a \#_k b \in \mathbb{X}_n \text{ is defined}$$

("$b \circ_k a$") usual composition notation

$\underbrace{c \cdots c}_{n-k} a$; $d^{(k)} b \in \mathbb{X}_k$

and

$$d(a \#_k b) = \begin{cases} da & \text{if } k = n-1 \\ da \#_k db & \text{if } k = n-1 \end{cases}$$

and similarly for c.

Ex's:

$n=1$, $k=0$:

$$\frac{da \xrightarrow{a} ca = db \xrightarrow{b} cb}{da \xrightarrow[a \#_0 b]{} cb}$$

$n=2$, $k=0$:

$$\frac{\begin{tikzcd} d^{(0)}a \arrow[r, bend left, "da"] \arrow[r, bend right, "ca"'] \arrow[r, phantom, "\Downarrow a"] & c^{(0)}a = d^{(0)}b \arrow[r, bend left, "db"] \arrow[r, bend right, "cb"'] \arrow[r, phantom, "\Downarrow b"] & c^{(0)}b \end{tikzcd}}{\begin{tikzcd} d^{(0)}a \arrow[r, bend left, "da \#_0 db"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow a \#_0 b"] & c^{(0)}b \end{tikzcd}}$$

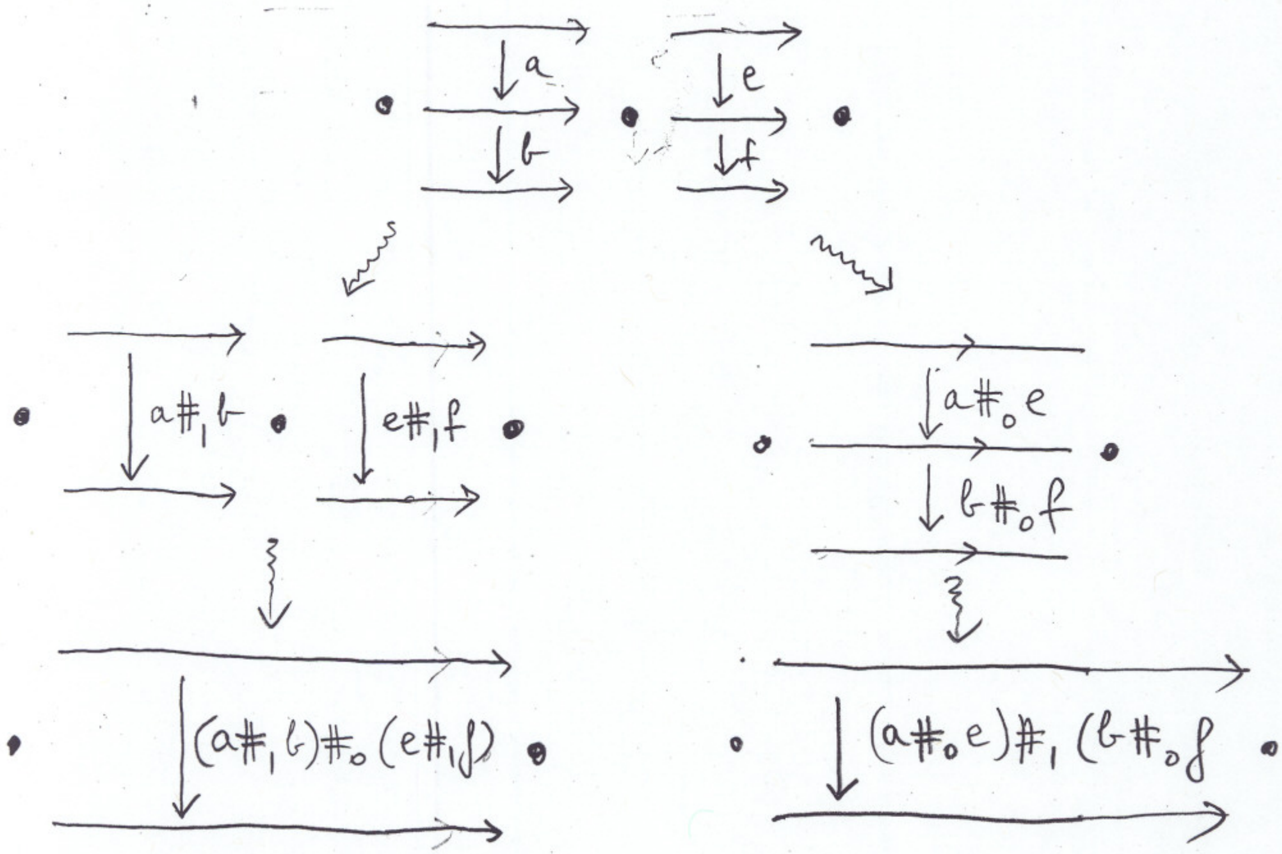
There are five further laws; the last one is:

(Middle) interchange law: [a 4-variable law!]

$$\boxed{(a \#_k b) \#_\ell (e \#_k f) = (a \#_\ell e) \#_k (b \#_\ell f)}$$

provided $k \neq \ell$, and the four simple composites are well-defined.

Ex: $n=2$, $k=1$, $\ell=0$:



Derived operations:

$a \in \mathbb{X}_m$, $b \in \mathbb{X}_n$, $0 < k < m, n$:

$N \underset{\text{def}}{=} \max(m, n)$;

$$a \#_k b \underset{\text{def}}{=} 1_a^{(N)} \#_k 1_b^{(N)}$$

(notation: $1_a^{(\ell)} = 1_{1_a^{(\ell-1)}}$ for $\ell > m$, $1_a^{(m)} = a$)

Ex: $m = 2$, $n = 1$, $k = 0$:

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow a"] & \cdot \arrow[r, "b"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "a\Downarrow"] & \cdot \arrow[r, bend left, "b"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow 1_b"] & \cdot \end{tikzcd}$$

"whiskering"

— · —

n-category : like ω-category, but without $\geq n+1$ - cells.

0-category : set
1-category : category

(small) 1-categories are the 0-cells of a 2-category 1-Cat

...

(small) n-categories are the 0-cells of an (n+1)-category n-Cat

Ex: in 1-Cat, a 2-category:

$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \end{tikzcd}$$

X, Y, Z : categories

F, G, H, K : functors

α, β : natural transformations

$\alpha \#_0 \beta \;:\; F \#_0 H \longrightarrow G \#_0 K$

natural transformation.

$\boxed{\text{Alternative definition of "}\omega\text{-category" (new!?):}}$

Use $(a \in \mathbb{X}_m, \; b \in \mathbb{X}_n) \longmapsto a \#_k b \underset{\text{def}}{=} a \cdot b$

for $k = k(a,b) \underset{\text{def}}{=} \min(m,n) - 1$ only,

together with $a \longmapsto 1_a$

as primitive.

We have:

domain/codomain laws

unit laws

and

associative	(3 variables)
distributive	(3 — " —)
commutative	(2 — " —) laws.

$a \in \mathbb{X}_m,\ b \in \mathbb{X}_n,\ c \in \mathbb{X}_p$:

associative law:

$$\boxed{a \cdot (b \cdot c) = (a \cdot b) \cdot c}$$

provided $m = n \leq p$ or $m \geq n = p$ or $m = p \leq n$

distributive laws:

$$\boxed{\begin{aligned} a \cdot (b \cdot c) &= (a \cdot b) \cdot (a \cdot c) \\ (a \cdot b) \cdot c &= (a \cdot c) \cdot (b \cdot c) \end{aligned}}$$

$(m < n,\ m < p)$

$(p < n,\ p < m)$

commutative law:

$$\boxed{(a \cdot \bar{d} b) \cdot ((\bar{c} a) \cdot b) = ((\bar{d} a) \cdot b) \cdot (a \cdot \bar{c} b)}$$

provided: for $k = k(a,b)$, $1 \leq k$ and $c^{(k-1)} a = d^{(k-1)} b$;

notation: $\bar{d} = d^{(k)}$, $\bar{c} = c^{(k)}$.

$\left.\begin{array}{l}\underline{\text{pre-normal}}\\ \underline{\text{expanded}}\end{array}\right\}$ $\underline{\text{form}}$:

$\mathbb{X}$: ω-category ; fix n ;

let $\mathcal{U} \subseteq \mathbb{X}_{n+1}$; suppose $\mathcal{U}$ generates $\mathbb{X}_{n+1}$ in the sense that $\mathbb{X}_{n+1}$ is the least set $X \subseteq \mathbb{X}_{n+1}$ such that

$$\mathcal{U} \subseteq X$$

$$b \in \mathbb{X}_n \Rightarrow 1_b \in X$$

$$a, b \in X \;\&\; a \#_k b \text{ well-defined} \Rightarrow a \#_k b \in X.$$

A $\mathcal{U}$-atom is a well-defined $(n+1)$-cell of the form

$$\boxed{b_n \cdot (b_{n-1} \cdot (\cdots (b_1 \cdot u \cdot e_1) \cdots) \cdot e_{n-1}) \cdot e_n}$$

where $b_i, e_i \in \mathbb{X}_i$ $(i = 1, \ldots, n)$

and $u \in \mathcal{U}$.

A $\mathcal{U}$-molecule is either $\boxed{1_a}$ for some $a \in \mathbb{X}_n$ or a well-defined composite

$$\boxed{\varphi_1 \cdot \varphi_2 \cdot \cdots \cdot \varphi_\ell} \quad (\ell \geq 1)$$

with each φ_i a $\mathcal{U}$-atom.

<u>Proposition</u> If $\mathcal{U}$ generates $\mathbb{X}_{n+1}$, then every $a \in \mathbb{X}_{n+1}$ is expressible as a $\mathcal{U}$-molecule.

<u>Rmk</u>: Analog of <u>disjunctive normal form</u>.

<u>Example</u>: The composite $(*)$ on p. 2, has <u>two</u> molecular expressions $(\mathcal{U} = \{a_1, a_2, a_3, a_4\})$; the two vertical composites:

$$
\begin{tikzcd}
 & g_1 g_2 f_3 f_6 \quad \leftarrow \text{1-cell} \arrow[d, "g_1 a_2 f_6 \qquad \leftarrow \text{2-cell}"] & \\
 & g_1 f_2 g_4 f_6 \arrow[dl, "a_1 g_4 f_6"'] \arrow[dr, "g_1 g_2 a_4"] & \\
f_1 g_3 g_4 f_6 \arrow[dr, "f_1 g_3 a_4"'] & & g_1 f_2 f_5 g_6 \arrow[dl, "a_1 f_5 g_6"] \\
 & f_1 g_3 g_5 g_6 \arrow[d, "f_1 a_3 g_6"] & \\
 & f_1 f_4 g_5 g_6 &
\end{tikzcd}
$$

2. Computads

Look at the diagram $(*)$ on p 3.

uninterpreted

as a single well-defined entity. It is like an uninterpreted logical expression. How should we construe this entity? Attempted answer: as a $\begin{Bmatrix} \text{3-cat} \\ (\omega\text{-cat}) \end{Bmatrix}$ freely generated by the named symbols

$$X_1, \dots, X_9,\ f_1, \dots, f_6,\ g_1, \dots, g_6,\ \Phi. \qquad (1)$$

Note, however, that the pieces of information:

$$d(a_1) = g_1 f_2, \quad c(a_1) = f_1 g_3 \qquad (2)$$

similarly for a_2, a_3, a_4, b

and, most importantly,

$$d(\Phi) = \text{composite}(a_1, a_2, a_3, a_4), \quad c(\Phi) = b \quad (3)$$

are parts of the diagram. Thus, it is not really possible to say, in one shot, that the diagram is free on the generators (1), because (2) and (3) are (further) constraints.

The correct answer is: a level-wise free 3-category, constructed in four steps:

Step 1 $\mathbb{X}^0 \overset{def}{=}$ 0-category (= set)

with elements $X_1, \dots, X_9$.

Step 2: $\mathbb{X}^1 \overset{def}{=} \mathbb{X}^0[f_1, \dots, f_6, g_1, \dots, g_6]$:

category obtained by freely adjoining

the indeterminates $f_1 : X_1 \to X_4, \dots, g_6 : X_8 \to X_9$

with prescribed domains and codomains as shown.

Step 3: $\mathbb{X}^2 = \mathbb{X}^1[a_1, a_2, a_3, a_4, b]$:

2-category obtained by freely adjoining

the indeterminates $a_1 : g_1 f_2 \to f_1 g_3, \dots, b : g_1 g_2 f_3 f_6 \to f_1 f_4 g_5 g_6$

with prescribed domains and codomains as shown;

note that this makes sense since the composites

$g_1 f_2,\ f_1 g_3,\ \dots,\ g_1 g_2 f_3 f_6,\ f_1 f_4 g_5 g_6$

are given in $\mathbb{X}^1$ already defined.

Step 4: $\mathbb{X}^3 = \mathbb{X}^2[\Phi]$: ...

— · —

Definition of computad [~~Ross~~ Street]

A computad is an ω-category $\mathbb{X}$ such that, for all $n = -1, 0, 1, 2, \ldots$, we have

$$\mathbb{X}\upharpoonright(n+1) = (\mathbb{X}\upharpoonright n)[\mathcal{U}_{n+1}] :$$

the $(n+1)^{st}$ truncation of $\mathbb{X}$ is obtained from the n^{th} truncation by adjoining some set $\mathcal{U}_{n+1}$ of indeterminate $(n+1)$-cells with prescribed domains and codomains.

Here we use the universal construction:

given $\mathbb{X}$, ω-category, and abstract set $\mathcal{U}$ with two maps

$$\mathcal{U} \underset{c}{\overset{d}{\rightrightarrows}} \underbrace{\|\mathbb{X}\|}_{\text{total set of all cells in } \mathbb{X}} \qquad (*)$$

such that $u \in \mathcal{U} \Rightarrow du \| cu$ ($ddu = dcu$ & $cdu = ccu$)

we construct $\mathbb{X}[\mathcal{U}]$, together with maps

$$\mathbb{X} \longrightarrow \mathbb{X}[\mathcal{U}]$$

$$\mathcal{U} \longrightarrow \|\mathbb{X}[\mathcal{U}]\|$$

$\mathbb{X}[\mathcal{U}]$ = ω-cat obtained from $\mathbb{X}$ by adjoining the elements of $\mathcal{U}$ subject to the constraint $(*)$.

The definition of $\mathbb{X}[U]$ is given by a universal property in the category <u>ωCat</u> of all ω-categories and their (ordinary structure-preserving) morphisms. It is similar to the one defining the polynomial ring $R[X, Y, \dots]$ out of the commutative ring R, with indeterminates $X, Y, \dots$. Also, similar to the construction of the group free on a given set of generators, or the group defined by given generators and relations.

<u>Important</u> to note that the <u>indeterminates</u> (<u>indets</u>) $U = U_0 \dot\cup U_1 \dot\cup U_2 \dot\cup \dots$ can be recovered as the (<u>indecomposable</u>) cells in the computad $\mathbb{X}$ and thus do not have to be kept track of.

<u>Thesis</u>: higher-dimensional diagram

=

computad.

$\mathbb{X}$ computad: $|\mathbb{X}| \underset{\text{def}}{=}$ set of $\underset{\text{(indets)}}{\underline{\text{indeterminates}}} = U_0 \dot\cup U_1 \dot\cup U_2 \cup \dots$

Just like a group presented by generators and relations, a computad can also be constructed as to consist of equivalence classes of suitable words formed of the generators; the equivalence in question is determined by the laws of "ω-category". Unlike the group case, now there is the issue of a word being well-formed, because of the conditional nature of the composition operations; note that the condition itself is an equality, which translates into an instance of the equivalence in a smaller dimension. The definition of the equivalence of words and the well-definedness of words is a simultaneous recursion on dimension.

$\boxed{\text{Theorem.}}$ The word problem for computads is (recursively) solvable.

The proof uses the expanded form and the content-function (see below).

Main inductive move:

$$\underbrace{\mathbb{X}\upharpoonright(n+1)}_{\mathbb{Y}} = \underbrace{(\mathbb{X}\upharpoonright n)}_{\mathbb{Z}}\,[\underbrace{U_{n+1}}_{U}]$$

$$\mathbb{Y} = \mathbb{Z}\,[U]$$

One 'reduces' equality in $\mathbb{Y}_{n+1}$ to equality in dimension n, as follows.

First, every element of $\mathbb{Y}_{n+1}$ is a U-molecule: either 1_a $(a \in \mathbb{Z}_n)$

or $\vec{\varphi} = \varphi_1 \cdots \varphi_\ell$

where each $\varphi = \varphi_i$ is a U-atom:

$$\varphi := \varphi[u] = b_n(b_{n-1}(\cdots(b_1 u e_1)\cdots)e_{n-1})e_n$$

where $b_i, e_i \in \mathbb{Z}_i$ $(i = 1, \dots, n)$

and $u \in U$.

Let $\bar{u}$ be a new n-indet with

$$d\bar{u} \stackrel{\text{def}}{=} ddu = d^{(n-1)}u$$

$$c\bar{u} \stackrel{\text{def}}{=} ccu = c^{(n-1)}u$$

& $\varphi[\bar{u}] \underset{\text{def}}{=} b_n(b_{n-1}(\cdots(b_1 \bar{u} e_1)\cdots)e_{n-1})e_n \in \mathbb{Z}[\{\bar{u}\}]$

n-computad!

Fact: $\varphi[u] = \varphi'[u]$

$$\Longleftrightarrow$$

$$\varphi[\bar{u}] = \varphi'[\bar{u}]$$

(reduction to n)

Let $\vec{\varphi} = \varphi_1 \cdots \varphi_i \cdot \varphi_{i+1} \cdots \varphi_\ell$,

$\vec{\psi} = \psi_1 \cdots \psi_i \cdot \psi_{i+1} \cdots \psi_\ell$.

Suppose, there are $\mathcal{U}$-atoms α, β such that

$$\underbrace{(\alpha \cdot d\beta)}_{=\,\varphi_i}\underbrace{(c\alpha \cdot \beta)}_{=\,\varphi_{i+1}} \overset{\checkmark}{=} \underbrace{(d\alpha \cdot \beta)}_{=\,\psi_i}\underbrace{(\alpha \cdot c\beta)}_{=\,\psi_{i+1}}$$

commutation law

or vice versa. (*) Then we write

$$\vec{\varphi} \; E_0 \; \vec{\psi}.$$

Let E be the transitive closure of E_0.

Fact: $\vec{\varphi} = \vec{\psi} \iff \vec{\varphi} \, E \, \vec{\psi}$

as elements of Ψ_{n+1} as expressions

(*) and $\varphi_j = \psi_j$ for $j \neq i, j \neq i+1$

$\mathbb{X}$: computad ; $|\mathbb{X}|$ = set of indets of $\mathbb{X}$

$\mathbb{Z}\cdot|\mathbb{X}|$ = free Abelian group on $|\mathbb{X}|$;

elements of $\mathbb{Z}\cdot|\mathbb{X}|$:

$$\sum_{\text{finite}} a_i x_i = a_1x_1 + \dots + a_nx_n = \begin{pmatrix} x_1 & \dots & x_n \\ a_1 & & a_n \end{pmatrix}$$

multiset of indeterminates;

multiplicity of x_i is a_i ($a_i \in \mathbb{Z}$;

possibly $a_i \le 0$ now!)

<u>Proposition</u> There is a (unique) function

$$\underbrace{\|\mathbb{X}\|}_{\text{all cells in } \mathbb{X}} \longrightarrow \mathbb{Z}\cdot|\mathbb{X}|$$

$a \longmapsto [a]$ = multiset of indet's in a = <u>content</u> of a

defined by:

$$[*] = 0 \qquad (* \in \mathbb{X}_{-1})$$

$$[x] = \begin{pmatrix} x \\ 1 \end{pmatrix} + [dx] + [cx] \qquad (x \in |\mathbb{X}|)$$

$$[1_a] = [a]$$

$$[a \#_k b] = [a] + [b] - [a \wedge_k b]$$

$$\left(a \wedge_k b \overset{\text{def}}{=} c^{(k)}(a) = d^{(k)}(b) \right)$$

<u>Point:</u> $[a]$ is <u>well-defined</u>!

We have:

$$[a] \geq 0$$

$$[da], [ca] \leq [a]$$

$$[a], [b] \leq [a \#_k b]$$

$$[a](x) > 0 \iff x \in \mathrm{supp}(a)$$

For any $F: \mathbb{X} \to \mathbb{Y}$ in Comp

$a \in \|\mathbb{X}\|$, $y \in |\mathbb{Y}|$:

$$[Fa]_{\mathbb{Y}}(y) = \sum_{\substack{x \in |\mathbb{X}| \\ Fx = y}} [a]_{\mathbb{X}}(x)$$

$$x \in \mathrm{supp}(a) \Rightarrow [x] \leq [a].$$

3. Type structures for computads

Example: R: commutative ring with 1

(e.g.: $R = \mathbb{Z}$: the integers)

$A, B, \dots$: denote R-modules (Abelian groups)

$(A, B) \xrightarrow{f} C$: bilinear map ($f: |A| \times |B| \to |C|$, …

$f(ar, b) = f(a, rb)$, …)

$(A, B, C) \xrightarrow{g} D$: trilinear map

Tensor product $A \otimes B$: comes with

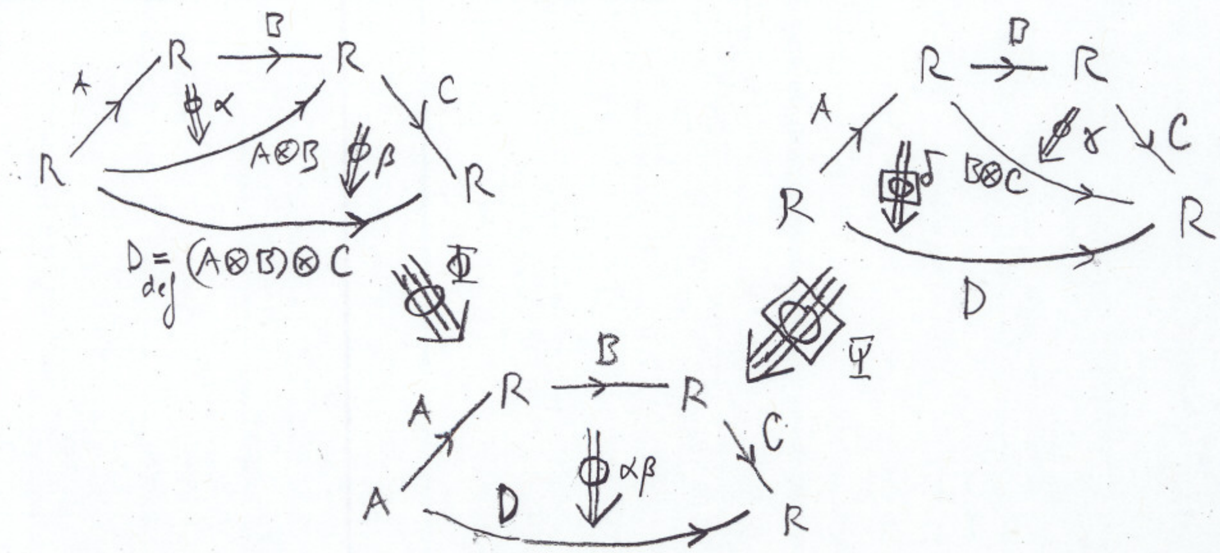
<u>universal</u> bilinear map $(A, B) \xrightarrow{\alpha} A \otimes B$

$$\begin{tikzcd}
(A,B) \arrow[r, "\alpha"] \arrow[dr, "f"'] & A\otimes B \arrow[d, dashed] \\
& C
\end{tikzcd}$$

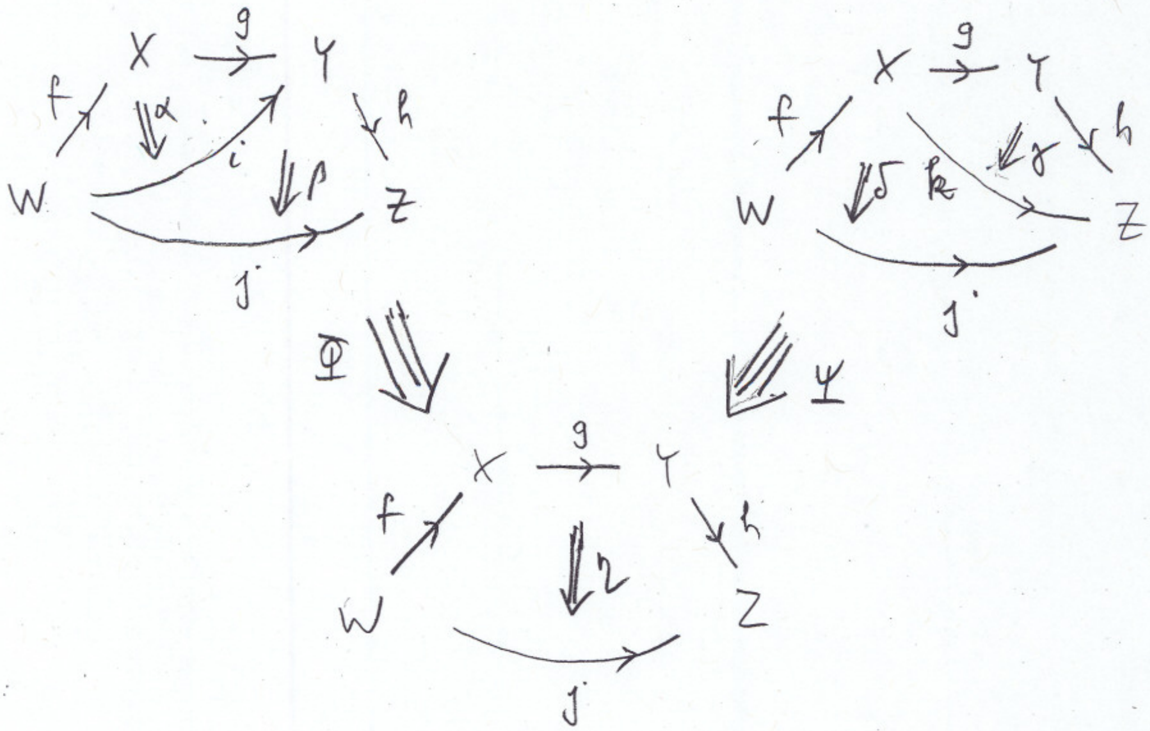
$A \otimes B$: defined up to $\cong$ <u>only</u>

<u>Fact:</u> $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

"<u>proof</u>":



A:



__Definition__ A computad $\mathbb{X}$ is many-to-one

if $x \in |\mathbb{X}|_{>0} \Rightarrow cx \in |\mathbb{X}|$.

Ex: A $\left\{\begin{matrix}\text{is}\\ \text{depicts}\end{matrix}\right\}$ a many-to-one computad.

M.M: "The multitopic cat of all multitopic cats" (1999/2003)

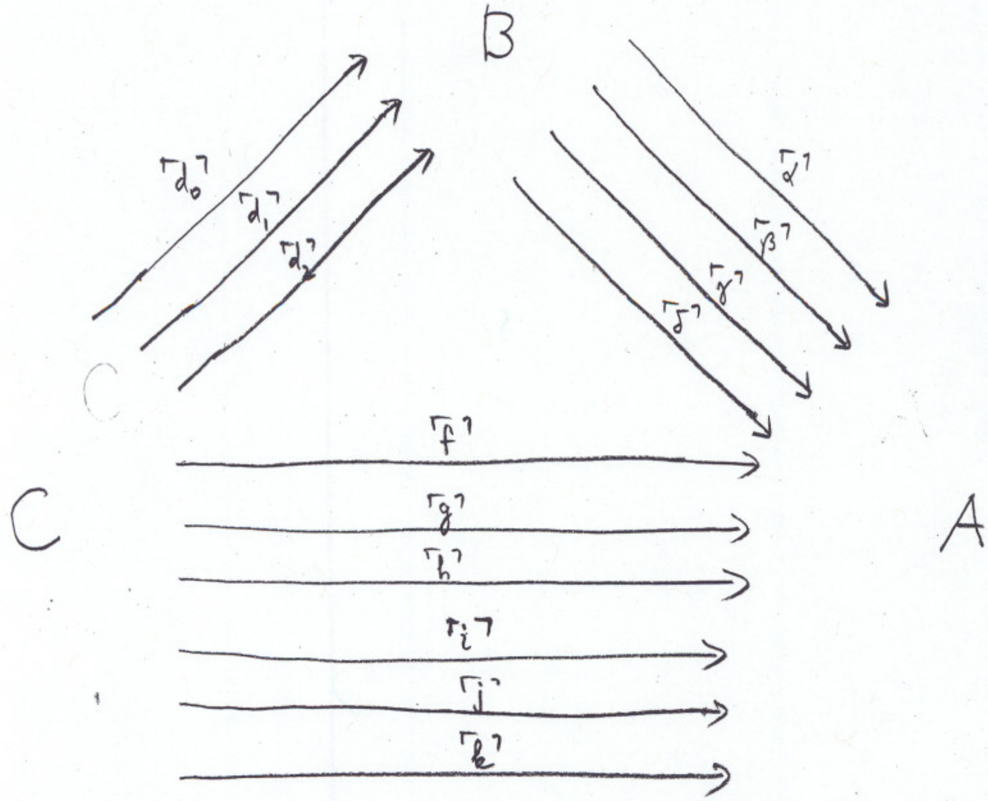
__Comp__ : the category of all small computads;

morphisms : ω-category morphisms

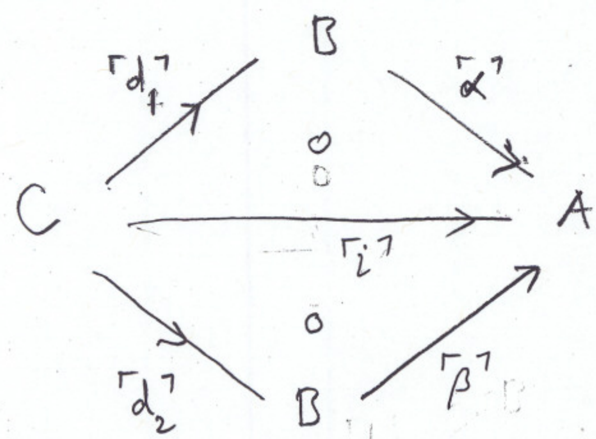
mapping indets to indets.

Ex: with

$$B :=: \begin{tikzcd} & 1 \arrow[dr, "d_0"] \arrow[d, phantom, "\Downarrow s"] & \\ 0 \arrow[ur, "d_2"] \arrow[rr, "d_1"'] & \, & 2 \end{tikzcd}, \quad C :=: \; 0 \xrightarrow{d} 1$$



and :



<u>Note</u> that of the 0-cells W, X, Y, Z one or more may coincide; the same for the 2-cells $\alpha, \beta, \gamma, \delta$ — but not for η or Φ or Ψ.

$\underline{\mathrm{Comp}}_{m/1}$: the category of many-to-one computads, a full subcategory of $\underline{\mathrm{Comp}}$.

Both $\underline{\mathrm{Comp}}$ and $\underline{\mathrm{Comp}}_{m/1}$ have a natural forgetful functor

$$\left.\begin{array}{ccc} \underline{\mathrm{Comp}} & \xrightarrow{|-|} & \mathrm{Set} \\ \mathbb{X} & \longmapsto & |\mathbb{X}| \end{array}\right\}$$

$$\left.\begin{array}{ccc} \underline{\mathrm{Comp}}_{m/1} & \xrightarrow{|-|} & \mathrm{Set} \\ \mathbb{X} & \longmapsto & |\mathbb{X}| \end{array}\right\}$$

Let $\mathbb{C}$ be a small category.

$\widehat{\mathbb{C}} \overset{\mathrm{def}}{=} \mathrm{Set}^{\mathbb{C}^{op}}$: the category of all functors $\mathbb{C}^{op} \xrightarrow{A} \mathrm{Set}$ (<u>presheaves</u> on $\mathbb{C}$) with arrows the natural transformations

$\widehat{\mathbb{C}}$ has the forgetful functor

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{|-|_{\mathbb{C}}} & \mathrm{Set} \\ A & \longmapsto & \coprod_{U \in ob(\mathbb{C})} A(U) \end{array}$$

A concrete category is a pair $(\mathbb{A}, U)$ with $U : \mathbb{A} \longrightarrow \mathrm{Set}$.

We say that the concrete categories $(\mathbb{A}, U)$, $(\mathbb{B}, V)$ are <u>equivalent</u>, $(\mathbb{A}, U) \simeq (\mathbb{B}, V)$, if there exist F and φ:

$$
\begin{tikzcd}
\mathbb{A} \arrow[rr, "F", "\simeq"'] \arrow[dr, "U"'] & \arrow[d, phantom, "\overset{\cong}{\varphi}"] & \mathbb{B} \arrow[dl, "V"] \\
& \mathrm{Set} &
\end{tikzcd}
$$

$$\varphi : U \xrightarrow[\cong]{} V \circ F$$

We are interested in when a given concrete category $(\mathbb{A}, U)$ is equivalent to some concrete presheaf category $(\widehat{\mathbb{C}}, |-|_{\mathbb{C}})$; if 'yes', the '<u>shape-category</u>' $\mathbb{C}$ is determined <u>up to isomorphism</u>.

We state an 'internal' criterion for $(\mathbb{A}, U)$ to be (equivalent to) a concrete presheaf category.

Given $(\mathbb{A}, U)$, let $\mathbb{E}$ be the "category of elements" of $(\mathbb{A}, U)$ $(\mathbb{E} = El(\mathbb{A}, U))$:

object of $\mathbb{E}$: pair (A, a)

where $A \in Ob(\mathbb{A})$, $a \in U(A)$

arrow of $\mathbb{E}$:

$$(A, a) \xrightarrow{f} (B, b)$$

$$A \xrightarrow{f} B$$

such that $\begin{cases} U(A) \xrightarrow{U(f)} U(B) \\ a \longmapsto b \end{cases}$

An object (A, a) of $\mathbb{E}$ is <u>principal</u> if it 'generated by a':

for any $(B, b) \xrightarrow{f} (A, a)$

if $B \xrightarrow{f} A$ is a monomorphism (in $\mathbb{A}$)

then f is necessarily an isomorphism.

(A, a) is <u>primitive</u> if it is principal, and every $(B, b) \xrightarrow{f} (A, a)$, with (B, b) principal, is an isomorphism.

<u>Proposition</u> $(\mathbb{A}, U)$ is a concrete presheaf category

if and only if (i) & (ii):

(i) $\mathbb{A}$ has all small colimits, U preserves them, and U is faithful. The class of isomorphism classes of primitive objects of $\mathbb{E}$ is small.

[(i) automatically holds in all cases when we are interested in the question whether (A, U) is a presheaf category.]

(ii) $\mathbb{E}$ $(= \mathrm{El}(\mathbb{A}, U))$, writing R, S, T for objects in $\mathbb{E}$:

(a) For every principal R, there exist:

primitive S

and arrow $S \longrightarrow R$.

(b) For S primitive, R principal

$$S \underset{g}{\overset{f}{\rightrightarrows}} R \quad \Rightarrow \quad f = g .$$

(c) For S, T primitive, R principal

$$\begin{tikzcd} S \arrow[dr] & \\ & R \\ T \arrow[ur] & \end{tikzcd} \quad \Rightarrow \quad S \cong T .$$

For $(\mathbb{A}, u) = (\underline{\text{Comp}}, 1\text{-}1)$,

a principal $(\mathbb{A}, a)$ has 'a' as the unique maximal dimensional indet, and all other indets in A are ones occurring in da or ca.

Can talk of A itself as principal (a is uniquely determined)

A computope is a primitive computad A (for some a, (A, a) is primitive).

For $(\mathbb{A}, U) = (\underline{\text{Comp}}_{m/1}, 1\text{-}1)$:

principal / primitive for $\underline{\text{Comp}}_{m/1}$

same as:

principal / primitive for $\underline{\text{Comp}}$ and belonging to $\underline{\text{Comp}}_{m/1}$

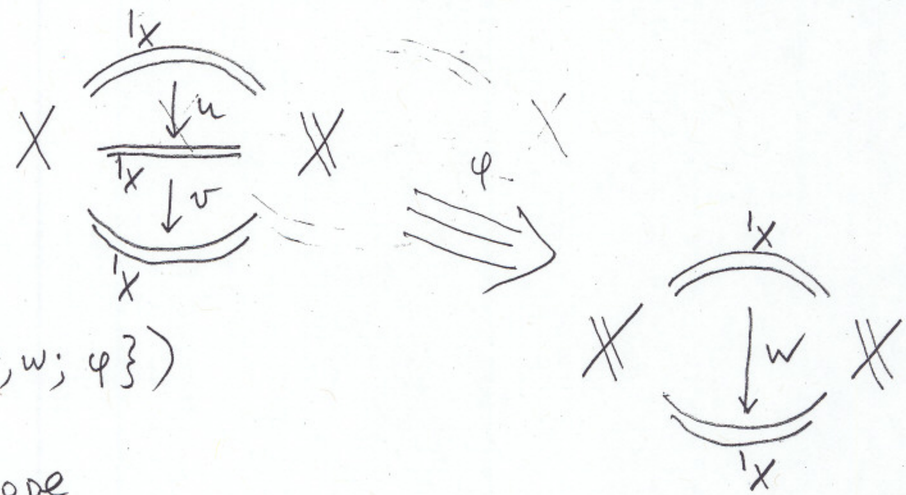
Theorem (C. Hermida, M.M., J. Power: JPAA 2000, 2001 & 2002

plus V. Harnik, M.M., M. Zawadowski: "Multitopic sets are the same as many-to-one computads" (www.math.mcgill.ca/makkai)

$\underline{\mathrm{Comp}}_{m/1}$ is a concrete presheaf category.

Example (M. Zawadowski, M.M.): the computad (not m-to-1)

$A :=$



$(|A| = \{X; u, v, w; \varphi\})$

is a computope

and it has a non-trivial automorphism $A \longrightarrow A$

$\therefore$ Comp is not a concrete presheaf category:
(ii)(b) is violated

$\boxed{\text{Theorem}}$ (ii)(a) holds for Comp: for every principal A there is $B \twoheadrightarrow A$, B primitive.

proof uses expanded form and content (as above)

(3.3)

Seminar talk on:

"The Word Problem for Computers"

paper

& "Logic in Hungary" talk

[('Sec II') because ('Sec I') is:

item 3.2, slides for the

Logic in Hungary talk]

$\boxed{\text{Sec II.1}}$ $\mathbb{Y}$: ω-category

Y_n : n-cells in $\mathbb{Y}$

$y \in Y_n$ is INDECOMPOSABLE if

1) $y \neq 1_b \quad \forall b \in Y_{n-1}$

2) $b, e \in Y_n$, $k < n$, $y = b \#_k e$

$$\Longrightarrow$$

$b = 1_a^{(n)}$ or $e = 1_a^{(n)}$

for some $a \in Y_k$.

$\mathbb{Y}$ is a COMPUTAD iff the set

$$\boxed{|\mathbb{Y}| = \{\text{all indecomposables in } \mathbb{Y}\}}$$

freely generates $\mathbb{Y}$;

'generates' (without 'freely') means:

the scheme

$$\ddot{y} \;\Big|\; 1_a \;\Big|\; a \#_k b$$

$y \in |\mathbb{Y}|$

generates all elements (cells) of $\mathbb{Y}$.

Morphism of computads:

$$\mathbb{X} \xrightarrow{F} \mathbb{Y}$$

$$x \in |\mathbb{X}| \Rightarrow Fx \in |\mathbb{Y}|$$

$$\text{indet} \longmapsto \text{indet}$$

$\shortparallel$

indecomparable

Given $\mathbb{X} \xrightarrow{F} \mathbb{Y}$, we get

$$|\mathbb{X}| \xrightarrow[|F|]{} |\mathbb{Y}|$$, a Set-map.

F is $\left\{\begin{matrix}\text{mono}\\ \text{epi}\\ \text{iso}\end{matrix}\right\}$ iff $|F|$ is $\left\{\begin{matrix}\text{mono}\\ \text{epi}\\ \text{iso}\end{matrix}\right\}$

Every subobject of (computad) $\mathbb{Y}$ is represented by a unique monomorphism which is an <u>inclusion</u>

$$|\mathbb{X}| \hookrightarrow |\mathbb{Y}|$$

on the level of indets (and also as a map $\|\mathbb{X}\| \longrightarrow \|\mathbb{Y}\|$) : subcomputad.

For $a \in \|\mathbb{Y}\|$ = set of all (cells) of ω-cat $\mathbb{Y}$:

$$\boxed{\operatorname{supp}_{\mathbb{Y}}(a)} \overset{\text{def}}{=} |\mathbb{X}|$$

for the least <u>subcomputad $\mathbb{X}$</u> of $\mathbb{Y}$ containing a; $\boxed{\mathbb{X} = \operatorname{Supp}_{\mathbb{Y}}(a)}$.

A subset $U \subseteq |\mathbb{Y}|$ is

$$U = |\mathbb{X}|$$

for a subcomputad $\mathbb{X}$ of $\mathbb{Y}$ iff

$$u \in U \implies \operatorname{supp}_{\mathbb{Y}}(u) \subseteq U.$$

Abbreviation:

$$a . b = a \#_k b$$

for k 'maximal possible',

i.e. $k = \min(\dim(a), \dim(b)) - 1$

Vocabulary (definitions of common terms)

1) (higher-dimensional) diagram

=

computad

2) pasting diagram (higher dimensional diagram that pasts (composes) in a definite way)

= pair

(A, a) s.t. $A = \mathrm{Supp}_A(a)$

computad

$(a \in \|A\|)$

pd = pasting diagram

3) <u>composable</u> <u>diagram</u>

= diagram (computad) A

such that $\exists$ a for which

(A, a) is a pd

4) <u>uniquely</u> <u>composable</u> <u>diagram</u>

A with unique a

s.t (A, a) is a pd.

A map of pd's:

$$(A, a) \xrightarrow{F} (B, b)$$

$$A \longrightarrow B$$

$$a \longmapsto b$$

Def'n: pd (B, b) is UNFOLDED if

for all $(A, a) \xrightarrow{F} (B, b)$,

map of pd's,

F is necessarily an isomorphism.

Examples: Ex1) (B, b) with B on p. I.1

b on p. I.2

Ex2) (A, Φ) : p. I.3

Ex3) (C, φ) : p. I.25

Ex4) : next page

Ex 4) Small Power example :

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\varphi", pos=0.3] & Z \arrow[d, "h"] \\ X \arrow[ur, "j"] \arrow[r, "i"'] \arrow[ur, phantom, "\Downarrow\psi", pos=0.7] & W \end{tikzcd} \quad \overset{\Phi}{\Rrightarrow} \quad \begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Leftarrow\sigma", pos=0.3] & Z \arrow[d, "h"] \arrow[dl, "k"] \\ X \arrow[r, "i'"'] \arrow[ur, phantom, "\Leftarrow\tau", pos=0.7] & W \end{tikzcd}$$

---

5) Computad A is <u>positive</u> if for all $x \in |A|$, dx and cx are not identities (1_b for some b)

A pd (A, a) is <u>positive</u> if A is.

<u>Conjecture</u> Any positive (unfolded) computad A, if composable, is uniquely composable

Ex's: 1) Both 'positive' and 'unfolded' are needed:

$$\begin{tikzcd} X \arrow[r, bend left, dashed, "1_X"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow u"] & X \end{tikzcd}$$

$|A| = \{X, f, u\}$

(A, u) : unfolded but not positive

$a = u$ & $b = u \cdot f$ have

$\mathrm{supp}_2(a) = \mathrm{supp}_2(b) = \{u\}$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f^2"'] \arrow[r, phantom, "u \cdot f \Downarrow"] & X \end{tikzcd}$$

2)

$$\begin{tikzcd} X \arrow[loop below, "f"] \end{tikzcd}$$

positive but not unfolded

$a = f$ and $b = f^2$ have

$\mathrm{supp}_1(a) = \mathrm{supp}_1(b) = \{f\}$

II. 6

An unfolded pd (A, a)

in which a is an indeterminate

is called a COMPUTOPE.

$\boxed{\text{Theorem}}$ Every pd can be unfolded:

for every pd (B, b), there is

an unfolded one (A, a), with

a map

$$(A, a) \xrightarrow{F} (B, b)$$

$\boxed{\text{Problem}}$: Is (A, a) unique up to iso?

Note: F is not unique:

ex 3) has a non-trivial

automorphism.

Rmk: this is one of the "normal forms" of the title ('unfolding') reference to "Lin H" talk

Repeat: I. 16.3, 16.4

$|a|$: natural content

defined similar, except:

$$|x| = \binom{x}{1} + E(x)$$

where, for any a,

$$E(a) \overset{\text{def}}{=} |da| + |ca| - |d^2 a| - |c^2 a| + |d^3 a| + |c^3 a| - \cdots$$

This is well-defined, but from the facts on p. I. 16.4, only one remains true:

$$|Fa|_{\mathbb{Y}}(y) = \sum_{\substack{x \in |\mathbb{X}| \\ Fx = y}} |a|_{\mathbb{X}}(x)$$

<u>Natural</u> multiplicity

of indeterminate x in pd (A, a):

$$\underset{\text{def}}{=} |a|(x).$$

<u>Note</u>: If $x \notin \mathrm{supp}(a)$, then $|a|(x) = [a](x) = 0$.

Unfolded pd (A, a) is called

<u>spherical</u>

if

$$x \in \mathrm{Supp}(a) \Rightarrow |a|(x) = 1$$

<u>Note</u>: 'unfolded' 'should' mean that

each indeterminate that occurs at all

occurs exactly once.

Ex's 1), 2) & 4) are spherical

but 3) is not; in fact, <u>u is not</u> (in 3))

$$\begin{tikzcd} X \arrow[r, bend left, dashed, "1_X"] \arrow[r, bend right, dashed, "1_X"'] \arrow[r, phantom, "\Downarrow u"] & X \end{tikzcd}$$: a pd

(a computope in fact)

$$|u| = \binom{u}{1} + |du| + |cu| - |d^2u| - |c^2u|$$

$$= u + X + X - X - X$$

$$= u$$

Thus, $X \in \mathrm{Supp}(u)$, but $|u|(X) = 0$,

rather than $= 1$.

It can get negative too ...

<u>Proposition</u> (partial sphericalness)

(A, a) unfolded pd, $\dim(a) = n$,

$x \in |a|$, $\dim(x) = n$

$$\Longrightarrow \quad |a|(x) = 1$$

Def'n:

(A, a) is an atom if, for

$n = \dim(a)$, $\underbrace{\mathrm{supp}(a) \cap \underbrace{|A|_n}_{n\text{-dim indet's}}}_{\mathrm{Supp}_n(a)} = \{x\}$

a singleton.

Proposition (second 'normal form' of title)

1) Every atom a, $\dim(a) = n+1$, is of the form:

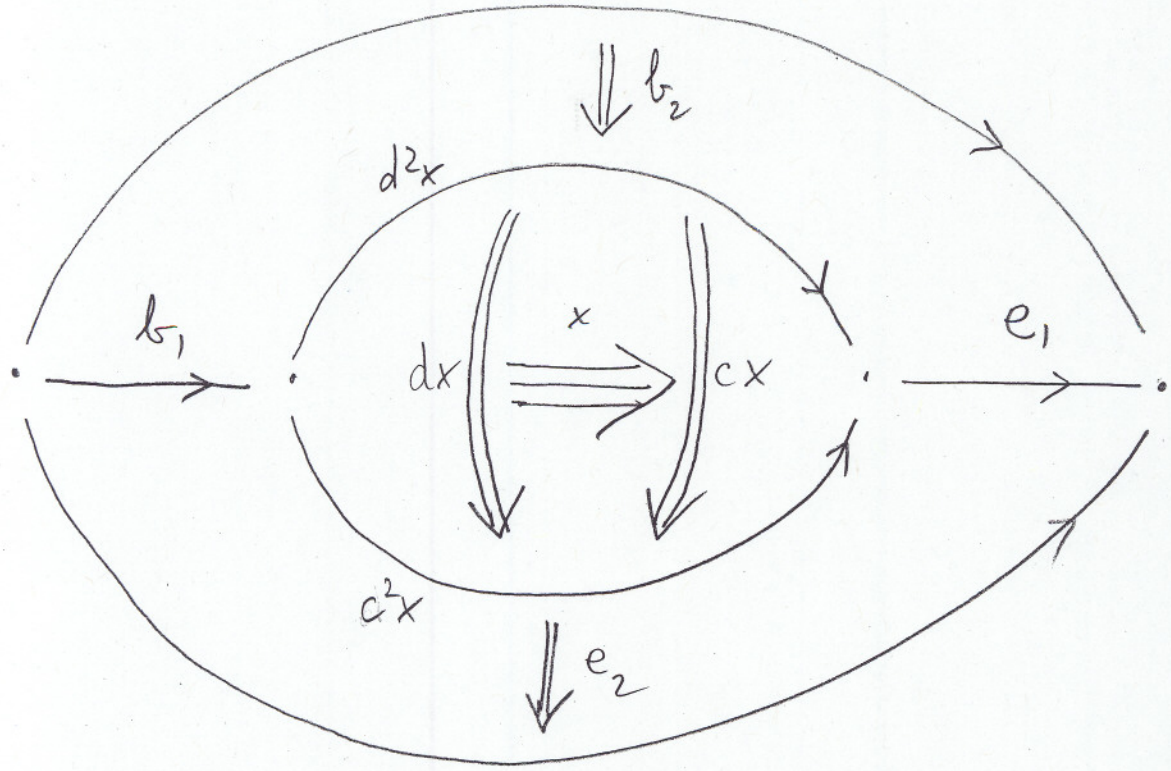
$$a = b_n(b_{n-1} \cdots (b_1\, x\, e_1) \cdots e_{n-1})\, e_n$$

where x is an indet of dim $= n+1$,

and b_i, e_i are of dim $= i$.

$n+1 = 5$:

$n+1 = 3$:



2) Every $(n+1)$-dimensional pd is of the form

$$a_1 \# a_2 \circ \cdots \circ a_k$$

with $k \geq 0$ & each a_i an atom of dim $= n+1$ (for $k=0$, we mean 1_b for some b of dim $= n$)

$\boxed{\mathrm{II}.12 = \mathrm{II}.13 = \mathrm{II}.14}$

I'll produce a
non-spherical
unfolded
positive pd,
the 'large' Power example

Let: A computad: ('cell' = indet)

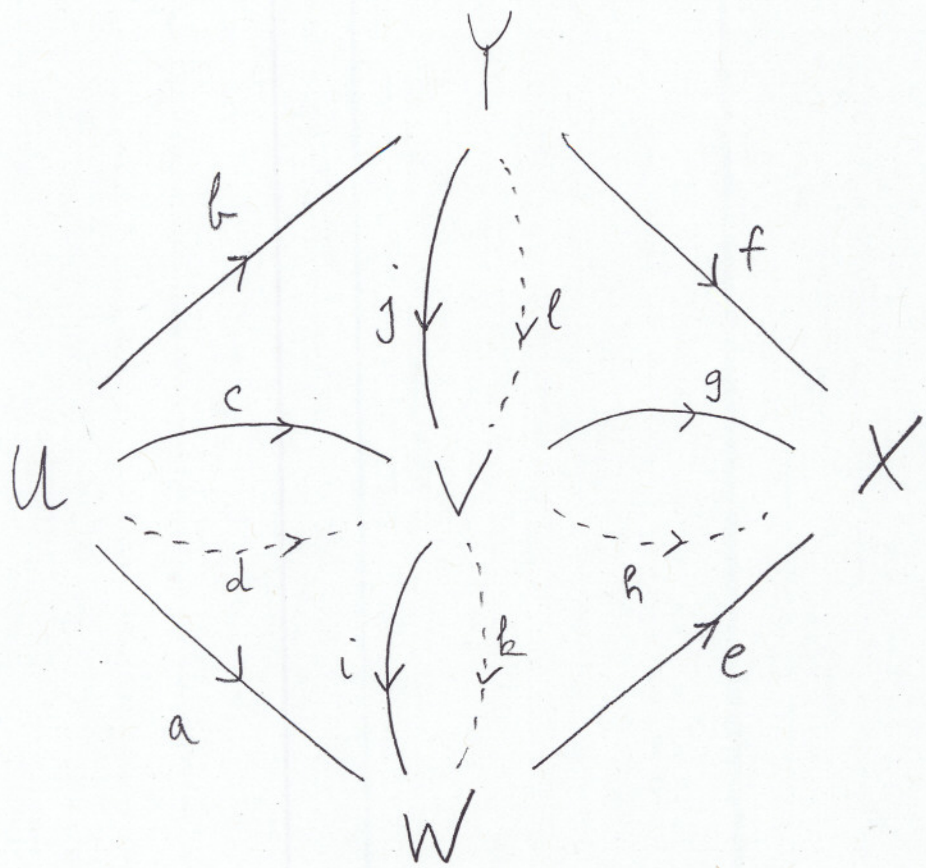
0-cells: u, V, W, X, Y (5)

1-cells: $a, b, c, d, e, f, g, h, i, j, k, l$ (12)

2-cells: $\alpha, \beta, \gamma, \delta, \varepsilon, \kappa, \lambda, \mu, \nu, \rho, \sigma, \theta, \tau, \omega$ (14)

3-cells: A, B, C, D, E, F, G, I (8)

0- and 1- cells:



2-cells:

uppers:	$a \xrightarrow{\alpha} \bar{c}\bar{i}$	$\bar{c} \xrightarrow{\beta} b\bar{j}$	$\bar{j}\bar{g} \xrightarrow{\nu} f$	$\bar{i}e \xrightarrow{\mu} \bar{g}$
crossers:	$a \xrightarrow{\gamma} \underline{d}\bar{i}$	$\bar{c} \xrightarrow{\varepsilon} b\underline{l}$	$\bar{j}\underline{h} \xrightarrow{\theta} f$	$\underline{k}e \xrightarrow{\rho} \bar{g}$
lowers:	$a \xrightarrow{\kappa} \underline{d}\,\underline{k}$	$\underline{d} \xrightarrow{\lambda} b\underline{l}$	$\underline{l}\,\underline{h} \xrightarrow{\omega} f$	$\underline{k}e \xrightarrow{\tau} \underline{h}$

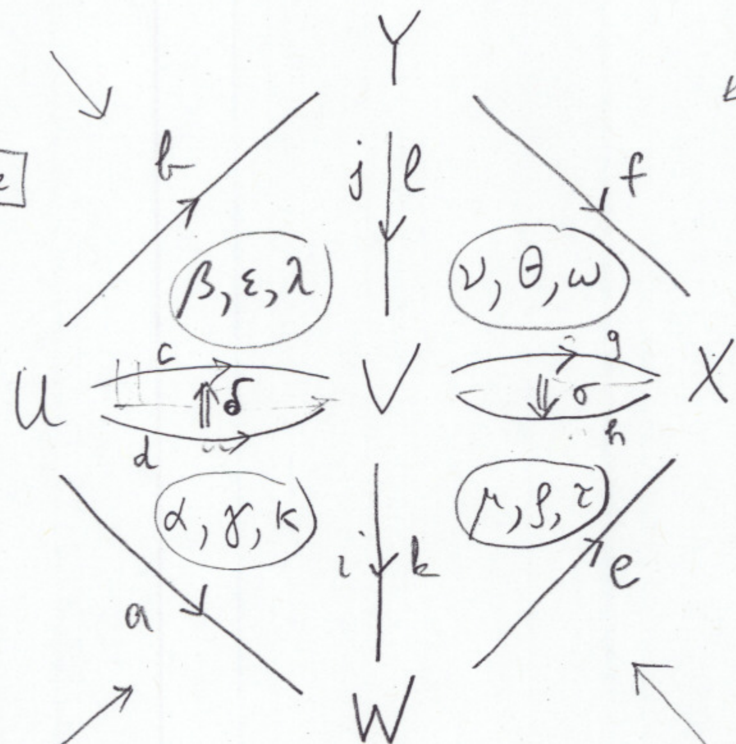
3-cells:

$\alpha\varepsilon \xrightarrow{A} \gamma\lambda$	$\rho\sigma \xrightarrow{E} \tau$
$\alpha \xrightarrow{B} \gamma\delta$	$\nu \xrightarrow{F} \sigma\theta$
$\delta\varepsilon \xrightarrow{C} \lambda$	$\gamma\mu \xrightarrow{G} \kappa\rho$
$\rho\nu \xrightarrow{D} \tau\theta$	$\rho\theta \xrightarrow{I} \varepsilon\omega$

Schematic:

$c, j : \beta$
$c, \ell : \varepsilon$
$d, \ell : \lambda$
$\boxed{d, j : \text{none}}$

$j, g : \nu$
$j, h : \theta$
$\boxed{\ell, g : \text{none}}$
$\ell, h : \omega$



$c, i : \alpha$
$\boxed{c, k : \text{none}}$
$d, i : \gamma$
$d, k : \kappa$

$i, g : \mu$
$\boxed{i, h : \text{none}}$
$k, g : \rho$
$k, h : \tau$

'Long' 2-molecules: of the form $\boxed{ae \Rightarrow bf}$

1. $\alpha\mu\nu\beta$
2. $\kappa\tau\omega\lambda$
3. $\gamma\delta\mu\nu\beta$
4. $\kappa\delta\rho\nu\beta$
5. $\kappa\delta\tau\theta\beta$
6. $\kappa\delta\tau\omega\varepsilon$
7. $\mu\alpha\beta\sigma\theta$
8. $\mu\alpha\varepsilon\sigma\omega$
9. $\mu\gamma\lambda\sigma\omega$
10. $\rho\kappa\lambda\sigma\omega$
11. $\gamma\mu\theta\beta\delta\sigma$
12. $\kappa\rho\theta\beta\delta\sigma$
13. $\gamma\mu\omega\varepsilon\delta\sigma$
14. $\kappa\rho\omega\varepsilon\delta\sigma$

II.19

3 atoms:

$$ae \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} bf$$

$\boxed{8} \xrightarrow{A} \boxed{9}$

$\boxed{1} \xrightarrow{B} \boxed{3}$

$\boxed{7} \xrightarrow{B} \boxed{11}$

$\boxed{8} \xrightarrow{B} \boxed{13}$

$\boxed{6} \xrightarrow{C} \boxed{2}$

$\boxed{13} \xrightarrow{C} \boxed{9}$

$\boxed{14} \xrightarrow{C} \boxed{10}$

$\boxed{4} \xrightarrow{D} \boxed{5}$

$\boxed{14} \xrightarrow{E} \boxed{6}$

$\boxed{10} \xrightarrow{E} \boxed{2}$

$\boxed{12} \xrightarrow{E} \boxed{5}$

$\boxed{1} \xrightarrow{F} \boxed{7}$

$\boxed{3} \xrightarrow{F} \boxed{11}$

$\boxed{14} \xrightarrow{F} \boxed{12}$

$\boxed{3} \xrightarrow{G} \boxed{4}$

$\boxed{11} \xrightarrow{G} \boxed{12}$

$\boxed{13} \xrightarrow{G} \boxed{14}$

$\boxed{9} \xrightarrow{G} \boxed{10}$

$\boxed{7} \xrightarrow{I} \boxed{8}$

$\boxed{5} \xrightarrow{I} \boxed{6}$

$\boxed{12} \xrightarrow{I} \boxed{14}$

$\boxed{11} \xrightarrow{I} \boxed{13}$

$$\boxed{\begin{aligned} &(2\times1) + (4\times3) + (2\times4) \\ &= 2+12+8 \\ &= 22 \end{aligned}}$$

φ: atom of dim 3 such that

$$d^2\varphi = ae$$

$$c^2\varphi = bf$$

$$ae \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow\varphi}} bf$$

Picture:

$$
\begin{tikzcd}
\boxed{1} \arrow[rr, "F"] \arrow[dr, "B"'] & & \boxed{7} \arrow[rr, "I"] \arrow[dr, "B"'] & & \boxed{8} \arrow[rr, "A"] \arrow[dr, "B"'] \arrow[rr, phantom, "\Downarrow \Phi"'] & & \boxed{9} \arrow[rr, "G"] & & \boxed{10} \arrow[rr, "E"] & & \boxed{2} \\
& \boxed{3} \arrow[rr, "F"'] \arrow[drr, "G"'] & & \boxed{11} \arrow[rr, "I"'] \arrow[drr, "G"] & & \boxed{13} \arrow[rr, "G"'] \arrow[ur, "C"'] & & \boxed{14} \arrow[rr, "E"'] \arrow[ur, "C"] & & \boxed{6} \arrow[ur, "C"'] & \\
& & & \boxed{4} \arrow[rr, "F"] \arrow[drr, "D"'] & & \boxed{12} \arrow[urr, "I"'] \arrow[d, "E"] \arrow[dl, phantom, "\Downarrow \Sigma"] & & & & & \\
& & & & & \boxed{5} \arrow[uurrrr, "I"'] & & & & &
\end{tikzcd}
$$

4-atoms:

$$\hat{\Phi} \overset{def}{=} F\, I\, \Phi\, G\, E$$

$$\hat{\Sigma} \overset{def}{=} B\, G\, \Sigma\, I\, C$$

$d\,\hat{\Phi} =$ F I A G E

$c\,\hat{\Phi} =$ F I B C G E

$d\,\hat{\Sigma} =$ B G F E I C

(=) !

$c\,\hat{\Sigma} =$ B G D I C

$\hat{\Phi} \cdot \hat{\Sigma}$ is well-defined.

Look at the domain and the codomain of the 4-pd

$\hat{\Phi} \circ \hat{\Sigma}$:

F I A G E & B G D I C

They share the 3-cells I & G

Now, let :

$$\Phi' \parallel \Phi$$

$$\Sigma' \parallel \Sigma$$

and let

5-cell : $\quad X : \hat{\Phi} \cdot \hat{\Sigma} \longrightarrow \hat{\Phi}' \cdot \hat{\Sigma}'$

Calculate :- but only on 3-cells:

$$|X| \overset{(3)}{\equiv} (X+) \, |\hat{\Phi} \cdot \hat{\Sigma}| + |\hat{\Phi}' \cdot \hat{\Sigma}'|$$

$$- \; |d(\hat{\Phi} \cdot \hat{\Sigma})| - |c(\hat{\Phi}' \cdot \hat{\Sigma}')|$$

$$\overset{(3)}{\equiv} \; 2\,|\hat{\Phi} \cdot \hat{\Sigma}| - |d(\hat{\Phi} \cdot \hat{\Sigma})| - |c(\hat{\Phi} \cdot \hat{\Sigma})|$$

$$\overset{(J)}{\equiv} 2(|\hat{\Phi}| + |\hat{\Sigma}| - |\hat{\Phi} \wedge \hat{\Sigma}|) - |d\,\hat{\Phi}| - |c\,\hat{\Sigma}|$$

$$\overset{(G)}{\equiv} \; 2(1 + 1 - 1) - 1 - 1 = 0$$