The invariance theorem

- Data:

$$\Delta = \wedge \neg \exists \qquad \text{or} \qquad \wedge \vee \rightarrow \exists \forall$$

(Boolean) (Heyting)

$$\underbrace{\begin{tikzcd} \mathbb{E} \arrow[d, "\mathcal{C}"'] \\ \mathbb{B} \end{tikzcd} , \quad \begin{tikzcd} \mathbb{F}_1 \arrow[d, "\mathcal{D}_1"'] \\ \mathbb{C}_1 \end{tikzcd} , \quad \begin{tikzcd} \mathbb{F}_2 \arrow[d, "\mathcal{D}_2"] \\ \mathbb{C}_2 \end{tikzcd} \qquad Q \subset \mathrm{Arr}(\mathbb{B})}_{\Delta\text{-}Q\text{-fibrations}}$$

$(\mathcal{D}_1, \mathcal{D}_2 : \text{full})$

interpretations:

$$\begin{tikzcd} & \mathcal{D}_1 \\ \mathcal{C} \arrow[ur, "I_1"] \arrow[dr, "I_2"'] & \\ & \mathcal{D}_2 \end{tikzcd} \quad : \quad \begin{tikzcd} \mathbb{E} \arrow[r, "I_1"] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\circ"] & \mathbb{F}_1 \arrow[d, "\mathcal{D}_1"] \\ \mathbb{B} \arrow[r, "I_1"] & \mathbb{C}_1 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B} \arrow[r, "I_2"] \arrow[dr, phantom, "\circ"] & \mathbb{C}_2 \\ \mathbb{E} \arrow[u, "\mathcal{C}"] \arrow[r, "I_2"'] & \mathbb{F}_2 \arrow[u, "\mathcal{D}_2"'] \end{tikzcd}$$

- **"Generic Q-equivalence of a $\mathcal{D}_1$-model and a $\mathcal{D}_2$-model":**

$\boxed{\text{Definition.}}$ • Let $\mathcal{E}$ be a (further) full Δ-fibration.

An $\mathcal{E}$-cocone (a cocone with vertex $\mathcal{E}$ and base $\mathcal{D}_1 \xleftarrow{I_1} \mathcal{C} \xrightarrow{I_2} \mathcal{D}_2$) is:

(M, N, P, m, n) s.t.:

$$
\begin{tikzcd}
\mathcal{D}_1 \arrow[drr, "M"] & & \\
\mathcal{C} \arrow[u, "I_1"] \arrow[d, "I_2"'] \arrow[rr, "P"] \arrow[r, phantom, "\Nearrow m"] & & \mathcal{E} \\
\mathcal{D}_2 \arrow[urr, "N"'] \arrow[u, phantom, "n \Searrow"] & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & MI_1 \\
P \arrow[ur, "m"] \arrow[dr, "n"'] & \\
 & NI_2
\end{tikzcd}
$$

- A morphism of $\mathcal{E}$-cocones:

$(M,N,P,m,n) \xrightarrow[(f,g,h)]{} (M',N',P',m',n')$: compatibility:

$$
\begin{tikzcd}
 & MI_1 \arrow[r, "f", "\cong"'] & M'I_1 \arrow[d, phantom, "\circ"] & \\
P \arrow[ur, "m"] \arrow[dr, "n"'] \arrow[rrr, "h \cong"'] & & \arrow[d, phantom, "\circ"] & P' \arrow[ul, "m'"'] \arrow[dl, "n'"] \\
 & NI_2 \arrow[r, "\cong", "g"'] & N'I_2 &
\end{tikzcd}
$$

• $\mathrm{Cocone}(\mathcal{E})$: category of $\mathcal{E}$-cocones and their morphisms.

$\boxed{\text{Definition}}$ Let $\mathcal{D}$, $\mathcal{E}$ be full Δ-fibs.

$\Delta^{\cong}(\mathcal{D}, \mathcal{E})$ is the $\left\{\begin{matrix}\text{groupoid}\\ \text{category}\end{matrix}\right\}$ of Δ-maps and their isomorphisms:

$$
\begin{tikzcd}
{[\mathcal{D}]} \arrow[r, shift left=2, "M"] \arrow[r, phantom, "h\Downarrow"] \arrow[r, shift right=2, "N"'] \arrow[d, "\mathcal{D}"'] & {[\mathcal{E}]} \arrow[d, "\mathcal{E}"] \\
{|\mathcal{D}|} \arrow[r, shift left=2, "M"] \arrow[r, phantom, "h\Downarrow"] \arrow[r, shift right=2, "N"'] & {|\mathcal{E}|}
\end{tikzcd}
$$

$h : M \to N$

— • —

Given $\mathcal{D}$-cocone $\gamma = (\tilde{M}, \tilde{N}, \bar{P}, \bar{m}, \bar{n})$

and $R : \mathcal{D} \to \mathcal{E} \in \Delta(\mathcal{D}, \mathcal{E})$

we have

$$(M, N, P, m, n) = (R\tilde{M}, R\tilde{N}, R\bar{P}, R\bar{m}, R\bar{n}) \in \mathrm{Cocone}(\mathcal{E}) :$$

$$\begin{tikzcd}
 & \mathcal{D}_1 \arrow[d, "\tilde{M}"] & \\
\mathcal{C} \arrow[ur, "I_1"] \arrow[r, "\bar{P}"] \arrow[dr, "I_2"'] \arrow[ur, phantom, shift right=4ex, "\bar{m}\,\Nearrow"] \arrow[dr, phantom, shift left=4ex, "\bar{n}\,\Downarrow"] & \mathcal{D} \arrow[r, "R"] & \mathcal{E} \\
 & \mathcal{D}_2 \arrow[u, "\tilde{N}"'] &
\end{tikzcd}$$

$$\rightsquigarrow \quad \gamma^*$$

$$\begin{tikzcd}
 & \mathcal{D}_1 \arrow[dr, "M"] & \\
\mathcal{C} \arrow[ur, "I_1"] \arrow[rr, "P"'] \arrow[dr, "I_2"'] \arrow[rr, phantom, shift left=3ex, "m\,\Nearrow"] \arrow[rr, phantom, shift right=4ex, "n\,\Downarrow"] & & \mathcal{E} \\
 & \mathcal{D}_2 \arrow[ur, "N"'] &
\end{tikzcd}$$

$$\boxed{\gamma^* : \Delta^{\cong}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Cocone}(\mathcal{E})}$$

$\boxed{\text{Definition}}$ (Generic cocone; generic Q-equivalence of a $\mathcal{D}_1$-model and a $\mathcal{D}_2$-model)

$(\mathcal{D}, \gamma)$ is generic if $\mathcal{D}$ is a Δ-fib, $\gamma \in \mathrm{Cocone}(\mathcal{D})$ and

$$\gamma^*: \Delta^{\cong}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Cocone}(\mathcal{E})$$

is an equivalence of categories for all Δ-fibs $\mathcal{E}$.

$\boxed{\text{Proposition}}$

$$\Big(\mathcal{D} = \mathcal{D}_1 \overset{\simeq Q}{\underset{\mathcal{C}}{+}} \mathcal{D}_2 \,,\ \gamma\Big) \quad \text{generic}$$

exists & is unique up to equivalence

• Interpolation property

$\boxed{\text{Definition}}$ A $\mathcal{D}$-cocone (M, N, P, m, n) has the **interpolation property** if:

$\forall$: $X \in \mathbb{B}$, $\Psi_1 \in \mathcal{D}_1^{I_1 X}$, $\Psi_2 \in \mathcal{D}_2^{I_2 X}$

$$\begin{tikzcd}
 & m^*_X M\Psi_1 \leq_{PX} n^*_X N\Psi_2 \arrow[dd, mapsto, "\mathcal{D}"] & \\
M\Psi_1 \arrow[dd, mapsto, "\mathcal{D}"'] & & N\Psi_2 \arrow[dd, mapsto, "\mathcal{D}"] \\
 & PX \arrow[dl, "m_X"'] \arrow[dr, "n_X"] & \\
MI_1X & & NI_2X
\end{tikzcd}$$

$\exists$: $\Phi \in \mathcal{C}^X$

$$\Psi_1 \leq_{I_1 X} I_1\Phi, \quad I_2\Phi \leq_{I_2 X} \Psi_2$$