RESIDUALLY TORSION-FREE NILPOTENT ONE RELATOR GROUPS

JOHN LABUTE

Abstract. We show that the group $G = \langle x_1, ..x_m, y_1, ..y_n \mid u = v\rangle$ is residually torsion-free nilpotent if $v \in \langle y_1, ..y_n\rangle$, $v \neq 1$, $u \in A = \langle x_1, ..x_m\rangle$, $u \in \gamma_d(A)$, u not a proper power mod $\gamma_{d+1}(A)$, where $\gamma_k(A)$ is the k-th term of the lower central series of A.

In memory of Gilbert Baumslag

In [1] Azarov proves that the group

$$G = \langle x_1, ..x_m, y_1, ..y_n \mid u = v\rangle$$

is residually a finite p-group for any prime p if $u \in A = \langle x_1, ..x_m\rangle$ is not a proper power and $v \in B = \langle y_1, ..y_n\rangle$, $v \neq 1$.

Let $\gamma_k(A)$ be the k-th term of the lower central series of A. If we strengthen the condition on u by requiring that $u \in \gamma_d(A)$ but not a proper power mod $\gamma_{d+1}(A)$ we obtain the following result which extends a result of Baumslag and Mikhailov (cf. [2], Theorem 5).

Theorem 1. *If $u \in \gamma_d(A)$ and u is not a proper power mod $\gamma_{d+1}(A)$ there exists a central series (G_i) in G such that the quotients G_i/G_{i+1} are torsion free and the intersection of the groups G_i is 1. In particular G is residually torsion-free nilpotent.*

A sequence of subgroups $G_i (i \geq 1)$ is said to be a central series for G if $G_1 = G$, $G_{i+1} \subseteq G_i$, $[G_i, G_j] \subseteq G_{i+j}$. To construct the required central series we will make use of the Magnus embedding of the free group $F = A * B$ into the Magnus algebra M of formal power series in the non-commuting variables $X_1, ..X_m, Y_1, ..Y_n$ with integer coefficients which sends x_i into $1 + X_i$ and y_i into $1 + Y_i$.

Let e be an integer greater than the integer d in Theorem 1. If

$$f = \sum a_{i_1 \dots i_k} Z_{i_1} \cdots Z_{i_k} \in M$$

is a sum of distinct monomials $Z_{i_1} \cdots Z_{i_k}$ with $a_{i_1 \dots i_k} \in \mathbb{Z}$ $(k \geq 0)$ and Z_i an element of $\{X_1, \dots, X_m, Y_1, \dots, Y_n\}$, we define a valuation v on M by

$$v(f) = \min\{a + eb \mid a_{i_1 \dots i_k} \neq 0\}$$

where a and b are respectively the number of the X_i and Y_i in $Z_{i_1} \cdots Z_{i_k}$. In particular we have $v(X_i) = 1, v(Y_i) = e$. By convention $v(0) = \infty$.

For $i \geq 1$ let $M_i = \{f \in M \mid v(f) \geq i\}$ and let $F_i = F \cap (1 + M_i)$ where we have identified F with its image in M. Then (F_i) is a central series for F (cf. [5], sect. 2).

Date: March 8, 2015.

1991 *Mathematics Subject Classification.* 12G10, 20F05, 20F14, 20F40.

I would like to thank Roman Mikhailov for pointing out the work of Azarov.

(Note that if $e = 1$ in the definition of v we have $F_i = \gamma_i(F)$ by a deep result of Magnus, cf. [5], section 3.) If G_i is the image of F_i in G we will show that (G_i) is the required central series.

Lemma 2. *For $i \geq e$ we have $F_i \subseteq \gamma_{[i/e]}(F)$ which shows that $G_i \subseteq \gamma_{[i/e]}(G)$ and hence that $\bigcap G_i = \bigcap \gamma_i(G) = 1$.*

This follows from the fact that if $i = a + eb$ then

$$a + b = (ea + eb)/b \geq (a + eb)/e = i/e$$

and the fact that G is residually nilpotent by Azarov's Theorem.

In order to show that the quotients G_i/G_{i+1} are torsion-free we have to bring into play the Lie ring structure on the graded ring $L(G) = \oplus_i(G_i/G_{i+1})$. The Lie ring $L(F) = \oplus_i(F_i/F_{i+1})$ is free on $\xi_1, ..\xi_m, \eta_1, ..\eta_n$ where ξ_i is the image of X_i in F_1/F_2 and η_j is the image of Y_j in F_e/F_{e+1} (cf. [5], section 3). The problem is to determine the kernel of the canonical surjection $L(F) \to L(G)$. Let ρ be the image of $r = uv^{-1}$ in F_d/F_{d+1}. Since $v \in F_{d+1}$, we have that ρ is also the image of u in F_d/F_{d+1}.

Theorem 3. *The Lie ring $\mathfrak{g} = L(F)/(\rho)$ is torsion-free.*

Theorem 4. *The kernel of $L(F) \to L(G)$ is $\mathfrak{r} = (\rho)$.*

To prove Theorem 3 we note that since ρ is not a proper multiple of an element of $L(F)$ the Lie algebra $(L(F)/(\rho)) \otimes \mathbb{F}_p$ is a graded Lie algebra over the finite field $\mathbb{F}_p$ defined by a single non-zero relator of degree d. In [3], théorème 2 we prove that the homogeneous component of degree n of this graded algebra has a finite dimension which depends only on n and d. Since p is an arbitrary prime this proves that the homogeneous components of $\mathfrak{g} = L(F)/(\rho)$ are torsion free. The Theorem of Birkhoff-Witt then shows that the enveloping algebra U of $\mathfrak{g}$ has no zero-divisors which can be used to prove that, via the adjoint representation, $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free U module generated by the image of ρ (cf. [3], théorème 1). This fact is the key to proving Theorem 4. The proof given in [4] can be easily be adapted to the valuation v used here. For details cf. [5], sections 2 and 3.

References

[1] D.N. Azarov, *On the Residual Nipotence of Free Products of Free Groups With Cylic Amalgamation*, Mathematical Notes, Vol. 64, No. 1, p. 3-7, (1998).

[2] Gilbert Baumslag and Roman Mikhailov, *Residual properties defined by basic commutators*, Groups Geom. Dyn. 8, p. 1-22 (2014).

[3] John P. Labute, *Algègres de Lie et pro-p-groupes définis par une seule relation*, Inv. Math. 4, p.142-158 (1967).

[4] John P. Labute, *On the Descending Central Series of Groups with a Single Defining Relation*, J. Alg. 14, p. 16-23, (1970).

[5] John P. Labute, *The Determination of the Lie Algebra Associated to the Lower Central Series of a Group*, Trans. AMS 288 p. 51-57 (1985)

Department of Mathematics and Statistics, McGill University, Burnside Hall, 805 Sherbrooke Street West, Montreal QC H3A 0B9, Canada

E-mail address: labute@math.mcgill.ca