1.1 (ω-cats and computads)

October 2004

Octoberfest talk

Oct 16/04 (Oktoberfest) 0

Quote from Bertrand Russell

'Logical Atomism' (1924)

(in: 'The Philosophy of Logical Atomism', pp 157-181)

"When I speak of 'simples', I ought to explain that I am speaking of something not experienced as such, but known only inferentially as the <u>limit of analysis</u>. It is quite possible that, by greater logical skill, the need for assuming them could be avoided."

...

But I confess it seems obvious to me (as it did to Leibniz) that what is complex must be composed of simples, though the number of constituents may be infinite."

(p 173)

'Concrete' 2-category $\mathcal{W}$ $(= \mathcal{W}^{(2)})$

0-cells: categories $\mathbb{X}, \mathbb{Y}, \dots$

1-cells: functors $\mathbb{X} \xrightarrow{F} \mathbb{Y}$, $\mathbb{X} \xrightarrow{G} \mathbb{Y}, \dots$

2-cells: natural transformations

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow h"] & \mathbb{Y} \end{tikzcd}$$

Concreteness, further:

objects X of $\mathbb{X}$, $X \in \mathbb{X}$ are (structured) sets;

arrows $X \xrightarrow{f} Y$ are (structure preserving) functions

$\mathbb{X} \in \mathcal{W}$ $\qquad$ $\mathbb{X}$: $\underline{0}$-cell of $\mathcal{W}$

$X \in \mathbb{X}$ $\qquad$ $\mathbb{1} \xrightarrow{\ulcorner X \urcorner} \mathbb{X}$ $\quad$ (special) $\underline{1\text{-cell}}$

$x \in X$ $\qquad$ $\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "\ulcorner 1_X \urcorner"] \arrow[r, bend right, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\Downarrow \ulcorner x \urcorner"] & \mathbb{X} \end{tikzcd}$ $\quad$ (special) $\underline{2\text{-cell}}$

$x \in X$: atoms ; the only primitive relation on elements of X is equality: $=_X$, a special case of equality of 2-cells in the 2-category $\mathcal{W}$.

—·—

May 'continue' the analysis of the 'world':

'atoms' are seen not to be indivisible:

they have 'parts' or 'structure'.

We 'resolve' $\mathcal{W}$ into a 3-dim category

$$\mathcal{W}^{(3)}$$

whose 2-truncation is identical to $\mathcal{W}^{(2)}$

and has (non-trivial)

3-cells:
$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "h\Downarrow \xrightarrow{\varphi} \Downarrow k"] & \mathbb{Y} \end{tikzcd}$$

Regarded as a 'concrete' 3-category

0-cells: 2-categories $\mathbb{X}, \mathbb{Y}, \dots$

1-cells: 2-functors $\mathbb{X} \xrightarrow{F} \mathbb{Y}$, $\mathbb{X} \xrightarrow{G} \mathbb{Y}, \dots$

2-cells: 2-natural transformations
$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow h"] & \mathbb{Y} \end{tikzcd}$$

3-cells: modifications
$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "h\Downarrow \xrightarrow{\varphi} \Downarrow k"] & \mathbb{Y} \end{tikzcd}$$

'Element'

$\mathbb{X} \in \mathbb{W}^{(3)}$ $\qquad$ $\mathbb{X}$: 0-cell of $\mathbb{W}^{(3)}$

$X \in \mathbb{X}$ $\qquad$ $\mathbb{1}_{\mathbb{X}} \xrightarrow{\ulcorner X \urcorner} \mathbb{X}$ $\qquad$ 1-cell

$x \in X$ $\qquad$ $\begin{tikzcd} \mathbb{1}_{\mathbb{W}} \arrow[r, bend left, "\ulcorner 1_{\mathbb{X}} \urcorner"] \arrow[r, bend right, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\Downarrow \ulcorner x \urcorner"] & \mathbb{X} \end{tikzcd}$ $\qquad$ 2-cell

$\xi \in x$ $\qquad$ $\begin{tikzcd} \mathbb{1} \arrow[r, bend left=60, "\ulcorner 1_{\mathbb{X}} \urcorner"] \arrow[r, bend right=60, "\ulcorner X \urcorner"'] \arrow[r, phantom, "\Downarrow\ulcorner 1_X \urcorner \xrightarrow{\ulcorner \xi \urcorner} \Downarrow\ulcorner x \urcorner"] & \mathbb{X} \end{tikzcd}$ $\qquad$ (special) 3-cell

now, the atoms: the ξ's.

In terms of "atomism", logical or physical, the analysis of the complex (Bertrand Russell's term) $\mathbb{W}$ yields successively "simpler" complexes:

$$\mathbb{W} \ni \mathbb{X} \ni X \ni x \ni \xi$$

and we may conclude that ξ is a simple (again: Russell's usage of the word as a noun).

From the categorical point of view:

the simpler is of higher dimension:

$$\mathcal{W} \ni \mathbb{X} \ni X \ni x \ni \xi$$

→ dimensions: $0 \quad 1 \quad 2 \quad 3$

ⓐ ω-graph $\mathbb{X}$

(·) $n \in \mathbb{N}$: $\mathbb{X}_n$ set of n(-dimensional) cells

conventionally $\mathbb{X}_{-1} = \{*\}$

(··) operations d (domain) and c (codomain):

$$\mathbb{X}_n \underset{c}{\overset{d}{\rightrightarrows}} \mathbb{X}_{n-1} \qquad (n \geq 0)$$

$$(d = d_n,\ c = c_n)$$

(···) subject to:

$$\mathbb{X}_{n+1} \underset{c}{\overset{d}{\rightrightarrows}} \mathbb{X}_n \underset{c}{\overset{d}{\rightrightarrows}} \mathbb{X}_{n-1}$$

$$dd = dc$$
$$cd = cc$$

notation: $s \xrightarrow{a} t$ means : $s = da,\ t = ca$

we have:

$$\begin{tikzcd} dda = dca \arrow[r, bend left=60, "da"] \arrow[r, bend right=60, "ca"'] \arrow[r, phantom, "\Downarrow a"] & cda = cca \end{tikzcd}$$

Fix ω-graph $\mathbb{X}$; $a, b, e, f, \ldots$: cells of $\mathbb{X}$ ($\in \bigsqcup_{n\in\mathbb{N}} \mathbb{X}_n$) ; write $a \in \mathbb{X}$.

$$a \in \mathbb{X}_m \underset{def}{\iff} \dim(a) = m$$

Convention in what follows:

$$\begin{array}{ccc} (m) & (n) & (p) \\ a & b & e \end{array} \quad \leftarrow \text{dimensions}$$

An ω-category is an ω-graph $\mathbb{X}$ together with:

① Operations:

∴ (identity) $\overset{(m)}{a} \longmapsto a \xrightarrow{\overset{(m+1)}{1_a}} a$

∴∴ partial operation of composition:

$$a, b \longmapsto a \cdot b \quad (= ab);$$

defined when: $m, n \geq 1$ and, with $\boxed{k \underset{def}{=} \min(m,n) - 1}$

$$c^{(k)} a = d^{(k)} b$$

$$\underbrace{\underset{m-k}{c} \circ\circ\circ \underset{1}{c}}(a) \qquad \underbrace{\underset{n-k}{d} \circ\circ\circ \underset{1}{d}}(b)$$

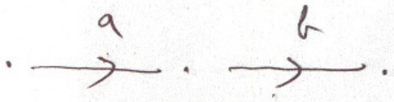
! Write $a \wedge b$ or even $a \wedge_k b$ for this

(under the condition!)

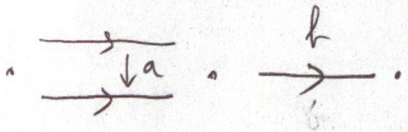
notation: $k = k(a,b)$

with: $\dim(ab) = \max(m,n)$

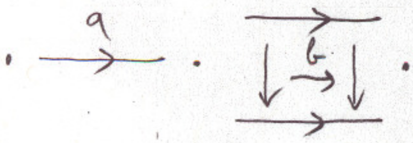
Examples:



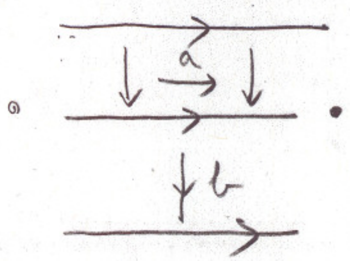
$m = n = 1;\ k = 0$



$m = 2,\ n = 1;\ k = 0$



$m = 1,\ n = 3;\ k = 0$



$m = 3,\ n = 2;\ k = 1$

"Non-example":

(*)

Here $m = n = 2$; but $k = 1$, not 0.

$c^{(0)}a = d^{(0)}b$, but $c^{(k)}a \neq d^{(k)}b$.

The 'composition' in (*) will be a <u>derived operation</u>.

(a) Laws:

(..) (domain / codomain)

$$\boxed{d(ab) = \begin{cases} (da)\,b & \text{if } m>n \\ a\,(db) & \text{if } m<n \\ da & \text{if } m=n \end{cases}}$$

(It follows: $d^{(k)}(ab) = d^{(k)}(a)$ if $k < m, n$

$d^{(k)}(ab) = d^{(k)}(a) = d^{(k)}(b)$ if $k < m-1, n-1$)

and <u>similarly for c in place of d</u>.

Ex's:

$m>n$: $$\begin{tikzcd} \cdot \arrow[r, bend left, "da"] \arrow[r, bend right, "ca"'] \arrow[r, phantom, "a\Downarrow"] & \cdot \arrow[r, "b"'] & \cdot \end{tikzcd} \;:=\; \begin{tikzcd} \cdot \arrow[r, bend left, "(da)b"] \arrow[r, bend right, "(ca)b"'] \arrow[r, phantom, "ab\Downarrow"] & \cdot \end{tikzcd}$$

$m<n$: $$\begin{tikzcd} \cdot \arrow[r, "a"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow b"] & \cdot \end{tikzcd}$$

$m=n$: $da \to ca = db \to cb \;:=\; da \xrightarrow{ab} cb$

annotation: see p.7/Jan8/05 | plus: pp4&5/ibid

annot.: pp 4/5 / Jan 8/05

$\odot$ (<u>identity</u>)

$$\boxed{a \cdot 1_b := \begin{cases} a & \text{if } m \geqslant n+1 \\ ab & \text{if } m < n+1 \end{cases}}$$

(proviso: LHS is $\downarrow$ (well-defined))

<u>Ex's:</u>

$m < n+1$:

$$\cdot \xrightarrow{a} \cdot \quad \cdot \quad \begin{array}{c} \xrightarrow{b} \\ \Big\downarrow 1_b \\ \xrightarrow[b]{} \end{array} \quad := \quad \cdot \xrightarrow{a} \cdot \xrightarrow{b} \cdot$$

$m = 1$ $\quad$ $n = 1$

how can you?

dimensions don't fit!

$m \geqslant n+1$:

$m = 3$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\xrightarrow{a}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "b"'] & \cdot \end{tikzcd} \quad \cdot \quad \begin{array}{c} \Big\downarrow 1_b \\ \xrightarrow[b]{} \end{array} \quad := \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\xrightarrow{a}"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

$n = 1$

Also, dually ...

(·_·) (<u>associative law</u>)

<u>if</u> either $m = n \leq p$ (Case 1)

or $m \geq n = p$ (Case 2)

or $m = p \leq n$ (Case 3)

$\Bigg\}$ (*)

<u>we have:</u>

$$a(be) :=: (ab)e$$

Ex's: Case 1:

Case 2:

Case 3:

<u>Remark.</u> (*) is equivalent to saying that

$$k(a, be) = k(a, b) = k(ab, e) = k(b, e).$$

In turn, this condition ensures that

$$(ab\Downarrow \;\&\; be\Downarrow) \Longrightarrow$$

and $a(be)\downarrow$ & $(ab)e\downarrow$ & $(d/c)(a(be)) = (d/c)((ab)e)$

(distributive law)

if $k(a,b) < k(b,e)$ ($\therefore$ $m<n,p$ & $k(a,e) = k(a,b)$):

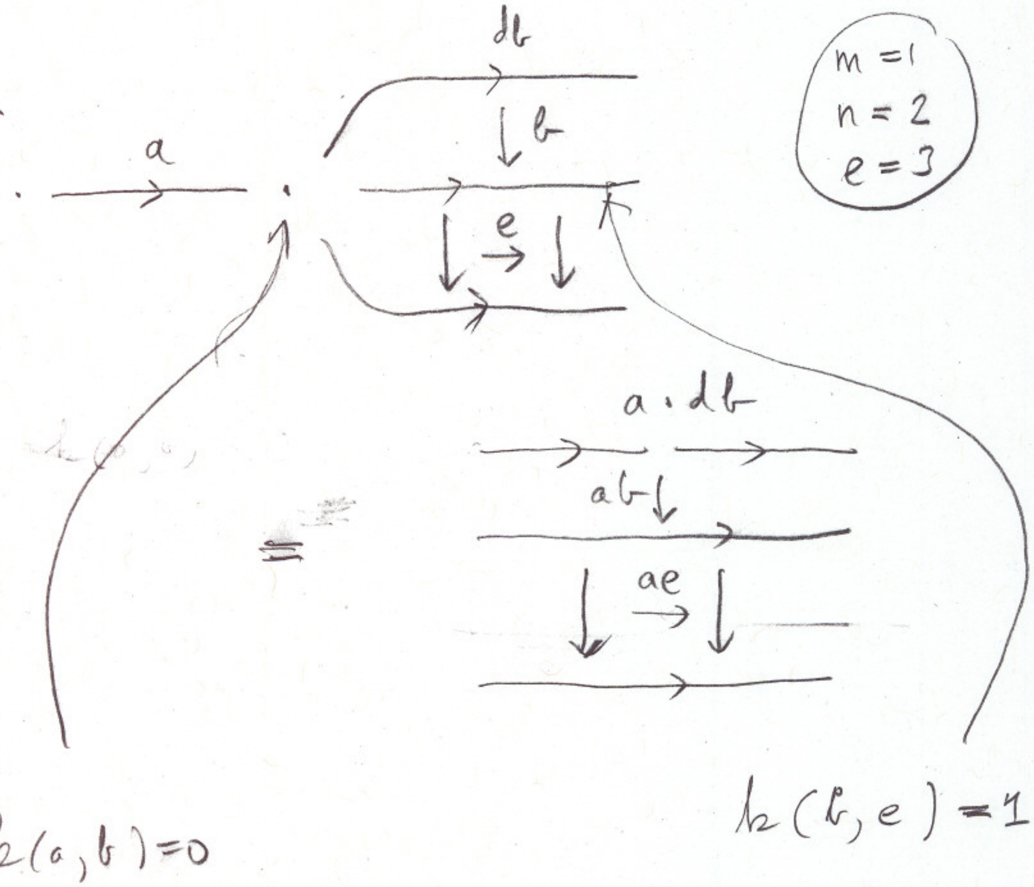
$$\boxed{a(be): = (ab)(ae)}$$

and dually:

$$(ab)e: = (ae)(be)$$

if $k(b,e) < k(a,b)$.

ex:



(:) ('commutative law'; exchange)

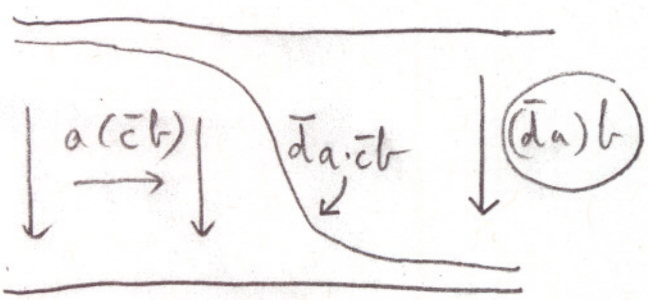
If $m, n \geq 2$, $\therefore k = k(a,b) = \min(m,n) - 1 \geq 1$

and
$$c^{(k-1)}a = d^{(k-1)}b \quad :$$

with the notation $\bar{d} = d^{(k)}$, $\bar{c} = c^{(k)}$:

$$\boxed{\begin{gathered}((\bar{d}a)b)(a(\bar{c}b)) \\ = \\ (a(\bar{d}b))((\bar{c}a)b)\end{gathered}}$$

Ex:
$$\begin{tikzcd} \bullet \arrow[r, bend left, "\bar{d}a"] \arrow[r, bend right, "\bar{c}a"'] \arrow[r, phantom, "\Downarrow a"] & \bullet \arrow[r, bend left, "\bar{d}b = db"] \arrow[r, bend right, "\bar{c}b = cb"'] \arrow[r, phantom, "\Downarrow b"] & \bullet \end{tikzcd} \quad :$$



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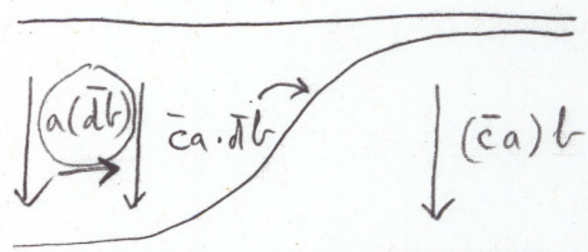


END OF DEF OF ω-cat

Derived operations:

$$\text{for } \ell < m, n; \text{ if } c^{(\ell)}a = d^{(\ell)}b$$

$$a \#_\ell b \quad \text{is defined}$$

recursively / inductively:

$$\text{For } \ell = k = k(a,b), \quad a \#_\ell b \underset{\text{def}}{=} a \cdot b$$

$$\text{For } \ell < k: \qquad \text{with } \bar{d} = d^{(\ell+1)}, \quad \bar{c} = c^{(\ell+1)}:$$

$$a \#_\ell b \overset{\text{def}}{=} ((\bar{d}a)\,b) \#_{\ell+1} (a(\bar{c}b))$$

$$= (a(\bar{d}b)) \#_{\ell+1} ((\bar{c}a)\,b)$$

(induct on $k - \ell$).

—·—

We are interested in

'all possible compositions'

a kind of 'positional geometry'.

General exchange ('theorem'):

(dim $f = q$)

$$(a \#_j b) \#_i (e \#_j f)$$

$$=$$

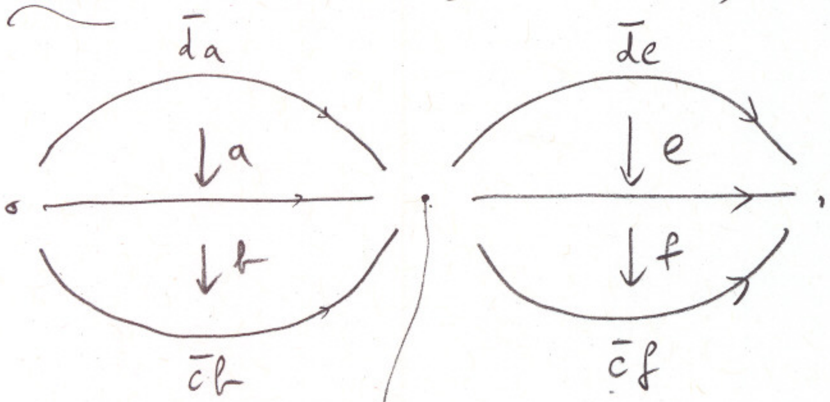
$$(a \#_i e) \#_j (b \#_i f)$$

provided: $i < j$, $j < m, n, p, q$ and

$c^{(j)}a = d^{(j)}b$, $c^{(j)}e = d^{(j)}f$

and $c^{(i)}a \; (= c^{(i)}b) = d^{(i)}e \; (= d^{(i)}f)$

Scheme: $\bar{c} = c^{(j)}$, $\bar{d} = d^{(j)}$; $\bar{\bar{c}} = c^{(i)}$, $\bar{\bar{d}} = d^{(i)}$



$\bar{\bar{c}}a = \bar{\bar{c}}b = \bar{\bar{d}}e = \bar{\bar{d}}f$

General associativity ('theorem'):

$$(a \#_i b) \#_i e = a \#_i (b \#_i e)$$

provided : $i < m, n, e$ and

$$c^{(i)} a = d^{(i)} b \quad \& \quad c^{(i)} b = d^{(i)} e$$

Example:

$$
\begin{tikzcd}
\cdot \arrow[r, "j"] & \cdot \arrow[r, "l"] & \cdot \\
\cdot \arrow[u, "i"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow A"] & \cdot \arrow[u, "k"'] \arrow[r, "m"] \arrow[ur, phantom, "\Downarrow B"] & \cdot \arrow[u, "n"'] \\
\cdot \arrow[u, "f"] \arrow[r, "e"'] \arrow[ur, phantom, "\Downarrow C"] & \cdot \arrow[u, "h"'] \arrow[r, "o"'] \arrow[ur, phantom, "\Downarrow D"] & \cdot \arrow[u, "p"']
\end{tikzcd}
$$

$$(i,j) \xrightarrow{A} (g,k)$$

$$(k,l) \xrightarrow{B} (m,n)$$

$$(f,g) \xrightarrow{C} (e,h)$$

$$(h,m) \xrightarrow{D} (o,p)$$

$$(i,j,l) \xrightarrow{A \#_k B} (g,m,n) \quad :$$

$$
\begin{tikzcd}
\cdot \arrow[r, "j"] & \cdot \arrow[r, "l"] & \cdot \\
\cdot \arrow[u, "i"] \arrow[r, "g"'] \arrow[urr, phantom, "\Downarrow A \#_k B"] & \cdot \arrow[r, "m"'] & \cdot \arrow[u]
\end{tikzcd}
$$

$\boxed{\text{new (temporary) notation: "glue A and B at k"}}$

$$(f, g, m) \xrightarrow{C \#_h D} (e, \sigma, p)$$

$$(A \#_k B) \#_{g \cdot m} (C \#_h D) \underset{def}{=} \Phi$$

$$\overset{?}{=}$$

$$(A \#_g C) \#_{h \cdot k} (B \#_m D) \underset{def}{=} \Psi$$

Yes: In fact

$$\Phi = f(\underbrace{(A\ell)(gB)}_{A \#_k B}))(((Cm)(eD))n)$$

$$= (fA\ell)(fgB)(Cmn)(eDn)$$

(assoc & dist.)

$$\Psi = (((\underbrace{(fA)(Ck)}_{A \#_g C}))\ell)(e(\underbrace{(hB)(Dn)}_{B \#_m D})))$$

$$= (fA\ell)(Ck\ell)(ehB)(eDm)$$

=

$\underset{\sim}{\text{and}}$

$$(\underbrace{fg}_{dC}\, B)(\cdot C\, \underbrace{mn}_{cB})$$

$$=$$

$$(C\, \underbrace{k\ell}_{dB})(\underbrace{eh}_{cC}\, B) \;.$$

<u>Computads</u> (Ross Street) =

= polygraphs (Albert Burroni)

Given n-cat $\mathbb{X}$, and a set

$\mathbb{U}$ of symbols $u \in \mathbb{U}$ ('indeterminates'

of dimension $(n+1)$), with

$$u \longmapsto du \| cu \quad \in \mathbb{X}_n$$

we can define

$$\mathbb{X}[\mathbb{U}]$$

'freely adjoin' each u to $\mathbb{X}$:

it is equipped with maps

$$\left\{ \begin{array}{l} \mathbb{X} \longrightarrow \mathbb{X}[\mathbb{U}] \\ \mathbb{U} \longrightarrow (\mathbb{X}[\mathbb{U}])_{n+1} \end{array} \right\}$$

without loss of generality: (inclusions)

such that for any

$$\mathbb{X} \xrightarrow{F} \mathbb{Y} \quad \omega\text{-cat map}$$

$$\mathbb{U} \xrightarrow{u \mapsto \hat{u}} \mathbb{Y}_{n+1}$$

such that

$$d\hat{u} = F(du) \qquad (u \in U)$$
$$c\hat{u} = F(cu)$$

there is unique G:

$$\begin{tikzcd} \mathbb{X} \arrow[r] \arrow[dr, "F"'] & \mathbb{X}[U] \arrow[d, "G"] \arrow[dl, phantom, "\circ"] \\ & \mathbb{Y} \end{tikzcd} \qquad \& \qquad \begin{tikzcd} u \arrow[d, mapsto, "G"] \\ \hat{u} \end{tikzcd} \quad (u \in U)$$

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & (\mathbb{X}[U])_{n+1} \arrow[d] \arrow[dl, phantom, "\circ"] \\ & \mathbb{Y}_{n} \end{tikzcd}$$

An ω-category $\mathbb{X}$ is a computad iff$_{\text{def}}$, for all $n \in \mathbb{N}$,

$$\mathbb{X}\restriction(n+1) \text{ is } (\mathbb{X}\restriction n)[U]$$

for suitable indet's U.

The indeterminates can be identified as the 'indecomposable' cells.

Morphism of computads:

$$\mathbb{X} \xrightarrow{F} \mathbb{Y}$$

ω-cat map, taking any indet. to an indet.

We have non-full inclusion

$$\mathrm{Comp} \xrightarrow[\langle - \rangle]{} \omega\mathrm{Cat}$$

with a right adjoint

$$\begin{tikzcd} \mathrm{Comp} \arrow[r, shift right, "\langle - \rangle"'] \arrow[r, phantom, "\top"] & \omega\mathrm{Cat} \arrow[l, shift right, "{[-]}"'] \end{tikzcd}$$

$[\mathbb{X}]$: nerve of $\mathbb{X}$.

<u>Example</u> (important) of structure in an ω-category:

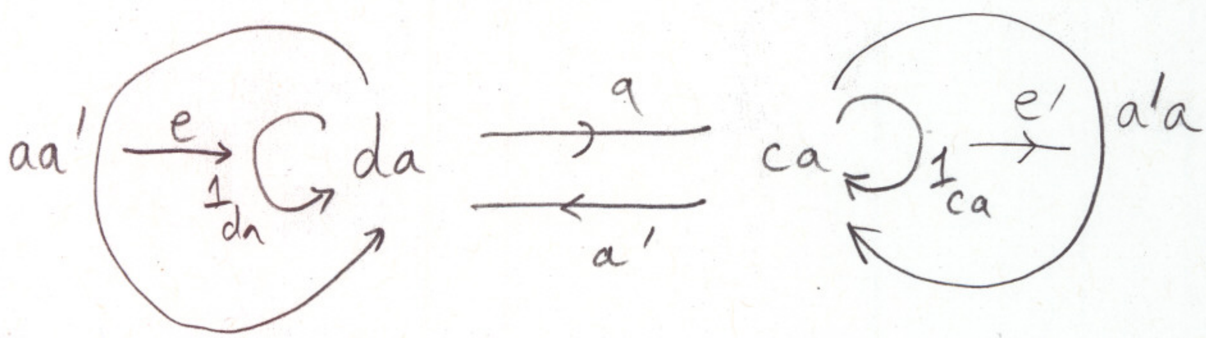
<u>equivalence</u> cells.

$\mathbb{X}$: ω-cat

Let: $E \subseteq \coprod_{n \geq 1} \mathbb{X}_n$.

E is <u>coinductive</u> if$_{def}$

$a \in E \implies \exists\ a', e, e'$:



<u>and</u> $e \in E$, $e' \in E$.

Any union of coinductive sets is coinductive. The largest coinductive set: $E_{\mathbb{X}}$ = union of all coind. set's is the set of all equivalence cells of $\mathbb{X}$.

Examples of facts on equivalences, (proved by coinduction):

$$e, a \in E_{\mathbb{X}} \Longrightarrow$$

$$e \#_k a \in E_{\mathbb{X}}$$

$$p = m$$

assume

$$e \in E_{\mathbb{X}} \Longrightarrow e \#_k a \in E_{\mathbb{X}}$$

$$p > m$$

assume

In particular, all 0-cells, plus all equivalence cells form a sub-ω-cat.

<u>Definition</u> A computad-weak-omega-category is a computad $\mathbb{Y}$ such that for every cell a in $\mathbb{Y}$ there are x and e ;

$$\begin{tikzcd} \cdot \arrow[r, bend left, "a"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "e\Downarrow"] & \cdot \end{tikzcd}$$

with x an indet., and e an equivalence.

<u>Proposition</u> (easy): For each $\mathbb{X} \in \omega\mathrm{Cat}$, $[\mathbb{X}]$ is a computad-weak-omega-cat.

Definition Computads $\mathbb{X}$ and $\mathbb{Y}$ are equivalent,

$$\mathbb{X} \simeq \mathbb{Y}$$

if there are:

$$\begin{tikzcd} & \mathbb{P} \arrow[dl, "F"'] \arrow[dr, "G"] & \\ \mathbb{X} & & \mathbb{Y} \end{tikzcd}$$

computad maps F and G which are fiberwise surjective (or: "trivial fibrations") in the sense: (for F):

indet

$$\begin{tikzcd} \delta \arrow[d, mapsto, "F"'] & & \gamma \arrow[d, mapsto, "F"] \\ dx \arrow[rr, "x"] & & cx \end{tikzcd} \quad \Rightarrow \exists \quad \begin{tikzcd} \delta \arrow[r, "y"] & \gamma \\ dx \arrow[r, "x"'] \arrow[u, phantom, ""] & cx \end{tikzcd}$$

indet in $\mathbb{P}$

$$y \overset{F}{\longmapsto} x$$

<u>Theorem</u> Any computad-weak-omega-category $\mathbb{Y}$ is equivalent to $[\mathbb{X}]$ for some ω-category; in fact for $\mathbb{X} = \mathbb{Y}$ (!)

This shows that this notion of weak-ω-cat is too strong: essentially the same as strict ω-cat.

1.2 Octobe 2004

Seminar talk; continuing
Octoberfest talk

$\mathbb{X}$: n-category $(n \geq 1)$

(cells of $\mathbb{X}$: $a, b, e, f, \dots$)

$\mathcal{U}$: set of $(n+1)$-indeterminates

$u \mapsto (du \| cu)$;

$du, cu \in \mathbb{X}_n$.

(elements of $\mathcal{U}$: $u, \dots, x, \dots$)

refers to: Oct 16/04 Oct Fest talk p. 14

We are aiming at "constructing" $\mathbb{X}[\mathcal{U}]$.

Atom: word (expression) of the form

$$\boxed{\varphi := b_n b_{n-1} \dots b_1 u e_1 \dots e_{n-1} e_n}$$

such that : $b_i, e_i \in \mathbb{X}_i$, u : indet $(\in \mathcal{U})$,

$$c b_i = b_{i-1}(b_{i-2}(\cdots(b_1(d^{(i-1)}u)e_1)\cdots)e_{i-2}e_{i-1} \qquad (1)_i$$

$$d e_i = b_{i-1}(b_{i-2}(\cdots(b_1(c^{(i-1)}u)e_1)\cdots)e_{i-2}e_{i-1} \qquad (2)_i$$

(in part.: $c b_1 = d^{(0)}u$, $d e_1 = c^{(0)}u$)

u: <u>nucleus</u> of atom φ.

[Of course, φ "stands for the cell

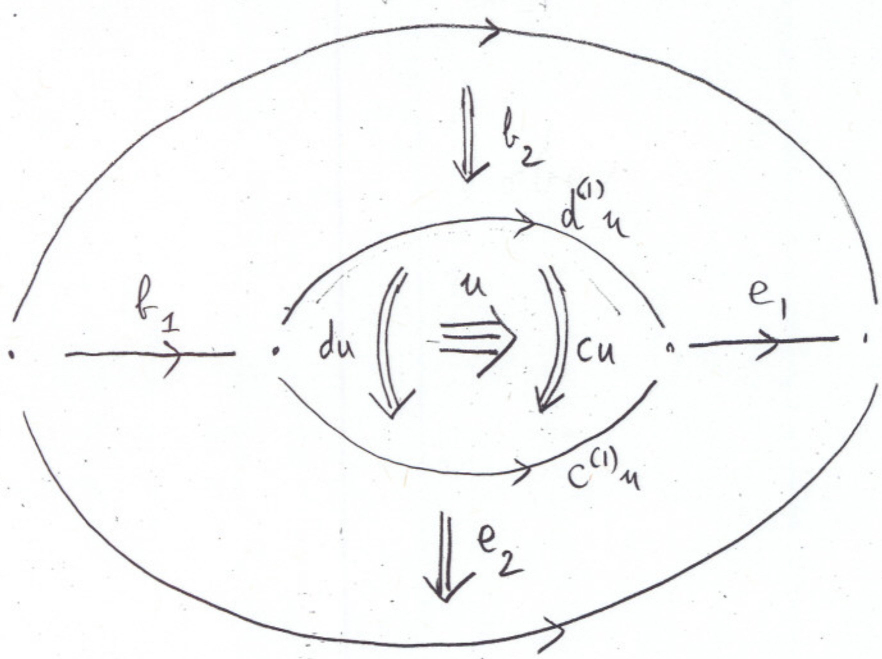
$$b_n(b_{n-1}(\cdots(b_1 u e_1)\cdots)b_{n-1})b_n \qquad \text{of } \mathbb{X}[\mathcal{U}];$$

then conditions (1), (2) ensure this is 'well-defined'

Note: the RHS of $(1)_i$ is <u>well-defined</u> (now, without quotes!) by the equalities $(1)_{i-1}$ & $(2)_{i-1}$ for $i \geq 2$; $(1)_2$ is well-defined because of $(1)_1$ and $(2)_1$.]

Picture, for $n=2$: $\qquad \varphi := b_2 b_1 u\, e_1 e_2$

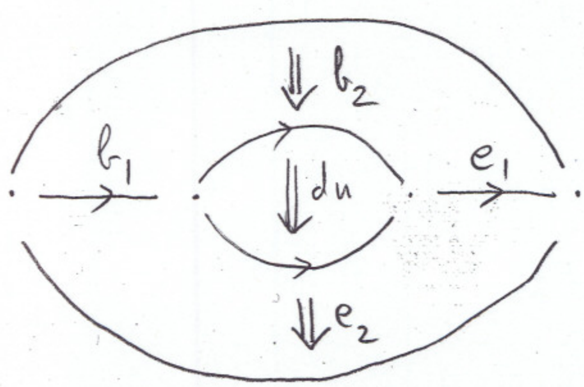
φ:



$$d\varphi \overset{\text{def}}{=} b_n(b_{n-1}(\cdots(b_1(du)e_1)\cdots)e_{n-1})e_n \quad \in X_n$$
$$c\varphi \overset{\text{def}}{=} b_n(b_{n-1}(\cdots(b_1(cu)e_1)\cdots)e_{n-1})e_n \quad \in X_n$$

$n=2$:

$d\varphi$:



du & cu are well-defined & du $\|$ cu

(Atoms : $\rho, \sigma, \varphi, \psi, \ldots$)

<u>Define:</u> for $0 \leq k < n$, $a \in X_{k+1}$: ($\underline{m = k+1}$)

$$a \cdot \varphi \overset{\text{def}}{=} (a \cdot b_n) \cdots (a \cdot b_k)\, b_{k-1} \cdots b_1\, u\, e_1 \cdots e_k\, (a \cdot e_{k+1}) \cdots (a \cdot e_n)$$

provided $c^{(k)} a = d^{(k)} \varphi$

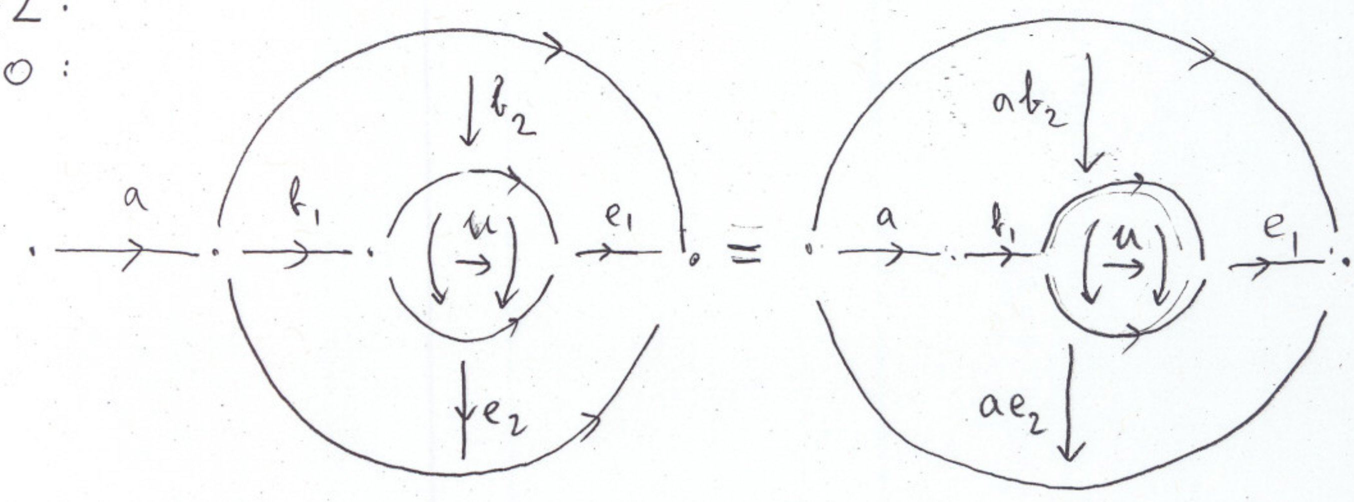
— · —

$$\varphi \cdot a \overset{\text{def}}{=} (b_n \cdot a) \cdots (b_{k+1} \cdot a)\, b_k \cdots b_1\, u\, e_1 \cdots e_{k-1}\, (e_k \cdot a) \cdots (e_n \cdot a)$$

provided $c^{(k)} \varphi = d^{(k)} a$

— · —

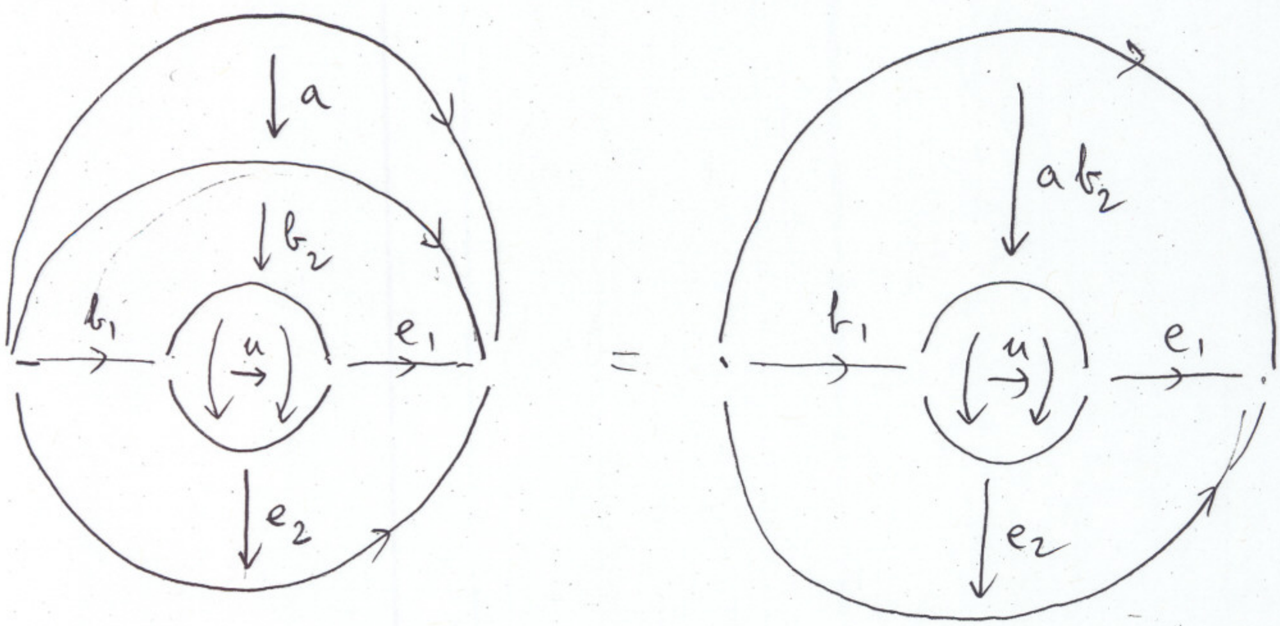
$n = 2$:
$k = 0$:



$$a\,(b_2 b_1 u e_1 e_2) = (ab_2)(ab_1)\,u\,e_1\,(ae_2)$$

$n=2$

$k=1$:



$$a(b_2 b_1 u e_1 e_2) = (ab_2) b_1 u e_1 e_2$$

Facts: $\overset{(k+1)}{a}$ & $\overset{(n+1)}{\varphi}$ as before:

(domain / codomain) $\quad d(a\varphi) = a(d\varphi)$; $d(\varphi a) = (d\varphi)a$;

$\quad c(a\varphi) = a(c\varphi)$; $c(\varphi a) = (c\varphi)a$.

(identity) $\quad 1_e \cdot \varphi = \varphi$; $\varphi \cdot 1_e = \varphi \quad (a = 1_e)$

(associative law) $\quad a\cdot(\varphi\cdot e) = (a\cdot\varphi)\cdot e \quad (m = p)$

(distributive law) $\begin{cases} a\cdot(\varphi\cdot e) = (a\cdot\varphi)(a\cdot e) & (m < p) \\ (a\cdot\varphi)\cdot e = (a\cdot e)(\varphi\cdot e) & (m > p) \end{cases}$

$m = \dim(a)$, $p = \dim(e)$ $(\le n)$ as before.

<u>Definition</u>. A <u>molecule</u> is a triple

$$(b, \vec{\varphi}, e)$$

where $\vec{\varphi} :=: \varphi_1 \cdots \varphi_N$ $\quad (N \geq 0)$

string of atoms

such that

$$c\varphi_i = d\varphi_{i+1} \quad (i = 1, \dots, N-1);$$

$$b, e \in \mathbb{X}_n$$

and, if $N = 0$: $\quad b = e$

(thus, the molecule is $(b, \emptyset, b)$)

if $N \geq 1$: $\quad b = d\varphi_1$ & $e = c\varphi_N$

(when $N \geq 1$, b & e are redundant)

Usually, we write $\vec{\varphi}$ for $(b, \vec{\varphi}, e)$, although for $N = 0$ this is not correct.

$$|\vec{\varphi}| \underset{\text{def}}{=} N$$

$$d\vec{\varphi} \overset{\text{def}}{=} b \quad (= d\varphi_1 \text{ if } N \geq 1)$$

$$c\vec{\varphi} \overset{\text{def}}{=} e \quad (= c\varphi_N \text{ if } N \geq 1)$$

<u>Define</u> : for $0 \leq k < n$,

$a \in \mathbb{X}_{k+1}$, $\vec{\varphi}$: molecule,

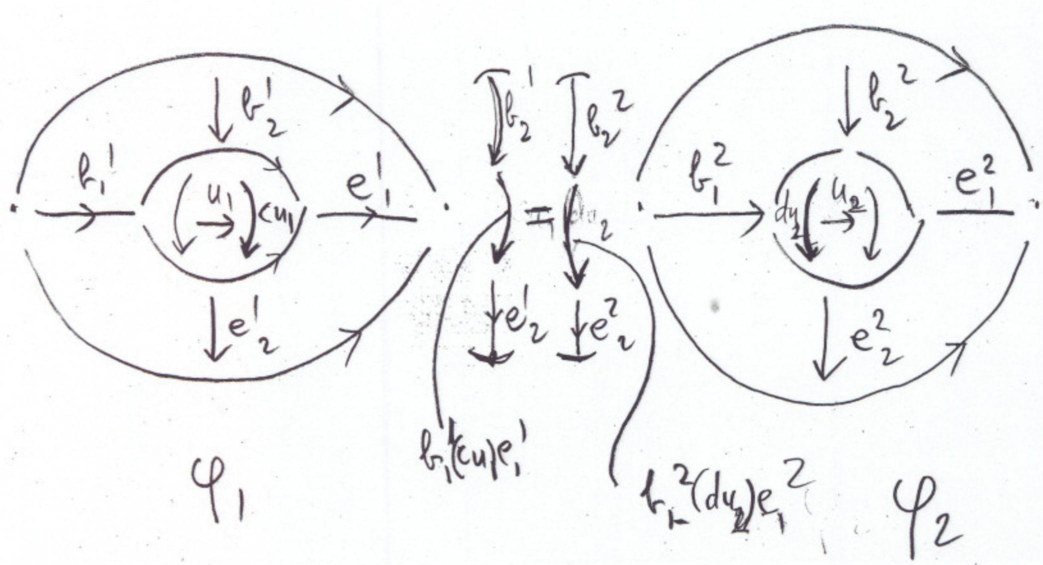
and $ca = d^{(k)}(\vec{\varphi})$:

$$a \cdot \vec{\varphi} \underset{\text{def}}{=} (a \cdot \varphi_1)(a \cdot \varphi_2) \cdots (a \cdot \varphi_N)$$

$$a(b, \emptyset, b) \underset{\text{def}}{=} (ab, \emptyset, ab)$$

<u>Example</u> : $N = 2$, $n = 2$, $k = 1$.

$$\vec{\varphi} = \varphi_1 \cdot \varphi_2$$



We have: $c\varphi_1 = d\varphi_2$

i.e.

$$b_2^1(b_1^1(cu_1)e_1^1)e_2^1 = b_2^2(b_1^2(du_2)e_1^2)e_2^2$$

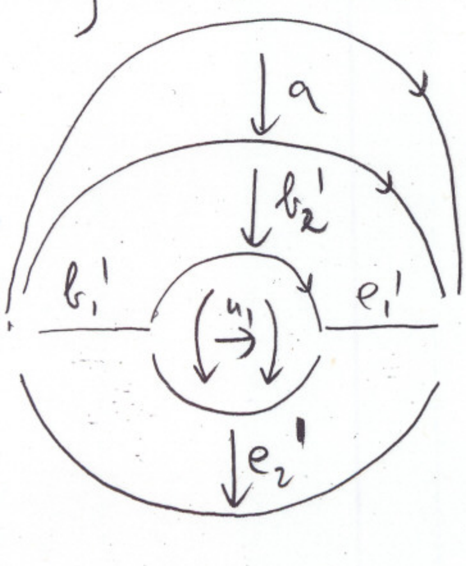
$$d^{(1)}\vec{\varphi} = \underline{d^{(1)}\varphi_1} = d(d\varphi_1) = d(c\varphi_2) = \underline{d^{(1)}\varphi_2}$$

and:

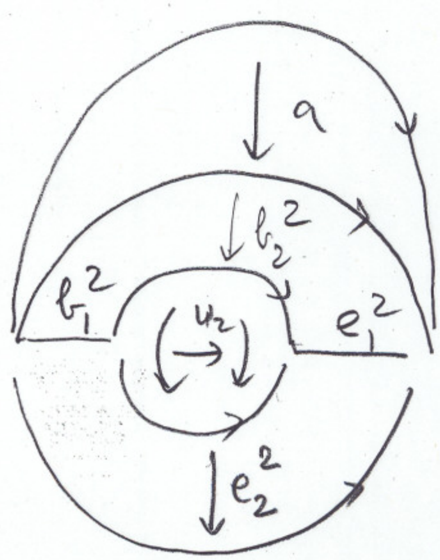
$$d^{(1)}\varphi_1 = d\big(\overbrace{b_2^1}^{2}(\overbrace{b_1^1(du_1)e_1^1}^{2})\overbrace{e_2^1}^{2}\big) = d\,b_2^1$$

$$d^{(1)}\varphi_2 = d\,b_2^2$$

Thus, both $a\cdot\varphi_1$ and $a\cdot\varphi_2$ are $\downarrow$ and they look like



$a\varphi_1$



$a\varphi_2$

$\therefore \quad (a\varphi_1)(a\varphi_2)\downarrow$

— . —

The laws on page 2.7 hold, with $\vec{\varphi}$ replacing φ.

— . —

<u>Definition</u>: For two pairs of atoms

$$(\rho, \sigma) \quad \text{and} \quad (\varphi, \psi)$$

read: "exchanging (ρ, σ) gives (φ, ψ)"

$$(\rho, \sigma) \rightsquigarrow (\varphi, \psi)$$

$\underset{\text{def}}{\Longleftrightarrow}$ there are atoms

α and β such that $c^{(n-1)}\alpha = d^{(n-1)}\beta$

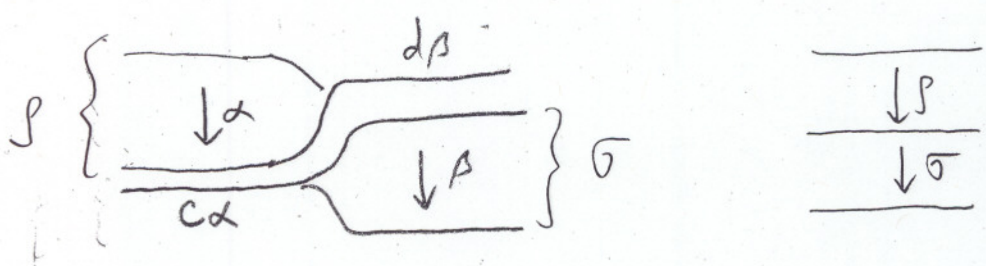
and

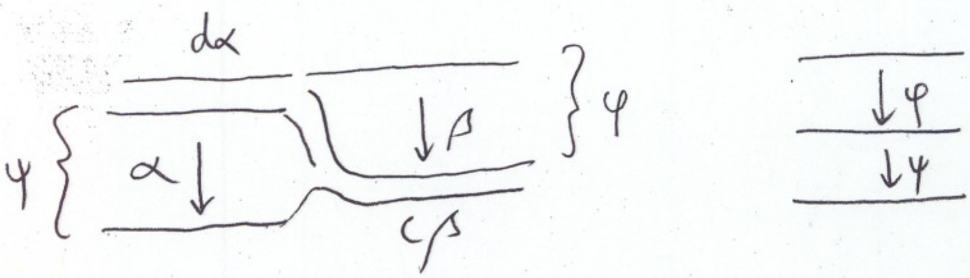
$$\rho = \alpha(d\beta), \quad \sigma = (c\alpha)\beta$$

$$\varphi = (d\alpha)\beta, \quad \psi = \alpha(c\beta)$$

Picture:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "d\alpha"] \arrow[r, bend right, "c\alpha"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, bend left, "d\beta"] \arrow[r, bend right, "c\beta"'] \arrow[r, phantom, "\Downarrow\beta"] & \cdot \end{tikzcd}$$





$$\begin{array}{ccc} L \quad R & & R \quad L \\ (\rho, \sigma) & \curvearrowright & (\varphi, \psi) \end{array}$$

implies $\underbrace{\rho \cdot \sigma}\downarrow$ & $\varphi \cdot \psi \downarrow$

i.e. $c\rho = d\sigma$ $\qquad c\varphi = d\psi$

<u>Notation:</u>

$$(\curvearrowleft) = (\curvearrowright)^{\text{conv.}}$$

For molecules $\vec{\varphi}$ and $\vec{\psi}$:

$$\vec{\varphi} \curvearrowright \vec{\psi} \underset{\text{def}}{\iff} |\vec{\varphi}| = |\vec{\psi}| = N \geq 2$$

and $\exists i \in \{1, \dots, N-1\} : (\varphi_i, \varphi_{i+1}) \curvearrowright (\psi_i, \psi_{i+1})$

Similarly $\curvearrowleft = (\curvearrowright)^{\text{conv.}}$

<u>Define</u> : $\sim$: the least equivalence on molecules such that $\curvearrowright \cup \curvearrowleft \subseteq \sim$.

Facts: $\vec{\varphi} \sim \vec{\psi} \Rightarrow |\vec{\varphi}| = |\vec{\psi}|$

$$\vec{\varphi} \sim \vec{\psi} \Rightarrow a \cdot \vec{\varphi} \sim a \cdot \vec{\psi}$$

$$\& \quad \vec{\varphi} \cdot a \sim \vec{\psi} \cdot a$$

$$\vec{\varphi} \sim \vec{\psi} \Rightarrow d\vec{\varphi} = d\vec{\psi} \ \& \ c\vec{\varphi} = d\vec{\psi}.$$

Define $\vec{\varphi} \cdot \vec{\psi}$ by concatenation

provided $c\vec{\varphi} = d\vec{\psi}$.

Define the $(n+1)$-category $\mathbb{Y}$:

$$\mathbb{Y} \upharpoonright n \underset{\text{def}}{=} \mathbb{X}$$

$$\mathbb{Y}_{n+1} = \{ [\vec{\varphi}]_\sim : \vec{\varphi} \text{ a molecule} \}$$

$$d([\vec{\varphi}]_\sim) \underset{\text{def}}{=} d\vec{\varphi}$$

$$c([\vec{\varphi}]_\sim) = c\vec{\varphi}$$

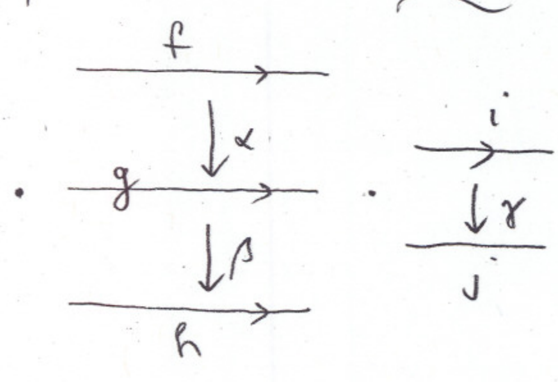
$$a \cdot [\vec{\varphi}]_{\sim} \stackrel{\text{def}}{=} [a \cdot \vec{\varphi}]_{\sim}$$

$$[\vec{\varphi}]_{\sim} \cdot [\vec{\psi}]_{\sim} \underset{\text{def}}{=} [\vec{\varphi} \cdot \vec{\psi}]_{\sim}$$

Proposition

$$\mathbb{Y} \cong \mathbb{X}[u].$$

Key point: Given

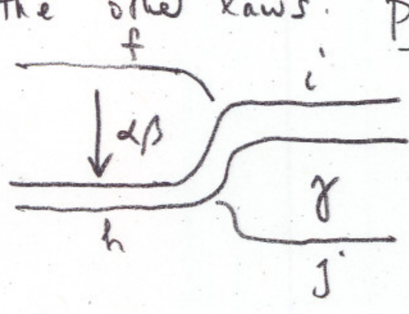


deduce $((\alpha\beta)i)(h\gamma) = (f\gamma)((\alpha\beta)j)$ (1)

from $(\alpha i)(g\gamma) = (f\gamma)(\alpha j)$ (2)

$(\beta i)(h\gamma) = (g\gamma)(\beta j)$ (3)

using the other laws. Picture:

(1): 

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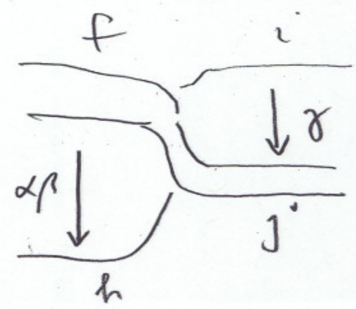

$$(\alpha\beta)i = (\alpha i)(\beta i)$$

$$(\alpha\beta)j = (\alpha j)(\beta j)$$

RHS in (1) $\Big\}$ $(g\gamma)((\alpha\beta)j) = (g\gamma)(\alpha j)(\beta j)$

$$\underset{(2)}{=} (\alpha i)(g\gamma)(\beta j)$$

$$\underset{(3)}{=} (\alpha i)(\beta i)(h\gamma)$$

$$= ((\alpha\beta)i)(h\gamma) \quad \text{LHS in (1)}$$

γ was shifted from below to the top in 2 steps, once past β, then past α.

To be continued; the really interesting part is yet to come!