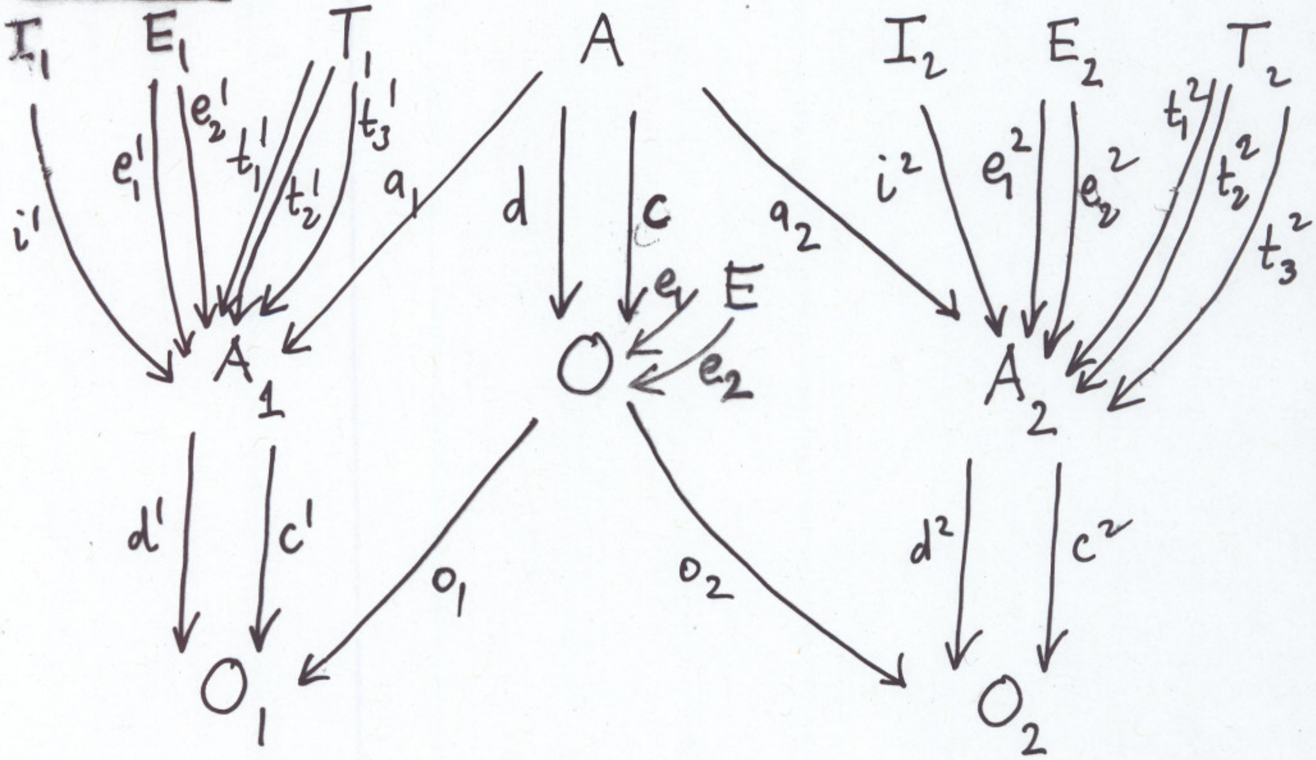
FOLDS signature for anafunctors:

L_{anafun}:



equations:

^{1}-copies & ^{2}-copies of equ's for L_{cat}

plus:

$$o_1 e_1 = o_1 e_2$$
$$o_2 e_1 = o_2 e_2$$

$$o_1 d = d^1 a_1$$
$$o_2 d = d^2 a_2$$

$$o_1 c = c^1 a_1$$
$$o_2 c = c^2 a_2$$

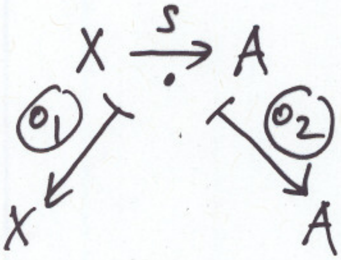
$\boxed{\text{Anafunctor } F: \mathbb{X} \xrightarrow{ana} \mathbb{A} \text{ as structure } \hat{F} \text{ for above signature:}}$

1-copy of L_{cat} : interpreted by $\mathbb{X}$

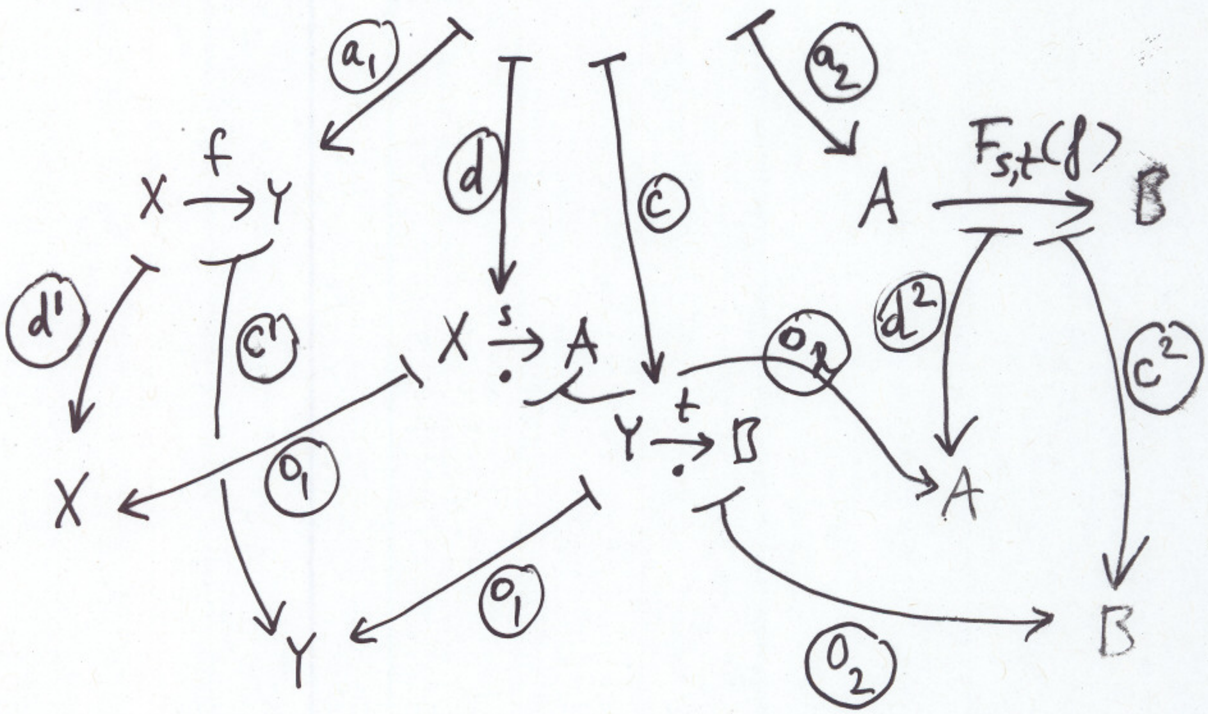
2-copy of L_{cat} : interpreted by $\mathbb{A}$

$$\hat{F}O = \coprod_{A \in O_2,\, X \in O_1} (F|(X,A) = \{ X \xrightarrow[\cdot]{s} A \}$$

= class of all specifications



$$\hat{F}A = \{ (X \xrightarrow[\cdot]{s} A \,,\, Y \xrightarrow[\cdot]{t} B \,,\, X \xrightarrow{f} Y) \}$$



$\boxed{\text{Equivalence}}$ of (free-living) functors:

$$\begin{tikzcd} \mathbb{X} \arrow[d, "F"'] \\ \mathbb{A} \end{tikzcd} \; , \; \begin{tikzcd} \mathbb{Y} \arrow[d, "G"] \\ \mathbb{B} \end{tikzcd} \; : \qquad F \simeq G \;\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$$

$$\exists: \begin{tikzcd} \mathbb{X} \arrow[r, shift left, "E_1"] \arrow[d, "F"'] & \mathbb{Y} \arrow[l, shift left, "\dot{E}_1"] \arrow[d, "G"] \\ \mathbb{A} \arrow[r, shift left, "E_2"] & \mathbb{B} \arrow[l, shift left, "\dot{E}_2"] \end{tikzcd} \;\&\; \begin{tikzcd} \cdot \arrow[r, "E_1"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}\cong"] & \cdot \arrow[d, "G"] \\ \cdot \arrow[r, "E_2"'] & \cdot \end{tikzcd} \;\&\; \begin{tikzcd} \cdot \arrow[d, "F"'] \arrow[dr, phantom, "\cong\overset{\beta}{\Searrow}"] & \cdot \arrow[l, "\dot{E}_1"'] \arrow[d, "G"] \\ \cdot & \cdot \arrow[l, "\dot{E}_2"] \end{tikzcd}$$

& $\gamma_2 : \dot{E}_2 E_2 \xrightarrow{\cong} \mathrm{Id}_{\mathbb{A}}$, $\delta_1 : E_1 \dot{E}_1 \xrightarrow{\cong} \mathrm{Id}_{\mathbb{Y}}$,

$\gamma_1 : \dot{E}_1 E_1 \xrightarrow{\cong} \mathrm{Id}_{\mathbb{X}}$, $\delta_2 : E_2 \dot{E}_2 \xrightarrow{\cong} \mathrm{Id}_{\mathbb{B}}$

such that (compatibility):

$$\begin{tikzcd} \dot{E}_2 G E_1 \dot{E}_1 \arrow[r, "\dot{E}_2 G \delta_1"] \arrow[d, "\dot{E}_2 \alpha \dot{E}_1"'] \arrow[dr, phantom, "\circ"] & \dot{E}_2 G \\ \dot{E}_2 E_2 F \dot{E}_1 \arrow[r, "\gamma_2 F \dot{E}_1"'] & F\dot{E}_1 \arrow[u, "\beta"'] \end{tikzcd} \quad \beta \,\&\, \alpha \quad \begin{tikzcd} G E_1 \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circ"] & E_2 \dot{E}_2 G E_1 \arrow[l, "\delta_2 G E_1"'] \\ E_2 F & E_2 F \dot{E}_1 E_1 \arrow[l, "E_2 F \gamma_1"] \arrow[u, "E_2 \beta E_1"'] \end{tikzcd}$$

$\boxed{\text{Saturation:}}$

$$\begin{tikzcd} \mathbb{X} \arrow[d, "F"', "\text{fun}"] \\ \mathbb{A} \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbb{X} \arrow[d, "F^{\#}"', "\text{saturated anafunctor}"] \\ \mathbb{A} \end{tikzcd}$$

$$|F^{\#}|(X,A) \underset{\text{def}}{=} \{ F(X) \xrightarrow[\cong]{s} A \}$$

and for

$$F(X) \xrightarrow[\cong]{s} A, \quad F(Y) \xrightarrow[\cong]{t} B, \quad X \xrightarrow{f} Y \ (\in \mathbb{X})$$

$F^{\#}_{s,t}(f)$ is defined by:

$$\begin{tikzcd} F(X) \arrow[r, "s", "\cong"'] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circ"] & A \arrow[d, "F^{\#}_{s,t}(f)"] \\ F(Y) \arrow[r, "\cong", "t"'] & B \end{tikzcd}$$

$F^{\#}$ is an L_{anafun} - structure

FACT:

$$\boxed{F \simeq G \iff F^{\#} \simeq_{L_{\text{anafun}}} G^{\#}}$$

above 'categorical' sense

$\boxed{\text{Theory of (free-living) functors}}$

Multisorted signature (graph): L_{fun}:

$$\begin{tikzcd}
I_1 \arrow[dr] & & T_1 \arrow[dl, shift left=2] \arrow[dl] \arrow[dl, shift right=2] & & I_2 \arrow[dr] & & T_2 \arrow[dl, shift left=2] \arrow[dl] \arrow[dl, shift right=2] \\
& A_1 \arrow[d, shift right] \arrow[d, shift left] \arrow[rrrr, "\text{'}F\text{'}"'] & & & & A_2 \arrow[d, shift right] \arrow[d, shift left] & \\
& O_1 \arrow[rrrr, "\text{'}F\text{'}"] & & & & O_2 &
\end{tikzcd}$$

Σ_{fun} : axioms (in first order logic with equality) for "functor".

$T_{fun} = (L_{fun}, \Sigma_{fun})$; $\mathbb{C}_{fun}$: Boolean category
(i.e., $\mathcal{P}(\mathbb{C}_{fun})$: full $\wedge\neg\exists$-fibration)
whose models are the models of T_{fun}

$L \underset{def}{=} L_{anafun}$

$$\boxed{(\mathcal{C}_{\wedge\neg\exists}(L, \emptyset), Q(L)) \xrightarrow{I} \mathcal{P}(\mathbb{C}_{fun})}$$

defined by

$$L \xrightarrow{I} \mathbb{C}$$

$$O \longmapsto [X:O_1, A:O_2, s\in A_2 \mid s : F(X) \overset{\cong}{\to} A]$$

$$A \longmapsto \Bigg[X,Y:O_1; A,B:O_2; f:A_1, s,t,g:A_2 \;\Bigg|\; \begin{tikzcd}
FX \arrow[r, "\cong"', "s"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circledcirc"] & A \arrow[d, "g"] \\
FY \arrow[r, "\cong", "t"'] & B
\end{tikzcd}\Bigg]$$

$\mathcal{I}$ is defined so that for

$$F : \mathcal{P}(\mathbb{C}_{fun}) \xrightarrow[\wedge \neg \exists]{} \mathcal{P}(Set)$$

that is, for F any free-living functor,

$$\boxed{F \restriction L \;=\; F^{\#}}$$

CONCLUSION: Sentence Φ in first-order logic with equality over L_{fun}

is preserved under equivalence of (free-living) functors

if and only if

for all functors F,

$$F \models \Phi \iff F \models \mathcal{I}(\varphi)$$

$\boxed{\text{for some FOLDS}/L_{anafun}\text{-sentence } \varphi.}$