MILD PRO-2-GROUPS AND 2-EXTENSIONS OF $\mathbb{Q}$ WITH RESTRICTED RAMIFICATION

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Abstract. Using the mixed Lie algebras of Lazard, we extend the results of the first author on mild groups to the case $p = 2$. In particular, we show that for any finite set S_0 of odd rational primes we can find a finite set S of odd rational primes containing S_0 such that the Galois group of the maximal 2-extension of $\mathbb{Q}$ unramified outside S is mild. We thus produce a projective system of such Galois groups which converge to the maximal pro-2-quotient of the absolute Galois group of $\mathbb{Q}$ unramified at 2 and ∞. Our results also allow results of Alexander Schmidt on pro-p-fundamental groups of marked arithmetic curves to be extended to the case $p = 2$ over a global field which is either a function field of characteristic $\neq 2$ or a totally imaginary number field.

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1. Introduction

In this paper we extend the theory of mild pro-p-groups developed in [8] to the case $p = 2$. In particular, we obtain the following result which is the missing ingredient in extending the results of Alexander Schmidt in [11] to the case $p = 2$ over a global field which is either a function field of characteristic $\neq 2$ or a totally imaginary number field. Let $H^i(G) = H^i(G, \mathbb{Z}/p\mathbb{Z})$.

Theorem 1.1. *Let G be a finitely generated pro-p-group. If $H^2(G) \neq 0$ and $H^1(G) = U \oplus V$ with the cup-product trivial on $U \times U$ and mapping $U \otimes V$ surjectively onto $H^2(G)$ then G is mild.*

For $p \neq 2$, Theorem 1.1 is a reformulation by Schmidt of a criterion for the mildness of a pro-p-group that was proven in [8]. We will show that mild pro-p-groups are also of cohomological dimension 2 when $p = 2$. To prove our results we have to further develop the theory of certain mixed Lie algebras of Lazard [9].

If S is a finite set of odd rational primes we let $G_S(2)$ be the Galois group of the maximal 2-extension of $\mathbb{Q}$ unramified outside S.

Theorem 1.2. *If S_0 is a finite set of odd rational primes there is a finite set S of odd rational primes containing S_0 such that $G_S(2)$ is mild.*

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Although the study of Galois groups of number fields with restricted ramification can be traced already to work of L. Kronecker and others in the 19-th century, the formal modern foundations were laid out by I.R. Šhafarevič. His work was influenced by geometrical considerations of finite coverings of Riemann surfaces ramified in a given finite set of primes, class field theory and a deep understanding of the Galois groups of local fields. His papers [13], [14] as well as his paper with E.S. Golod [15] demonstrated the extraordinary power of his vision. Koch's monograph [5], first published in 1970, summarized the important contributions to the subject. For example, information of the cohomological dimension of $G_S(p)$ was obtained when p was odd and in S. When p was not in S, nothing was known about $G_S(p)$, other that it could be infinite by the work of Golod and Shafarevich, until the recent work of the first author [8] where it was shown that for p odd this group was of cohomological dimension 2 for certain S. The more difficult case $p = 2$ was left open. This work finally extends these results to the case $p = 2$.

2. Mixed Lie Algebras

Let G be a pro-2-group and let G_n $(n \geq 1)$ be the n-th term of the lower 2-central series of G. We have

$$G_1 = G, \quad G_{n+1} = G_n^2[G, G_n]$$

where, for subgroups H, K of G, $[H, K]$ is the closed subgroup generated by the commutators $[h, k] = h^{-1}k^{-1}hk$ with $h \in H, k \in K$ and H^2 is the subset of squares h^2 of elements of H. Let $L(G)$ be the Lie algebra associated to the lower 2-central series of G. We have

$$L(G) = \oplus_{n\geq 1} L_n(G)$$

where $L_n(G) = G_n/G_{n+1}$ is denoted additively. This defines L(G) as a graded vector space over $\mathbb{F}_2$. If l_n is the canonical homomorphism $G_n \to L_n(G)$, the Lie bracket $[\xi, \eta]$ of $\xi = l_m(x)$, $\eta = l_n(y)$ is $l_{m+n}([x, y])$. To the homogeneous element $\xi = l_n(x)$ we associate the homogeneous element $P\xi = l_{n+1}(x^2)$. If $\xi, \eta \in L_n(G)$ then

$$P(\xi + \eta) = \begin{cases} P\xi + P\eta & \text{if } n > 1, \\ P\xi + P\eta + [\xi, \eta] & \text{if } n = 1. \end{cases}$$

If $\xi \in L_m(G)$, $\eta \in L_n(G)$ we have

$$[P\xi, \eta] = \begin{cases} P[\xi, \eta] & \text{if } m > 1, \\ P[\xi, \eta] + [\xi, [\xi, \eta]] & \text{if } m = 1. \end{cases}$$

Thus the operator P defines a mixed Lie algebra structure on $L(G)$ in the terminology of Lazard, cf. [9], Ch.2, §1.2. The operator P extends to a linear operator on the Lie algebra

$$L^+(G) = \oplus_{n>1} \mathrm{L}_n(G).$$

It follows that $L^+(G)$ is a module over the polynomial ring $\mathbb{F}_2[\pi]$ where $\pi u = P(u)$.

If $A = \sum_{n\geq 0} A_n$ is a graded associative algebra over the graded algebra $\mathbb{F}_2[\pi]$, where multiplication by π on homogeneous elements increases the degree by 1, then $A_+ = \sum_{n>0} A_n$ has the structure of a mixed Lie algebra where

$$P\xi = \begin{cases} \pi\xi & \text{if } \xi \text{ is of degree} > 1, \\ \pi\xi + \xi^2 & \text{if } \xi \text{ is of degree } 1. \end{cases}$$

Every mixed Lie algebra $\mathfrak{g}$ has an enveloping algebra $U_{\mathrm{mix}}(\mathfrak{g})$. This is graded associative algebra U over $\mathbb{F}_2[\pi]$ together with and a mixed Lie algebra homomorphism f of $\mathfrak{g}$ into U_+ such that, for every graded associative algebra B over $\mathbb{F}_2[\pi]$ and mixed Lie algebra homomorphism φ_0 of $\mathfrak{g}$ into B_+, there is a unique algebra homomorphism φ of U into B satisfying $\varphi \circ f = \varphi_0$. The existence of $U_{\mathrm{mix}}(\mathfrak{g})$ is proven in [9], Th. 1.2.8. It is also shown there that the canonical mapping of $\mathfrak{g}$ into $U(\mathfrak{g})$ is injective; this fact is referred to as the Birkhoff-Witt Theorem for mixed Lie algebras. If $X = \{x_1, \ldots, x_d\}$ is a weighted set, the enveloping algebra of the free mixed Lie algebra $L_{\mathrm{mix}}(X)$ on the weighted set X is the free associative algebra $A(X)$ over $\mathbb{F}_2[\pi]$ on X. Indeed, giving a mixed Lie algebra homomorphism $f : L_{\mathrm{mix}}(X) \to B_+$ is the same as giving a graded map of X into B_+ which is the same as giving a homomorphism of the graded algebra $A(X)$ into B. It is now a straight-forward argument to verify the following Proposition.

Proposition 2.1. *If $0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{h} \to 0$ is an exact sequence of mixed Lie algebras, we have*

$$U_{\mathrm{mix}}(\mathfrak{h}) = U_{\mathrm{mix}}(\mathfrak{g})/\mathfrak{R}$$

where $\mathfrak{R}$ is the ideal of $U_{\mathrm{mix}}(\mathfrak{g})$ generated by the image of $\mathfrak{r}$.

Let $X = \{x_1, \ldots, x_d\}$ be a set and let $F = F(X)$ be the free pro-2-group on X. The completed group algebra $\Lambda = \mathbb{Z}_2[[F]]$ over the 2-adic integers $\mathbb{Z}_2$ is isomorphic to the Magnus algebra of formal power series in the non-commuting indeterminates $X_1, \ldots, X_d$ over $\mathbb{Z}_2$. Identifying F with its image in Λ, we have $x_i = 1 + X_i$ (cf. [12], Ch. I, §1.5).

The lower 2-cental series of F can be obtained by means of a valuation on Λ. More generally, if $\tau_1, \ldots, \tau_d$ are integers > 0, we define a valuation w in the sense of Lazard by setting

$$w(\sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}) = \inf_{i_1,\ldots,i_k} (v(a_{i_1,\ldots,i_k}) + \tau_{i_1} + \cdots + \tau_{i_k}),$$

where v is the 2-adic valuation of $\mathbb{Z}_2$ with $v(2) = 1$. Let $\Lambda_n = \{u \in A \mid w(u) \geq n\}$. Then $(\Lambda_n)_{n\geq 0}$ is a filtration of Λ by ideals and the associated graded algebra $\mathrm{gr}(\Lambda)$ is a graded algebra over the graded ring $\mathbb{F}_2[\pi] = \mathrm{gr}(\mathbb{Z}_2)$ with π the image of 2 in $2\mathbb{Z}_2/4\mathbb{Z}_2$. If ξ_i is the image of X_i in $\mathrm{gr}_{\tau_i}(\Lambda)$ then $\mathrm{gr}(\Lambda)$ is the free associative $\mathbb{F}_2[\pi]$-algebra on $\xi_1, \ldots, \xi_d$ with a grading in which ξ_i is of degree τ_i and multiplication by π increases the degree by 1. The Lie subalgebra L of $\mathrm{gr}(\Lambda)$ generated by the ξ_i is the free mixed Lie algebra over $\mathbb{F}_2[\pi]$ on $\xi_1, \ldots, \xi_d$ by the Birkhoff-Witt Theorem. Note that when $\tau_i = 1$ for all i we have $\Lambda_n = I^n$, where I is the augmentation ideal $(2, X_1, \ldots, X_d)$ of Λ.

For $n \geq 1$, let $F_n = (1 + \Lambda_n) \cap F$ and for $x \in F$ let $\omega(x) = w(x-1)$ be the filtration degree of x. Then (F_n) is a decreasing sequence of closed subgroups of F with the following properties:

$$F_1 = F,\ [F_n, F_k] \subseteq F_{n+k},\ F_n^2 \subseteq F_{n+1}.$$

It is called the (x, τ)-filtration of F. Such a sequence of subgroups of a pro-2-group G is called a 2-central series of G. If $\tau_i = 1$ for all i then (F_n) is the lower 2-central series of F.

If (G_n) is a 2-central series of G, let $\mathrm{gr}_n(G) = G_n/G_{n+1}$ with the group operation denoted additively. Then $\mathrm{gr}(G) = \oplus_{n\geq 1}\mathrm{gr}_n(G)$ is a graded vector space over $\mathbb{F}_2$ with a bracket operation $[\xi, \eta]$ which is defined for $\xi \in G_n$, $\eta \in G_k$ to be the image in $\mathrm{gr}_{n+k}(F)$ of $[x, y]$ where x, y are representatives of ξ, η in $\mathrm{gr}_n(G), \mathrm{gr}_k(G)$ respectively. Under this bracket operation, $\mathrm{gr}(G)$ is a Lie algebra over $\mathbb{F}_2$. The mapping $x \mapsto x^2$ induces an operator P on $\mathrm{gr}(G)$ sending $\mathrm{gr}_n(G)$ into $\mathrm{gr}_{n+1}(G)$. For homogeneous ξ, η of degree m, n respectively, we have

$$\begin{aligned}
P(\xi + \eta) &= P(\xi) + P(\eta) + [\xi, \eta] \text{ if } m = n = 1,\\
P(\xi + \eta) &= P(\xi) + P(\eta) \text{ if } m = n > 1,\\
[P(\xi), \eta] &= P([\xi, \eta]) + [\xi, [\xi, \eta]] \text{ if } m = 1,\\
[P(\xi), \eta] &= P([\xi, \eta]) \text{ if } m > 1.
\end{aligned}$$

Hence $\mathrm{gr}(G)$ is a mixed Lie algebra.

In the case $F = F(X)$ and $F_n = (1+\Lambda_n)\cap F$, the mapping $x \mapsto x-1$ induces an injective Lie algebra homomorphism of $\mathrm{gr}(F)$ into $\mathrm{gr}(\Lambda)$. Identifying $\mathrm{gr}(F)$ with its image in $\mathrm{gr}(\Lambda)$, we have $P(\xi) = \pi\xi$ unless $\xi \in \mathrm{gr}_1(F)$ in which case

$$P(\xi) = \xi^2 + \pi\xi.$$

The Lie algebra $\mathrm{gr}(F)$ is the smallest $\mathbb{F}_2$-subalgebra of $\mathrm{gr}(\Lambda)$ which contains $\xi_1, \ldots \xi_d$ and is stable under P. To see this, let X_n be the set of elements x_i with $\tau_i = n$ and define subsets T_n inductively as follows: $T_1 = X_1$ and, for $n > 1$, $T_n = T_n' \cup T_n''$ where

$$T_n' = \{x^2 \mid x \in T_{n-1}\}, \quad T_n'' = X_n \cup \{[x, y] \mid x \in T_r'', y \in T_s'',\ r + s = n\}.$$

If F_n' is the closed subgroup of F generated by the T_k with $k \geq n$, then (F_n') is a 2-central series of F (cf. [9], §1.2). If $\mathrm{gr}'(F)$ is the associated graded Lie-algebra, the inclusions $F_n' \subseteq F_n$ induce a mixed Lie algebra homomorphism $\mathrm{gr}'(F) \to \mathrm{gr}(F)$. We obtain a sequence of mixed Lie algebra homomorphisms

$$L_{\mathrm{mix}}(X) \to \mathrm{gr}'(F) \to \mathrm{gr}(F) \to \mathrm{gr}(\Lambda),$$

where the homomorphism $L_{\mathrm{mix}}(X) \to \mathrm{gr}'(F)$ sends ξ_i to ξ_i', the image of ξ_i in $\mathrm{gr}'_{\tau_i}(F)$, and hence is surjective since the ξ_i' generate $\mathrm{gr}'(F)$ as a mixed Lie algebra over $\mathbb{F}_2[\pi]$. The composite of these homomorphisms sends ξ_i to ξ_i and hence is injective. Thus $\mathrm{gr}'(F) \to \mathrm{gr}(F)$ is injective from which it follows inductively that $F_n' = F_n$ for all n. Hence we obtain that $\mathrm{gr}(F(X)) = L_{\mathrm{mix}}(X)$. The above 2-filtration (F_n) of F is called the (x, τ)-filtration of F. If $\tau_i = 1$ for all i then (F_n) is the lower 2-central series of F. Thus we have shown the following result.

Theorem 2.2. *If $L(F(X))$ is the Lie algebra associated to the (x,τ)-filtration of the free pro-2-group $F(X)$ on the weighted set $X = \{x_1, \ldots, x_d\}$, with x_i of weight τ_i, then $L(F(X)) = L_{\mathrm{mix}}(X)$, the free mixed Lie algebra on $X = \{\xi_1, \ldots, \xi_d\}$, where ξ_i is the image of x_i in $L_{\tau_i}(F(X))$.*

Theorem 2.3. *$L^+(X)$ is a free Lie algebra over $\mathbb{F}_2[\pi]$. If $\xi_1, \ldots, \xi_m$ are the elements of X of weight 1 then, as a free Lie algebra, $L^+(X)$ has a basis Y consisting of*

(1) *the $\binom{m+1}{2}$ elements*

$$P\xi_1, \ldots, P\xi_m, \ [\xi_i, \xi_j] \ (1 \le i < j \le m),$$

(2) *the elements*

$$\xi_{m+1}, \ldots, \xi_d, \ [\xi_i, \xi_j] \ (1 \le i \le m, \ m+1 \le j \le d),$$

(3) *for $3 \le k$, the $(k-1)\binom{m}{k-1}$ commutators*

$$\mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2}) \cdots \mathrm{ad}(\xi_{i_{k-3}})\mathrm{ad}(\xi_j)^2(\xi_{i_{k-2}}),$$

where $m \ge i_1 > i_2 > \cdots > i_{k-2} \ge 1$, $1 \le j \le m$, $j \ne i_1, \ldots, i_{k-2}$,

(4) *for $3 \le k$, the $(k-1)\binom{m}{k}$ commutators*

$$\mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2}) \cdots \mathrm{ad}(\xi_{i_{k-2}})\mathrm{ad}(\xi_{i_{k-1}})(\xi_{i_k}),$$

where $m \ge i_1 > i_2 > \cdots > i_{k-1} \ge 1$, $i_{k-1} < i_k \le m$, $i_k \ne i_1, \ldots, i_{k-2}$,

(5) *for $3 \le k$, the $\binom{m}{k-1}(d-m)$ commutators*

$$\mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2}) \cdots \mathrm{ad}(\xi_{i_{k-2}})\mathrm{ad}(\xi_{i_{k-1}})(\xi_{i_k}),$$

where $m \ge i_1 > i_2 > \cdots > i_{k-1} \ge 1$, $i_k > m$.

If $A = A(X)$ is the free associative $\mathbb{F}_2[\pi]$-algebra on X and B is the subalgebra of A generated by Y then B is the free associative algebra over $\mathbb{F}_2[\pi]$ on the weighted set Y. Moreover, A is a free B-module with basis $\xi_1^{e_1} \cdots \xi_s^{e_s}$ ($e_i = 0, 1$).

Proof. Let A be the free associative algebra on $X = \{\xi_1, \ldots, \xi_d\}$ over $\mathbb{F}_2[\pi]$ and let $\bar{L}$ be the Lie subalgebra over $\mathbb{F}_2$ generated by X. Then $\bar{L}$ is the free Lie algebra over $\mathbb{F}_2$ generated by X. If $L = L_{\mathrm{mix}}(X)$ we have

$$L_1 = \bar{L}_1 = \sum_{i=1}^{m} \mathbb{F}_2 \xi_i,$$

$$L_n = \pi^{n-2} \sum_{i=1}^{m} \mathbb{F}_2 P\xi_i + \pi^{n-2}\bar{L}_2 + \cdots + \pi \bar{L}_{n-1} + \bar{L}_n \ (n \ge 2).$$

Let Z be a homogeneous basis of $\bar{L}$ containing X with $\xi_1, \ldots, \xi_m$ the elements of Z of degree 1. If Z^+ is the set of elements of Z of degree > 1 then

$$Z^* = \{P\xi_1, P\xi_2, \ldots, P\xi_m\} \cup Z^+$$

is an $\mathbb{F}_2$-basis for L^+ modulo πL^+ and hence is an $\mathbb{F}_2[\pi]$-basis for the free $\mathbb{F}_2[\pi]$-module L^+. If $Z = \{\eta_i \mid i \ge 1\}$ is linearly ordered so that $\eta_i \le \eta_{i+1}$ and degree(η_i) $\leq$ degree(η_{i+1}) then, by the Birkhoff-Witt theorem for Lie algebras over $\mathbb{F}_2$, the elements

$$\eta^\alpha = \prod_{i\geq 1} \eta_i^{\alpha_i},$$

where $\alpha = (\alpha_i)_{i\geq 1}$ with $\alpha_i = 0$ for almost all i, form a $\mathbb{F}_2$-basis of $\bar{A} = A/\pi A$, the enveloping algebra of $\bar{L}$. It follows that the elements

$$\prod_{i=1}^m \eta_i^{\beta_i} \prod_{i=1}^m \eta_i^{2\gamma_i} \prod_{i>m} \eta_i^{\alpha_i},$$

where $\beta_i = 0, 1$ and $\gamma_i, \alpha_i \in \mathbb{N}$, are also an $\mathbb{F}_2$-basis of $\bar{A}$. Note that, in our convention, $0 \in \mathbb{N}$. Hence the elements

$$\prod_{i=1}^m \eta_i^{\beta_i} \prod_{i=1}^m P\eta_i^{\gamma_i} \prod_{i>m} \eta_i^{\alpha_i},$$

where $\beta_i = 0, 1$ and $\gamma_i, \alpha_i \in \mathbb{N}$, are a $\mathbb{F}_2[\pi]$-basis for A. In particular, the elements

$$\prod_{i=1}^m P\eta_i^{\gamma_i} \prod_{i>m} \eta_i^{\alpha_i},$$

where $\gamma_i, \alpha_i \in \mathbb{N}$, are an $\mathbb{F}_2[\pi]$-basis for the $\mathbb{F}_2[\pi]$-subalgebra B of A generated by Z^*. This implies that A is a free B-module with basis

$$\xi_1^{i_1} \cdots \xi_m^{i_m} \quad (i_k = 0, 1).$$

Let a_n be the number of elements of Z of degree n. Then

$$\prod_{n\geq 1} (1 - t^n)^{-a_n} = \frac{1}{1 - \sum_i m_i t^{e_i}},$$

where $e_1 < e_2 < \cdots < e_r$ are the possible values of the $\tau_i = \deg(\xi_i)$ and m_i is the number of j with $\tau_j = e_i$. We can rewrite this equation in the form $(1+t)^m P(t) = (1 - \sum_i m_i t^{e_i})^{-1}$ where

$$\begin{aligned}
P(t) &= (1 - t^2)^{-m} \prod_{n\geq 2} (1 - t^n)^{-a_n} \\
&= (1 - t^2)^{-(a_2+m)} \prod_{n\geq 3} (1 - t^n)^{-a_n} \\
&= \frac{1}{1 - (c_2t^2 + c_3t^3 + \cdots + c_{m+1}t^{m+1} + \sum_{k\geq 1} q_k(t))},
\end{aligned}$$

where

$$\begin{aligned}
c_k &= (k-1)\binom{m+1}{k} \\
&= (k-1)\binom{m}{k-1} + (k-1)\binom{m}{k}, \\
q_k(t) &= \sum_{j\geq 2} \binom{m}{k-1} m_j t^{k-1+e_j}.
\end{aligned}$$

The power series $P(t)$ is the Poincaré series of $\bar{B} = B/\pi B$; the Poincaré series of B is $P(t)/(1-t)$.

To show that the elements of Y generate L^+ it suffices to show that they generate L^+ as a vector space over $\mathbb{F}_2$ modulo $\pi L^+ + [L^+, L^+]$. For $k > 2$, we have $L_k^+ = \bar{L}_k$ modulo πL^+. For $k \geq 2$, every element of $\bar{L}_k$ can be uniquely written modulo $[\bar{L}, \bar{L}]$ as a linear combination of the sequence S of elements of the form

$$\mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2})\cdots\mathrm{ad}(\xi_{i_{k-2}})\mathrm{ad}(\xi_{i_{k-1}})(\xi_{i_k})$$

with $d \geq i_1 \geq i_2 \geq \cdots \geq i_{k-1} \geq 1$ and $i_{k-1} < i_k$. Modulo πL^+ we have

$$[P(\xi_i), P(\xi_j)] = \mathrm{ad}(\xi_i)\mathrm{ad}(\xi_j)^2(\xi_i),$$
$$[P(\xi_i), u] = \mathrm{ad}(\xi_i)^2(u) \text{ if } u \in L^+$$

and $\mathrm{ad}(\xi_i)\mathrm{ad}(\xi_j)(u) = \mathrm{ad}(\xi_j)\mathrm{ad}(\xi_i)(u)$ modulo $[L^+, L^+]$ if $u \in L^+$. If follows that the only terms of the sequence S which possibly do not lie in $\pi L^+ + [L^+, L^+]$ are the terms of the subsequence T of elements of the form

$$\text{(A)} \qquad \mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2})\cdots\mathrm{ad}(\xi_{i_{k-2}})\mathrm{ad}(\xi_{i_{k-1}})(\xi_{i_k})$$

with $m \geq i_1 > i_2 > \cdots > i_{k-1} \geq 1$ and $i_{k-1} < i_k$, or of the form

$$\text{(B)} \qquad \mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2})\cdots\mathrm{ad}(\xi_{i_{k-3}})\mathrm{ad}(\xi_{i_{k-2}})^2(\xi_{i_{k-1}})$$

with $m \geq i_1 > i_2 > \cdots > i_{k-2} \geq 1$, $i_{k-2} < i_{k-1} \leq m$, or of the form

$$\text{(C)} \qquad \mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2})\cdots\mathrm{ad}(\xi_{i_{k-2}})\mathrm{ad}(\xi_{i_{k-1}})(\xi_{i_k})$$

with $m \geq i_1 > i_2 > \cdots \geq i_{k-1} \geq 1$ and $i_{k-1} < i_k = i_1$. Working modulo $\pi L^+ + [L^+, L^+]$, this last element is equal to

$$\mathrm{ad}(\xi_{i_2})\cdots\mathrm{ad}(\xi_{i_{k-2}})\mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_{k-1}})(\xi_{i_k}) = \mathrm{ad}(\xi_{i_2})\cdots\mathrm{ad}(\xi_{i_{k-2}})\mathrm{ad}(\xi_{i_1})^2(\xi_{i_{k-1}})$$

which is an element in the family (3) in the statement of the theorem. Using the identity

$$\mathrm{ad}(x)\mathrm{ad}(y)^2\mathrm{ad}(z) = \mathrm{ad}(z)\mathrm{ad}(y)^2\mathrm{ad}(x) \pmod{\pi L^+ + [L^+, L^+]},$$

the elements of the form (B) can be also written in the form (3). The elements in (A) with $i_k \leq m$ account for the elements in (4) and the elements in (A) with $i_k > m$ account for the elements in (5). The later account for the terms $q_k(t)$ in $P(t)$. Thus Y generates $L^+(X)$ and so the canonical mapping of $L(Y)$, the free Lie algebra over $\mathbb{F}_2[\pi]$ on the weighted set Y, into L^+ is surjective. It is injective since $L(Y)$ and L^+ have the same Poincaré series. □

Corollary 2.4. *Let $\tilde{L}_{\mathrm{mix}}(X) = L_{\mathrm{mix}}(X)/\pi L_{\mathrm{mix}}(X)^+$ and let Y be as in Theorem 2.3. Then $\tilde{L}_{\mathrm{mix}}(X)^+ = \bar{L}(Y)$, the free Lie algebra over $\mathbb{F}_2$ on Y. Its enveloping algebra $\bar{B}$ is the subalgebra of $\bar{A} = \bar{A}(X)$ (the free associative $\mathbb{F}_2$-algebra on X) generated by $\tilde{L}(X)^+$. The $\bar{B}$-module $\bar{A}$ is free with basis consisting of the elements $\xi_1^{i_1}\cdots\xi_m^{i_m}$ $(i_k = 0, 1)$.*

This follows immediately from the fact that A is a free B-module with basis $\xi_1^{i_1}\cdots\xi_m^{i_m}$ $(i_k = 0, 1)$.

3. Quadratic Lie Algebras

If $\mathfrak{g}$ is a mixed Lie algebra we let $\tilde{\mathfrak{g}} = \mathfrak{g}/\pi\mathfrak{g}^+$. Then $\tilde{\mathfrak{g}}$ is a Lie algebra over $\mathbb{F}_2$ which we call the reduced algebra of $\mathfrak{g}$. The operator P on $\mathfrak{g}$ induces an operator on $\tilde{\mathfrak{g}}$, also denoted by P, which is zero in degree > 1 and which, for homogeneous elements ξ, η, satisfies

(QL1) $P(\xi + \eta) = P(\xi) + P(\eta) + [\xi, \eta]$ if ξ, η are of degree 1,
(QL2) $[P\xi, \eta] = [\xi, [\xi, \eta]]$ if ξ is of degree 1.

Thus $\tilde{\mathfrak{g}}$ satisfies the axioms for a mixed Lie algebra where $P(\xi) = 0$ if ξ is homogeneous of degree > 1. It is an example of what we call a quadratic Lie algebra.

Definition 3.1 ((Quadratic Lie Algebra)). *A quadratic Lie algebra is a graded Lie algebra $\mathfrak{h} = \oplus_{i\geq 1}\mathfrak{h}_i$ over $\mathbb{F}_2$ together with a mapping $P : \mathfrak{h}_1 \to \mathfrak{h}_2$ satisfying* (QL1) *and* (QL2)*.*

A homomorphism $f : \mathfrak{h} \to \mathfrak{h}'$ of quadratic Lie algebras is a homomorphism of graded Lie algebras (over $\mathbb{F}_2$) such that $f(P(s)) = P(f(s))$ for every homogenous element s of degree 1. By an ideal of $\mathfrak{h}$ we mean an ideal $\mathfrak{a}$ of $\mathfrak{h}$ as a Lie algebra over $\mathbb{F}_2$ such that $P(s) \in \mathfrak{a}$ for every element s of $\mathfrak{a}$ of degree 1. Every quadratic Lie algebra is a mixed Lie algebra if we set $P\xi = 0$ for every homogeneous element ξ of degree 1. In this way Quadratic Lie algebras form a full subcategory of the category of mixed Lie algebras.

If $A = \oplus_{i\geq 0}A_i$ is a graded associative algebra over $\mathbb{F}_2$ then the mapping $P : x \mapsto x^2$ of A_1 into A_2 together with the bracket $[x, y] = xy+yx$ defines the structure of a quadratic Lie algebra on $A_+ = \oplus_{i>0}A_i$. Indeed, we have $(x+y)^2 = x^2+y^2+xy+yx$ and

$$[x, [x, y]] = [x, xy + yx] = x^2y + xyx + xyx + yx^2 = [x^2, y].$$

Definition 3.2 ((Derivation of a quadratic Lie algebra)). *If $\mathfrak{h}$ is a quadratic Lie algebra then by a derivation of $\mathfrak{h}$ we mean an additive mapping $D : \mathfrak{h} \to \mathfrak{h}$ that*

(Der 1) *There is an integer $s \geq 1$ such that $D(\mathfrak{h}_n) \subseteq \mathfrak{h}_{n+s}$ (s is the degree of D),*
(Der 2) *$D(P(\xi)) = [\xi, D(\xi)]$ if ξ is homogeneous of degree 1,*
(Der 3) *$D[\xi, \eta] = [D(\xi), \eta] + [\xi, D(\eta)]$.*

The set $\mathrm{Der}_{\mathrm{quad}}(\mathfrak{h})$ of derivations of the quadratic Lie algebra $\mathfrak{h}$ is a quadratic Lie algebra under the operations of addition and Lie bracket $[D_1, D_2] = D_1D_2 + D_2D_1$ with $P(D) = D^2$ if D is of degree 1. The grading is defined by the degree of a derivation.

If $\mathfrak{a}$ and $\mathfrak{h}$ are Lie algebras over $\mathbb{F}_2$ and f is a homomorphism of $\mathfrak{h}$ into the Lie algebra of derivations of $\mathfrak{a}$, the semi-direct product of $\mathfrak{a}$ and $\mathfrak{h}$ is the direct product $\mathfrak{a} \times \mathfrak{h}$ as vector spaces with the Lie algebra structure given by

$$[(\xi, \sigma), (\xi', \sigma')] = ([\xi, \xi'] + f(\sigma)(\xi') + f(\sigma')(\xi), [\sigma, \sigma']).$$

We denote this Lie algebra by $\mathfrak{a} \times_f \mathfrak{h}$. We will agree to identify $\mathfrak{a}$ and $\mathfrak{h}$ with their canonical images in $\mathfrak{a} \times_f \mathfrak{h}$. If $\mathfrak{a}$ and $\mathfrak{h}$ are graded then so is $\mathfrak{a} \times_f \mathfrak{h}$ with n-th homogeneous component $\mathfrak{a}_n \times \mathfrak{h}_n = \mathfrak{a}_n + \mathfrak{h}_n$.

Theorem 3.3. *Let $\mathfrak{a}$ and $\mathfrak{h}$ be quadratic Lie algebras and f is a homomorphism of $\mathfrak{h}$ into $\mathrm{Der}_{\mathrm{quad}}(\mathfrak{a})$. If (ξ, σ) is an element of $\mathfrak{a} \times \mathfrak{h}$ of degree 1 then*

$$P(\xi, \sigma) = (P(\xi) + f(\sigma)(\xi), P(\sigma))$$

defines the structure of a quadratic Lie algebra on $\mathfrak{a} \times_f \mathfrak{h}$.

Proof. Let $\xi + \sigma$, $\xi' + \sigma'$ be elements of $\mathfrak{a} \times \mathfrak{h}$ of degree 1. Then

$$\begin{aligned}
&P(\xi + \sigma) + \xi' + \sigma') = P(\xi + \xi' + \sigma + \sigma') = \\
&P(\xi + \xi') + f(\sigma + \sigma')(\xi + \xi') + P(\sigma + \sigma') = \\
&P(\xi) + P(\xi') + [\xi, \xi'] + f(\sigma)(\xi) + f(\sigma)(\xi') + f(\sigma')(\xi) + f(\sigma')(\xi') + \\
&\qquad P(\sigma) + P(\sigma') + [\sigma, \sigma'] = \\
&P(\xi + \sigma) + P(\xi' + \sigma') + [\xi + \sigma, \xi' + \sigma'].
\end{aligned}$$

If $\xi + \sigma$ is of degree 1 we have

$$\begin{aligned}
&[P(\xi + \sigma), \xi' + \sigma'] = [P(\xi) + f(\sigma)(\xi) + P(\sigma), \xi' + \sigma'] = \\
&[P(\xi) + f(\sigma)(\xi), \xi'] + f(P(\sigma))(\xi') + f(\sigma')((P(\xi) + f(\sigma)(\xi) + [P(\sigma), \sigma'] = \\
&[P(\xi), \xi'] + [f(\sigma)\xi, \xi'] + f(\sigma)^2\xi' + [\xi, f(\sigma')(\xi)] + f(\sigma')f(\sigma)(\xi) + [P(\sigma), \sigma'] = \\
&[\xi, [\xi, \xi']] + [f(\sigma)(\xi, \xi'] + f(\sigma)^2(\xi') + [\xi, f(\sigma')(\xi) + f(\sigma')f(\sigma)(\xi) + [\sigma, [\sigma, \sigma']] = \\
&[\xi, [\xi, \xi']] + [\xi, f(\sigma)(\xi')] + [\xi, f(\sigma')(\xi)] + f(\sigma)([\xi, \xi'] + f(\sigma)^2(\xi') + f(\sigma)f(\sigma')(\xi) \\
&\quad + f([\sigma, \sigma'])(\xi) + [\sigma, [\sigma, \sigma]] = \\
&[\xi + \sigma, [\xi, \xi'] + f(\sigma)(\xi') + f(\sigma')(\xi) + [\sigma, \sigma']] = [\xi + \sigma, [\xi + \sigma, \xi' + \sigma']].
\end{aligned}$$

□

If X is a homogeneous subset of the quadratic Lie algebra $\mathfrak{h}$ then the quadratic subalgebra of $\mathfrak{h}$ generated by X is the smallest Lie subalgebra $\mathfrak{a}$ of $\mathfrak{h}$ which contains X and which contains $P(x)$ for every $x \in X$ of degree 1. Let $\mathfrak{h}^* = P(\mathfrak{h}_1) + [\mathfrak{h}, \mathfrak{h}]$. Then $\mathfrak{h}^*$ is a vector subspace of $\mathfrak{h}$ by (QL1). The proof of the following result is left to the reader.

Proposition 3.4. *The subset X generates the quadratic Lie algebra $\mathfrak{h}$ if and only its image in the vector space $\mathfrak{h}/\mathfrak{h}^*$ is a generating set.*

If X is a weighted set then the natural map of $\tilde{L}_{\mathrm{mix}}(X) = L_{\mathrm{mix}}(X)/\pi L_{\mathrm{mix}}(X)^+$ into $\bar{A}(X) = A(X)/\pi A(X)$ is injective map of quadratic Lie algebras. We use this to identify $\tilde{L}_{\mathrm{mix}}(X)$ with the quadratic subalgebra of the free associative algebra $\bar{A}(X)$ over $\mathbb{F}_2$ generated by X. If $\bar{L}(X)$ is the Lie subalgebra of $\bar{A}(X)$ generated by X we have

$$\tilde{L}_{\mathrm{mix}}(X) = \bar{L}(X) + \sum_{s \in S} \mathbb{F}_2 s^2,$$

where S is the set of elements of X of degree 1 and $P(s) = s^2$ for $s \in S$. The Lie algebra $\bar{L}(X)$ is the free Lie algebra over $\mathbb{F}_2$ on X. Note that $\tilde{L}_{\mathrm{mix}}(X)/\tilde{L}_{\mathrm{mix}}(X)^* = \bar{L}(X)/[\bar{L}(X), \bar{L}(X)]$.

Proposition 3.5. *The Lie algebra $\tilde{L}_{\mathrm{mix}}(X)$ is the free quadratic Lie algebra on the set X.*

Proof. Let f be a weight preserving map of X into a quadratic Lie algebra $\mathfrak{h}$. Then f extends uniquely to a Lie algebra homomorphism φ_0 of $\bar{L}(X)$ into $\mathfrak{h}$. The only way to extend φ_0 to a quadratic Lie algebra homomorphism φ of $\tilde{L}_{\mathrm{mix}}(X)$ into $\mathfrak{h}$ is to define $\varphi(P(s)) = P(\varphi(s))$ for any $s \in S$ and to extend by linearity to all of $\tilde{L}_{\mathrm{mix}}(X)$. A straightforward verification yields that $\varphi([P(s), y]) = [\varphi(P(s)), \varphi(y)]$ for any $y \in \bar{L}(X)$ and that $\varphi([P(s), P(t)] = [\varphi(P(s)), \varphi(P(t))]$ for any $s, t \in S$ and hence that φ is a homomorphism of quadratic Lie algebras. □

Every quadratic Lie algebra $\mathfrak{h}$ has a universal enveloping algebra $U = U_{\mathrm{quad}}(\mathfrak{h})$. More precisely, there is a graded associative algebra U over $\mathbb{F}_2$ and a quadratic Lie algebra homomorphism f of $\mathfrak{h}$ into U_+ such that for every quadratic Lie algebra homomorphism φ_0 of $\mathfrak{h}$ into an associative algebra B over $\mathbb{F}_2$ there is a unique algebra homomorphism φ of U into B satisfying $\varphi \circ f = \varphi_0$. We have $U_{\mathrm{quad}}(\tilde{L}_{\mathrm{mix}}(X)) = \bar{A}(X)$ since $\bar{A}(X)$ has the correct universal property. More generally, we have

Proposition 3.6. *Let $\mathfrak{g} = \tilde{L}_{\mathrm{mix}}(X)/\mathfrak{r}$ be a presentation of a quadratic Lie algebra $\mathfrak{g}$ and let $\mathfrak{R}$ be the ideal of $\bar{A}(X) = U_{quad}(\tilde{L}_{\mathrm{mix}}(X))$ generated by the image of $\mathfrak{r}$. Then*

$$\bar{A}(X)/\mathfrak{R} = U_{\mathrm{quad}}(\mathfrak{g}).$$

Proposition 3.7. *Let $\mathfrak{g}$ be a mixed Lie algebra and $\tilde{\mathfrak{g}} = \mathfrak{g}/\pi\mathfrak{g}^+$ the reduced algebra of $\mathfrak{g}$. If $U = U_{\mathrm{mix}}(\mathfrak{g})$ then $U_{\mathrm{quad}}(\tilde{\mathfrak{g}}) = U/\pi U$.*

Proof. If $\mathfrak{g} = L_{\mathrm{mix}}(X)/\mathfrak{r}$ then $\tilde{\mathfrak{g}} = \tilde{L}_{\mathrm{mix}}(X)/\tilde{\mathfrak{r}}$, where $\tilde{\mathfrak{r}}$ is the image of $\mathfrak{r}$ in $\tilde{L}_{\mathrm{mix}}(X)$. Then

$$U_{\mathrm{quad}}(\tilde{\mathfrak{g}}) = \bar{A}(X)/\tilde{\mathfrak{R}},$$

where $\tilde{\mathfrak{R}}$ is the image of $\mathfrak{R}$ in $\bar{A}(X)$. □

4. Strongly Free Sequences

Let $\rho_1, \dots, \rho_m \in L = L_{\mathrm{mix}}(X)$ with ρ_i homogeneous of degree $h_i > 1$ and let $\mathfrak{r}$ be the ideal of the free mixed Lie algebra L generated by $\rho_1, \dots, \rho_m$. Let $\mathfrak{g} = L/\mathfrak{r}$. Then $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a module over the enveloping algebra $U = U_{\mathrm{mix}}(\mathfrak{g})$ via the adjoint representation.

Definition 4.1. *The sequence $\rho_1, \dots, \rho_m$ is said to be strongly free in L if the following conditions hold.*

(i) *The $\mathbb{F}_2[\pi]$-module U is torsion free.*
(ii) *The U-module M is free on the images of $\rho_1, \dots, \rho_m$.*

Let $\tilde{\rho}_i$ be the image of ρ_i in $\tilde{L} = \tilde{L}_{\text{mix}}(X)$ and let $\tilde{\mathfrak{r}}$ be the ideal of $\tilde{L}$ generated by $\tilde{\rho}_1, \ldots, \tilde{\rho}_m$. Let $\tilde{\mathfrak{g}} = \tilde{L}/\tilde{\mathfrak{r}}$. Then $\tilde{M} = \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$ is a module over the enveloping algebra $\tilde{U} = U_{\text{quad}}(\tilde{\mathfrak{g}})$ via the adjoint representation.

Definition 4.2. *The sequence $\tilde{\rho}_1, \ldots, \tilde{\rho}_m$ is said to be a strongly free in $\tilde{L}$ if the $\tilde{U}$-module $\tilde{M}$ is free on the images of $\tilde{\rho}_1, \ldots, \tilde{\rho}_m$.*

Let $X = \{\xi_1, \ldots, \xi_d\}$ with ξ_i of weight e_i.

Theorem 4.3. *The sequence $\tilde{\rho}_1, \ldots, \tilde{\rho}_m$ is strongly free in $\tilde{L}$ if and only if the Poincaré series of $\tilde{U}$ is*

$$1/(1 - (t^{e_1} + \cdots + t^{e_d}) + t^{h_1} + \cdots + t^{h_m}).$$

Proof. Let $\mathfrak{R}$ be the ideal of $\bar{A}(X)$ generated by $\tilde{\mathfrak{r}}$. Then $\bar{A}(X)/\mathfrak{R} = U_{\text{quad}}(\tilde{\mathfrak{g}}) = \tilde{U}$. If I is the augmentation ideal of $V = \bar{A}(X)$ and J is the augmentation ideal of $W = U_{\text{quad}}(\tilde{\mathfrak{r}})$ then, by tensoring the exact sequence $0 \to I \to V \to \mathbb{F}_2 \to 0$ with $\mathbb{F}_2 = W/J$ over W, we obtain the exact sequence

$$\mathrm{Tor}_1^W(\mathbb{F}_2, V) \to \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}] \to I/\mathfrak{R}I \to V/\mathfrak{R} \to \mathbb{F}_2 \to 0$$

using the fact that

1. If M is a W-module then $M \otimes_W (W/J) = M/JM$;
2. $\mathfrak{R} = \tilde{\mathfrak{r}}V = V\tilde{\mathfrak{r}}$;
3. $\mathrm{Tor}_1^W(\mathbb{F}_2, \mathbb{F}_2) = \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$ (cf. [3], Ch. XIII, §2).

The map $\tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}] \to I/\mathfrak{R}I$ is induced by the inclusion $\tilde{\mathfrak{r}} \subseteq I$. Since I is the direct sum of the left ideals $V\xi_i$. The $\tilde{U}$-module $I/\mathfrak{R}I$ is the direct sum of the free $\tilde{U}$-submodules Ug_i where g_i is the image of ξ_i in $U = \bar{A}(X)/\mathfrak{R}$. Since $\tilde{\mathfrak{r}} \subset \tilde{L}$ the algebra $V = \bar{A}(X)$ is a free W-module by Corollary 2.4 and the Birkhoff-Witt Theorem for Lie algebras over $\mathbb{F}_2$. In this case we have the exact sequence

$$0 \to \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}] \to I/\mathfrak{R}I \to \bar{A}(X)/\mathfrak{R} \to \mathbb{F}_2 \to 0.$$

Expressing $\tilde{M} = \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$ as a quotient $\tilde{U}^m/N$ using the relators $\tilde{\rho}_i$, we obtain the exact sequence of graded modules whose homogeneous components are finitely generated free $\mathbb{F}_2$-modules

$$0 \to N \to \oplus_{j=1}^m \tilde{U}[h_j] \to \oplus_{j=1}^d \tilde{U}[e_j] \to U \to \mathbb{F}_2 \to 0$$

where $\tilde{U}[n] = \tilde{U}$ but with degrees shifted by n; by definition, $\tilde{U}[n](t) = t^n \tilde{U}(t)$. We have $N = 0$ if and only if $\tilde{M}$ is a free $\tilde{U}$-module on the images of the $\tilde{\rho}_i$.

Taking Poincaré series in the above long exact sequence, we get

$$N(t) - (t^{h_1} + \cdots + t^{h_m})\tilde{U}(t) + (t^{e_1} + \cdots + t^{e_d})\tilde{U}(t) - \tilde{U}(t) + 1 = 0.$$

Solving for $\tilde{U}(t)$, we get $\tilde{U}(t) = P(t) + N(t)P(t)$, where

$$P(t) = \frac{1}{1 - (t^{e_1} + \cdots + t^{e_d}) + t^{h_1} + \cdots + t^{h_m}}.$$

Hence $N(t) = 0 \iff \tilde{U}(t) = P(t)$. $\square$

Theorem 4.4. *The sequence $\rho_1, \dots, \rho_m$ is strongly free in $L = L_{\mathrm{mix}}(X)$ if and only if the sequence $\tilde\rho_1, \dots, \tilde\rho_m$ is strongly free in $\tilde L$.*

Proof. If $\rho_1, \dots, \rho_m$ is a strongly free sequence then the enveloping algebra U of the mixed Lie algebra $\mathfrak{g} = L/\mathfrak{r}$ is a torsion free $\mathbb{F}_2[\pi]$-module. By the Birkhoff-Witt Theorem for mixed Lie algebras, the canonical mapping of $\mathfrak{g}$ into U is injective. Hence $\mathfrak{g}^+ = L^+/\mathfrak{r}$ is a torsion free $\mathbb{F}_2[\pi]$-module. If B is the subalgebra of $A = A(X)$ generated by L^+ then B is the enveloping algebra of L^+. By Birkhoff-Witt the canonical mapping of the enveloping algebra W of $\mathfrak{r}$ into B is injective and B is a free W-module. Since A is a free B-module it follows that A is a free W-module. Thus, if $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ and $\mathfrak{R}$ the ideal of A generated by $\mathfrak{r}$ and I the augmentation ideal of A, we have an exact sequence

$$0 \to M \to I/\mathfrak{R}I \to A/\mathfrak{R} \to \mathbb{F}_2[\pi] \to 0.$$

As in the proof of Theorem 4.3 we obtain that the Poincaré series of U is

$$Q(t) = \frac{1}{(1-t)(1-(t^{e_1} + \cdots + t^{e_d}) + t^{h_1} + \cdots + t^{h_m})}.$$

If $\tilde U$ is the enveloping algebra of $\tilde L/(\tilde\rho_1, \dots, \tilde\rho_m)$ we have $\tilde U = U/\pi U = U \otimes_{\mathbb{F}_2} \mathbb{F}_2[\pi]$. Since U is torsion free over $\mathbb{F}_2[\pi]$ the Poincaré series of $\bar U$ is $(1-t)Q(t)$ which proves that the sequence $\tilde\rho_1, \dots, \tilde\rho_m$ is strongly free.

Conversely, suppose that the sequence $\tilde\rho_1, \dots, \tilde\rho_m$ is strongly free in $\tilde L$. We have the exact sequence of graded vector spaces over $\mathbb{F}_2$

$$0 \to K \to M \to U[e_1] \oplus \cdots \oplus U[e_d] \to U \to \mathbb{F}_2 \to 0.$$

Taking Poincaré series we get

$$K(t) - M(t) + (t^{e_1} + \cdots + t^{e_d})U(t) - U(t) + \frac{1}{1-t} = 0$$

from which we get $M(t) = K(t) - (1 - (t^{e_1} + \cdots + t^{e_d}))U(t) + 1/(1-t)$. Hence

$$\frac{M(t)}{1-(t^{e_1} + \cdots + t^{e_d})} = \frac{K(t)}{1-(t^{e_1} + \cdots + t^{e_d})} + \frac{1}{(1-t)(1-(t^{e_1} + \cdots + t^{e_d}))} - U(t).$$

Now suppose that $\tilde\rho_1, \dots, \tilde\rho_m$ is strongly free. Then, if $\tilde{\mathfrak{r}}$ is the ideal of $\tilde L$ generated by $\tilde\rho_1, \dots, \tilde\rho_m$ and $\tilde M = \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$, we have surjections

$$\tilde U[h_1] \oplus \cdots \oplus \tilde U[h_m] \to \tilde M \to \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$$

whose composite is an isomorphism. It follows that

$$\tilde M \cong \tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}] \cong \tilde U[h_1] \oplus \cdots \oplus \tilde U[h_m],$$

$$M(t) \le \frac{\tilde M(t)}{1-t} = \frac{1}{1-t} \cdot \frac{t^{h_1} + \cdots + t^{h_m}}{1-(t^{e_1} + \cdots + t^{e_d}) + t^{h_1} + \cdots + t^{h_m}},$$

$$U(t) \le \frac{\tilde U(t)}{1-t} = \frac{1}{1-t} \cdot \frac{1}{1-(t^{e_1} + \cdots + t^{e_d}) + t^{h_1} + \cdots + t^{h_m}}.$$

Using the fact that $K(t) \geq 0$, we get

$$\frac{M(t)}{1-(t^{e_1}+\cdots+t^{e_d})} \geq \frac{1}{(1-t)(1-(t^{e_1}+\cdots+t^{e_d}))} - \frac{\tilde{U}(t)}{1-t} =$$
$$\frac{1}{1-t}\left(\frac{1}{(1-(t^{e_1}+\cdots+t^{e_d})} - \frac{1}{1-(t^{e_1}+\cdots+t^{e_md})+t^{h_1}+\cdots+t^{h_m}}\right) =$$
$$\frac{\tilde{M}(t)}{(1-t)(1-(t^{e_1}+\cdots+t^{e_d}))} \geq \frac{M(t)}{1-(t^{e_1}+\cdots+t^{e_d})}.$$

It follows that $K(t)=0$, $U(t)=\tilde{U}(t)/(1-t)$ and $M(t)=\tilde{M}/(1-t)$. Hence U is a free $\mathbb{F}_2[\pi]$-module and M is a free U-module since we have a natural surjection

$$U[h_1]\oplus\cdots U[h_m]\to M$$

with both sides having the same Poincaré series. □

In general it is very difficult to determine whether a sequence in $\tilde{L}$ is strongly free but we can construct a large supply using the following elimination theorem for free quadratic Lie algebras.

Theorem 4.5 ((Elimination Theorem))**.** *Let S be a subset of the weighted set X and let $\mathfrak{a}$ be the ideal of the free quadratic Lie algebra $\tilde{L}_{\mathrm{mix}}(X)$ generated by $X-S$. Then $\mathfrak{a}$ is a free quadratic Lie algebra with basis*

$$\mathrm{ad}(\sigma_1)\mathrm{ad}(\sigma_2)\cdots\mathrm{ad}(\sigma_n)(\xi), \quad (n\geq 0,\ \sigma_i\in S,\ \xi\in X-S).$$

Proof. We first show that the quadratic Lie algebra $\tilde{L}_{\mathrm{mix}}(X)$ is the semi-direct product of the quadratic Lie algebras $\mathfrak{a}$ and $\tilde{L}_{\mathrm{mix}}(S)$. Let f be the adjoint representation of $\tilde{L}_{\mathrm{mix}}(S)$ on $\mathfrak{a}$. Then f is a homomorphism of the quadratic Lie algebra $\tilde{L}_{\mathrm{mix}}(S)$ into the quadratic Lie algebra $\mathrm{Der}_{\mathrm{quad}}(\mathfrak{a})$ of derivations of the quadratic Lie algebra $\mathfrak{a}$. More precisely, if $f(\sigma)=D$ then $f(P(\sigma))=D^2$ and $D(P(\xi))=[\xi,D(\xi)]$ if σ,ξ are homogeneous of degree 1. Every element of $\tilde{L}_{\mathrm{mix}}(X)$ can be uniquely written in the form $\xi+\sigma$ with $\xi\in\mathfrak{a},\sigma\in\tilde{L}_{\mathrm{mix}}(S)$. We have

$$[\xi_1+\sigma_1,\xi_2+\sigma_2]=[\xi_1,\xi_2]+f(\sigma_1)(\xi_2)+f(\sigma_2)(\xi_1)+[\sigma_1,\sigma_2]$$

and $P(\xi+\sigma)=P(\xi)+f(\sigma)(\xi)+P(\sigma)$ if ξ,σ are of degree 1. As a quadratic Lie algebra, $\mathfrak{a}$ is generated by the family of elements

$$\mathrm{ad}(\sigma_1)\mathrm{ad}(\sigma_2)\cdots\mathrm{ad}(\sigma_n)(\xi), \quad (n\geq 0,\sigma_i\in S,\xi\in X-S).$$

If $\sigma\in S$ and $f(\sigma)=D$ then

$$D(\mathrm{ad}(\sigma_1)\mathrm{ad}(\sigma_2)\cdots\mathrm{ad}(\sigma_n)(\xi))=\mathrm{ad}(\sigma)\mathrm{ad}(\sigma_1)\mathrm{ad}(\sigma_2)\cdots\mathrm{ad}(\sigma_n)(\xi).$$

Let T be the family of elements $(\sigma_1,\sigma_2,\ldots,\sigma_n,\xi)$ with $n\geq 0,\sigma_i\in S,\xi\in X-S$ and weight equal to the sum of the weights of the components σ_i,ξ. Let φ_1 be the quadratic Lie algebra homomorphism of $\tilde{L}_{\mathrm{mix}}(T)$ into $\mathfrak{a}$ such that

$$\varphi_1(\sigma_1,\sigma_2,\ldots,\sigma_n,\xi)=\mathrm{ad}(\sigma_1)\mathrm{ad}(\sigma_2)\cdots\mathrm{ad}(\sigma_n)(\xi).$$

Since φ_1 is surjective it suffices to prove φ_1 is injective. Let g be the quadratic Lie algebra homomorphism of $\tilde{L}_{\mathrm{mix}}(S)$ into $\mathrm{Der}_{\mathrm{quad}}(\tilde{L}_{\mathrm{mix}}(T))$ where, for $\sigma\in S$, we define $g(\sigma)$ be the derivation which takes $(\sigma_1,\sigma_2,\ldots,\sigma_n,\xi)$ into $(\sigma,\sigma_1,\sigma_2,\ldots,\sigma_n,\xi)$.

That such a derivation exists follows from the fact that the derivations D of the free Lie algebra $\bar{L}(T)$ can be assigned arbitrarily and can be uniquely extended to derivations of the quadratic Lie algebra $\tilde{L}(T)$ by defining $D(\xi^2) = [\xi, D(\xi)]$ if ξ is an element of T of degree 1. Let L be the semi-direct product of $\tilde{L}_{\text{mix}}(T)$ and $\tilde{L}_{\text{mix}}(S)$ with respect to the homomorphism g. Every element of L can be uniquely written in the form $\xi + \sigma$ with $\xi \in \tilde{L}_{\text{mix}}(T), \sigma \in \tilde{L}_{\text{mix}}(S)$. Then

$$[\xi_1 + \sigma_1, \xi_2 + \sigma_2] = [\xi_1, \xi_2] + g(\sigma_1)(\xi_2) + g(\sigma_2)(\xi_1) + [\sigma_1, \sigma_2].$$

and $P(\xi+\sigma) = P(\xi) + g(\sigma)(\xi) + P(\sigma)$ if ξ, σ are of degree 1. Since $\varphi_1(g(\sigma)(\xi)) = f(\sigma)(\varphi_1(\xi))$ we see that there is a unique homomorphism φ of L into $\tilde{L}_{\text{mix}}(X)$ which restricts to φ_1 and is the identity on $\tilde{L}_{\text{mix}}(S)$. If ψ is the homomorphism of $\tilde{L}(X)$ into L which is the identity on X we have $\varphi \circ \psi$ and $\psi \circ \varphi$ identity maps so that φ and hence φ_1 is bijective. □

Corollary 4.6. *If B is the enveloping algebra of $\tilde{L}_{\text{mix}}(S) = \tilde{L}_{\text{mix}}(X)/\mathfrak{a}$ then, via the adjoint representation, $\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]$ is a free B-module with basis the images of the elements $\xi \in X - S$.*

Let X be a finite weighted set and let $S \subset X$. Let $\mathfrak{a}$ be the ideal of $\tilde{L} = \tilde{L}_{\text{mix}}(X)$ generated by $X - S$ and let B be the enveloping algebra of $\tilde{L}/\mathfrak{a}$.

Theorem 4.7. *Let $T = \{\tau_1, \dots, \tau_t\} \subset \mathfrak{a}$ whose elements are homogeneous of degree ≥ 2 and B-independent modulo $\mathfrak{a}^*$. If $\rho_1, \dots, \rho_m$ are homogeneous elements of $\mathfrak{a}$ which lie in the $\mathbb{F}_2$-span of T modulo $\mathfrak{a}^*$ and which are linearly independent over $\mathbb{F}_2$ modulo $\mathfrak{a}^*$ then the sequence $\rho_1, \dots, \rho_m$ is strongly free in $\tilde{L}$.*

Proof. Let $\mathfrak{r}$ is the ideal of $\tilde{L}$ generated by $\rho_1, \dots, \rho_m$ and let $U = U_{\text{quad}}$ be the enveloping algebra of $\tilde{L}/\mathfrak{r}$. The elements

$$\text{ad}(\sigma_1)\text{ad}(\sigma_2)\cdots\text{ad}(\sigma_n)(\rho_j)$$

with $1 \leq j \leq m$, $n \geq 0$, $\sigma_i \in S$ generate $\mathfrak{r}$ as an ideal of the quadratic Lie algebra $\mathfrak{a}$. Suppose that these elements form part of a basis of the free quadratic Lie algebra $\mathfrak{a}$. The elimination theorem then shows that $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free module over the enveloping algebra C of $\mathfrak{a}/\mathfrak{r}$ with the images of these elements as basis. Now let μ_i be the image of ρ_i in M and suppose that $\sum_i u_i \mu_i = 0$ with $u_i \in U$. Then, since every u_i can be written in the form

$$u_i = \sum \bar{c}_{ij} w_j$$

where the w_j are distinct products of elements of S and $c_{ij} \in C$ with $\bar{c}_{ij}$ its image in U, the dependence relation

$$0 = \sum_i u_i \mu_i = \sum_{i,j} (\bar{c}_{ij}\, w_j)\mu_i = \sum_{i,j} c_{ij}(w_j \mu_i)$$

implies that all c_{ij} are zero and hence that each u_i is zero.

To show that the elements of the form $\text{ad}(\sigma_1)\text{ad}(\sigma_2)\cdots\text{ad}(\sigma_n)(\rho_j)$ are part of a Lie algebra basis of $\mathfrak{a}$ it suffices to show that $\rho_1, \dots, \rho_m$ are B-independent modulo

$\mathfrak{a}^*$. We now work modulo $\mathfrak{a}^*$. If H is the $\mathbb{F}_2$-span of $\rho_1, \ldots, \rho_m$, we can find a basis $\gamma_1, \ldots, \gamma_m$ of H such that

$$\gamma_i = a_i\alpha_i + \sum_{j=1}^{s} a_{ij}\beta_j$$

where $a_i, a_{ij} \in \mathbb{F}_2$, $a_i \neq 0$, $m+s = t$, $T = \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_s\}$. If $u_1, \ldots, u_m \in B$, we have

$$\sum_{i=1}^{m} u_i\gamma_i = \sum_{i=1}^{m} a_iu_i\alpha_i + \sum_{j=1}^{s}(\sum_{i=1}^{m} a_{ij}u_i)\beta_j.$$

If $\sum_{i=1}^{m} u_i\gamma_i = 0 \mod \mathfrak{a}^*$ then by the B-independence of the elements of T we $a_iu_i = 0$ so that $u_i = 0$ for all i which implies the B-independence of $\gamma_1, \ldots, \gamma_m$ and hence of $\rho_1, \ldots, \rho_m$. □

Corollary 4.8. *Let $X = \{\xi_1, \ldots, \xi_d\}$ with $d \geq 4$ even and let $\rho_1, \ldots, \rho_d \in \tilde{L}_{\mathrm{mix}}(X)$ with*

$$\rho_i = a_i\xi_i^2 + \sum_{j=1}^{d} \ell_{ij}[\xi_i, \xi_j],$$

where (a) $a_i = 0$ *if i is odd,* (b) $\ell_{ij} = 0$ *if i, j odd,* (c) $\ell_{12} = \ell_{23} = \ldots = \ell_{d-1,d} = \ell_{d1} = 1$ *and* (d) $\ell_{1d}\ell_{d,d-1}\cdots\ell_{32}\ell_{21} = 0$. *Then the sequence $\rho_1, \ldots, \rho_d$ is strongly free.*

Proof. Let $\mathfrak{a}$ be the ideal of $\tilde{L}_{\mathrm{mix}}(X)$ generated by the ξ_i with i even and let $\mathfrak{b}$ be the subspace of $\mathfrak{a}_2$ generated by the ξ_i^2, $[\xi_i, \xi_j]$ with i, j even. Then the ρ_i are in $\mathfrak{a}$ and their images in $V = (\mathfrak{a}/\mathfrak{a}^*)_2 = \mathfrak{a}_2/\mathfrak{b}$ are linearly independent. Indeed, the images in V of the elements $[\xi_i, \xi_j]$ with i odd, j even $i < j$ form a basis for V which we order lexicographically. If A is the matrix representation of $\rho_1, \ldots, \rho_d$ with respect to this basis, the d columns $(1,2), (2,3), (3,4), \ldots, (1,d)$ of A form the matrix

$$\begin{bmatrix} \ell_{12} & 0 & 0 & \cdots & 0 & -\ell_{1m} \\ \ell_{21} & \ell_{23} & 0 & \cdots & 0 & 0 \\ 0 & \ell_{32} & \ell_{34} & \cdots & 0 & 0 \\ 0 & 0 & \ell_{43} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \ell_{m,m-1} & 0 \\ 0 & 0 & 0 & \cdots & \ell_{m,m-1} & \ell_{m1} \end{bmatrix}$$

which has determinant $\ell_{12}\ell_{23}\cdots\ell_{m-1,m}\ell_{m1} + \ell_{1m}\ell_{21}\ell_{32}\cdots\ell_{m,m-1} = 1$. □

Example 4.9. If $d \geq 4$ is even then

$$a_1\xi_1^2 + [\xi_1, \xi_2], a_2\xi_2^2 + [\xi_2, \xi_3], \ldots, a_d\xi_d^2 + [\xi_d, \xi_1]$$

is a strongly free sequence if $a_i = 0$ for i odd.

5. Mild Groups

Let $F = F(x_1, \dots, x_d)$ be the free pro-2-group on $x_1, \dots, x_d$ and let $G = F/R$ with R the closed normal subgroup of F generated by $r_1, \dots, r_m$. Let (F_n) be the filtration of F induced by the (x, τ)-filtration of F. It is induced by the (x, τ)-filtration of $\Lambda = \mathbb{Z}_2[[F]]$. Let G_n be the image of F_n in G and let Γ_n be the image of Λ_n in $\Gamma = \mathbb{Z}_2[[G]]$.

Let ρ_i be the initial form of r_i with respect to the (x, τ)-filtration of F ; by definition, if $r \in F_k,\ r \notin F_{k+1}$, the initial form of r is the image of r in $L_k(F) = \mathrm{gr}_k(F)$. We assume that the degree h_i of ρ_i is > 1.

Definition 5.1 ((Strongly Free Presentation))**.** *The presentation $G = F/R$ is strongly free if $\rho_1, \dots, \rho_m$ is strongly free in $L_{\mathrm{mix}}(F)$.*

Definition 5.2 ((Mild Group))**.** *A pro-2-group G is said to be* **weakly mild** *if it has a minimal presentation $G = F/R$ of finite type which is strongly free with respect to some (x, τ)-filtration of F. It is called* **mild** *if the $\tau_i = 1$ for all i in which case the (x, τ)-filtration is the lower 2-central series of F.*

Theorem 5.3. *Let F/R be a strongly free presentation of G with $R = (r_1, \dots, r_m)$. Let $\mathfrak{r}$ is the ideal of $L(F(X))$ generated by the initial forms $\rho_1, \dots, \rho_m$ of the defining relators $r_1, \dots, r_m$. Then*

(a) $L(G) = L(F)/\mathfrak{r}$.
(b) *The group $R/[R, R]$ is a free $\mathbb{Z}_2[[G]]$-module on the images of $r_1, \dots, r_m$.*
(c) *The presentation $G = F/R$ is minimal and* $\mathrm{cd}(G) = 2$.
(d) *The enveloping algebra of $L(G)$ is the graded algebra associated to the filtration (Γ_n) of $\Gamma = \mathbb{Z}_2[[G]]$, where Γ_n is the image of Λ_n in G.*
(e) *The filtration (G_n) of G is induced by the filtration (Γ_n) of Γ.*
(f) *The Poincaré series of* $\mathrm{gr}(\Gamma)$ *is* $1/(1-t)(1-(t^{\tau_1}+\dots+t^{\tau_d})+t^{h_1}+\ldots+t^{h_m}))$.
(g) *If $b_n = \dim \tilde{L}_n$ then the Poincaré series of* $\mathrm{gr}(\Gamma)/\pi\,\mathrm{gr}(\Gamma)$ *is equal to*

$$(1+t)^r \prod_{n \geq 2} (1-t^n)^{-b_n},$$

where $r = b_1$ is the number of i with $\tau_i = 1$.
(h) *If b_n, r are as in (g) and*

$$1 - (t^{\tau_1} + \dots + t^{\tau_d}) + t^{h_1} + \ldots + t^{h_m}) = (1-\alpha_1 t) \cdots (1-\alpha_s t)$$

then $\dim \mathrm{gr}_n(G)^+ = \sum_{k=2}^n b_k$ *with*

$$b_n = \frac{1}{n} \sum_{\ell | n} \mu(\frac{n}{\ell})(\alpha_1^\ell + \dots + \alpha_s^\ell + (-1)^\ell r).$$

Except for (g) and (h), the proof this theorem is the same as the proof of Theorem 4.1 in [8] except that the freeness of the Lie algebra $\mathfrak{r}$ over $\mathbb{F}_2[\pi]$ is deduced from the fact that $\mathfrak{r}$ is an ideal of the free Lie algebra $L_{\mathrm{mix}}(X)^+$ and that $L_{\mathrm{mix}}(X)^+/\mathfrak{r}$ a torsion free $\mathbb{F}_2[\pi]$-module.

To prove (g) and (h) let A be the enveloping algebra of the mixed Lie algebra $L = L_{\text{mix}}(F(X))$ and let B be the enveloping algebra of the $\mathbb{F}_2[\pi]$-Lie algebra L^+. Then L is the free mixed Lie algebra on $\xi_1, \ldots, \xi_d$, where ξ_i is the image of x_i in $\text{gr}_{\tau_i}(F)$. By Theorem 2.3, L^+ is a free Lie algebra over $\mathbb{F}_2[\pi]$ and the canonical map of B into A is injective. Moreover, assuming that $\xi_1, \ldots, \xi_s$ the ξ_i of degree 1, then A is a free B-module with basis $\xi_1^{e_1} \cdots \xi_s^{e_s}$ $(e_i = 0, 1)$. If $\mathfrak{r}_B$ be the ideal of B generated by $\mathfrak{r}$ then

$$\mathfrak{r}_A = \sum_{e_i=0,1} \xi_1^{e_1} \cdots \xi_s^{e_s} \mathfrak{r}_B$$

is the ideal of A generated by $\mathfrak{r}$. It follows that the canonical map of $B/\mathfrak{r}_B$ into $A/\mathfrak{r}_A$ is injective and that $A/\mathfrak{r}_A$ is a free $B/\mathfrak{r}_B$-module with basis $\xi_1^{e_1} \cdots \xi_s^{e_s}$ $(e_i = 0, 1)$. The algebra $U = A/\mathfrak{r}_A$ is the enveloping algebra of the mixed Lie algebra $\mathfrak{g} = L/\mathfrak{r}$ and $V = B/\mathfrak{r}_B$, the enveloping algebra of the Lie algebra $L^+/\mathfrak{r}$ over $\mathbb{F}_2[\pi]$. If $\bar{U} = U/\pi U$ and $\bar{V} = \bar{V}/\pi\bar{V}$ we obtain that the canonical map of $\bar{V}$ into $\bar{U}$ is injective and that $\bar{U}$ is a free $\bar{V}$-module with basis $\xi_1^{e_1} \cdots \xi_s^{e_s}$ $(e_i = 0, 1)$. The algebra $\bar{U}$ is the enveloping algebra of the quadratic Lie algebra $\tilde{\mathfrak{g}}$ and $\bar{V}$ is the enveloping algebra of the Lie algebra $\tilde{\mathfrak{g}}^+$ over $\mathbb{F}_2$.

We now use the fact that $\tilde{L}/\mathfrak{r}$, where $\tilde{\mathfrak{r}}$ is the image of $\mathfrak{r}$ in $\tilde{L}$, is a strongly free presentation to deduce that $P(\xi) \notin \tilde{\mathfrak{r}}$ for every non-zero element ξ of $\tilde{L}$ of degree 1. Indeed, if $P(\xi)$ lies in $\tilde{\mathfrak{r}}$ then, if $\bar{\xi}$ is the image of $P(\xi)$ in $\tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$ and $\tilde{\xi}$ the image of ξ in $\tilde{\mathfrak{g}}$, we would have $\bar{\xi}, \tilde{\xi} \neq 0$

$$ad(\tilde{\xi})(\bar{\xi}) = 0$$

which contradicts the fact that $\tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}}, \tilde{\mathfrak{r}}]$ is a free $\bar{V}$-module via the adjoint representation and the fact that $\bar{V}$ is an integral domain. Thus multiplication by $P(\xi) = \xi^2$ maps $\bar{V}$ injectively in to $\bar{V}$ which implies that multiplication by ξ is injective on $\bar{V}$. This in turn implies that

$$P_{\xi\bar{V}}(t) = tP_{\bar{V}}(t).$$

We thus obtain that $P_{\bar{U}}(t) = (1 + t)^s P_{\bar{V}}(t)$. This implies (g) since

$$P_{\bar{V}}(t) = \prod_{n \geq 2} (1 - t^n)^{-b_n}$$

and $U_{mix}(\text{gr}(G)) = U$. The assertion (h) follows form the fact that $\text{gr}(G)^+$ is a free $\mathbb{F}_2[\pi]$-module and a standard argument to compute b_n using the formula

$$(1 + t)^r \prod_{n \geq 2} (1 - t^n)^{-b_n} = \frac{1}{(1 - \alpha_1 t) \cdots (1 - \alpha_s t)}$$

6. Zassenhaus Filtrations

Theorem 5.3 can be extended under certain conditions to filtrations induced by valuations of the completed group ring $\mathbb{F}_2[[F]]$. The Lie algebras associated to these filtrations are restricted Lie algebras in the sense of Jacobson [4]. A sufficient condition is that the initial forms of the relators lie in a Lie subalgebra over $\mathbb{F}_2$ which is quadratic and that these initial forms are strongly free. This will give a second proof that the pro-2-group with these relators is of cohomological dimension 2.

Let F be the free pro-2-group on $x_1, \dots, x_d$. The completed group algebra $\bar{\Lambda} = \mathbb{F}_2[[F]]$ over the finite field $\mathbb{F}_2$ is isomorphic to the algebra of formal power series in the non-commuting indeterminates $X_1, \dots, X_d$ over $\mathbb{F}_2$. Identifying F with its image in $\bar{A}$, we have $x_i = 1 + X_i$.

If $\tau_1, \dots, \tau_d$ are integers > 0, we define a valuation $\bar{w}$ of $\bar{\Lambda}$ by setting

$$\bar{w}(\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} X_{i_1} \cdots X_{i_k}) = \inf_{i_1,\dots,i_k} (\tau_{i_1} + \cdots + \tau_{i_k}).$$

Let

$$\bar{\Lambda}_n = \{u \in \bar{\Lambda} \mid \bar{w}(u) \geq n\},\ \mathrm{gr}_n(\bar{\Lambda}) = \bar{\Lambda}_n/\bar{\Lambda}_{n+1},\ \mathrm{gr}(\bar{\Lambda}) = \oplus_{n\geq 0}\mathrm{gr}_n(\bar{\Lambda}).$$

Then $\mathrm{gr}(\bar{\Lambda})$ is a graded $\mathbb{F}_2$-algebra. If ξ_i is the image of X_i in $\mathrm{gr}_{\tau_i}(\bar{\Lambda})$ then $\mathrm{gr}(\bar{\Lambda})$ is the free associative $\mathbb{F}_2$-algebra $\bar{A}$ on $\xi_1, \dots, \xi_d$ with a grading in which ξ_i is of degree τ_i . Note that when $\tau_i = 1$ for all i we have $\bar{\Lambda}_n = \bar{I}^n$, where $\bar{I}$ is the augmentation ideal $(X_1, \dots, X_d)$ of $\bar{\Lambda}$.

The Lie subalgebra $\bar{L}$ of $\bar{A}$ generated by the ξ_i is the free Lie algebra over $\mathbb{F}_2$ on $\xi_1, \dots, \xi_m$ by the Birkhoff-Witt Theorem. The Lie subalgebra $\tilde{L}$ generated by $\xi_1, \dots, \xi_d$ and the ξ_i^2 where ξ_i is of degree 1 is the free quadratic Lie algebra on $\xi_1, \dots, \xi_d$.

A decreasing sequence (G_n) of closed subgroups of a pro-2-group G which satisfies

$$[G_i, G_j] \subseteq G_{i+j}, \quad G_i^2 \subseteq G_{2i}.$$

is called a called, after Lazard [9], a 2-restricted filtration of G.

For $n \geq 1$, let $F_n = (1 + \bar{\Lambda}_n) \cap F$. Then (F_n) is a 2-restricted filtration of F. This filtration is also called the Zassenhaus (x, τ)-fitration of F. The mapping $x \mapsto x^2$ induces an operator P on $\mathrm{gr}(F)$ sending $\mathrm{gr}_n(F)$ into $\mathrm{gr}_{2n}(F)$. With this operator, $\mathrm{gr}(F)$ is a restricted Lie algebra over $\mathbb{F}_2$. If $\tau_i = 1$ for all i, the subgroups F_n are the so-called dimension subgroups mod 2. They can be defined by

$$F_n = \langle [y_1, [\cdots [y_{r-1}, y_r] \cdots]]^{2^s} \mid y_1, \dots, y_r \in F,\ r2^s \geq n \rangle.$$

Let $r_1, \dots, r_m \in F^2[F, F]$ and let $R = (r_1, \dots, r_m)$ be the closed normal subgroup of F generated by $r_1, \dots, r_m$. Let $\rho_i \in \mathrm{gr}_{h_i}(F)$ be the initial form of r_i with respect to the Zassenhaus (x, τ)-filtration (F_n) of F. If $G = F/R$ and G_n is the image of F_n in $G = F/R$ then $(G_n)_{n\geq 1}$ is a 2-restricted filtration of G. Let $\bar{\Gamma}_n$ be the image of $\bar{\Lambda}_n$ in $\bar{\Gamma} = \mathbb{F}_2[[G]]$.

Theorem 6.1. *Suppose that the initial forms $\rho_1, \dots, \rho_m$ of $r_1, \dots, r_m$ are in $\tilde{L}$ and are strongly free. Then*

(a) *We have* $\mathrm{gr}(G) = \mathrm{gr}(F)/(\rho_1, \dots, \rho_m)$*,*
(b) *The group $R/R^2[R, R]$ is a free $\mathbb{F}_2[[G]]$-module on the images of $r_1, \dots, r_m$,*
(c) *The presentation $G = F/R$ is minimal and* $\mathrm{cd}(G) = 2$.
(d) *The enveloping algebra of* $\mathrm{gr}(G)$ *is the graded algebra associated to the filtration* $(\bar{\Gamma}_n)$.

(e) *The filtration* $(\bar{\Gamma}_n)$ *of* $\bar{\Gamma}$ *induces the filtration* (G_n) *of* G.
(f) *The Poincaré series of* $\mathrm{gr}(\bar{\Gamma})$ *is* $1/(1-(t^{\tau_1}+\cdots+t^{\tau_d})+t^{h_1}+\ldots+t^{h_m})$.
(g) *If* $\tau_i = 1$ *for all* i *and* $a_n = \dim \mathrm{gr}_n(G)$ *then*

$$\prod_{n\geq 1}(1+t^n)^{a_n} = \frac{1}{1-dt+mt^2}.$$

Proof. In [6], Koch proves that if $\bar{\mathcal{R}}/\bar{\mathcal{R}}\bar{I}$ is a free $\bar{A}/\bar{\mathcal{R}}$ module on the images of $\rho_1,\ldots,\rho_m$ then $\mathrm{gr}(\bar{\Gamma}) = \bar{A}/\bar{\mathcal{R}}$, where $\bar{\mathcal{R}}$ is the ideal of $\bar{A} = \mathrm{gr}(\bar{\Lambda})$ generated by $\rho_1,\ldots,\rho_m$. The former is true if $\rho_1,\ldots,\rho_m$ lie in $\tilde{L}$ and are strongly free since $\bar{\mathcal{R}}/\bar{\mathcal{R}}\bar{I}$ is the image of the free $\bar{A}/\bar{\mathcal{R}}$-module $\tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}},\tilde{\mathfrak{r}}]$ under the injective mapping

$$\tilde{\mathfrak{r}}/[\tilde{\mathfrak{r}},\tilde{\mathfrak{r}}] \to \bar{I}/\bar{\mathcal{R}}\bar{I}$$

where $\tilde{\mathfrak{r}}$ is the ideal of the quadratic Lie algebra $\tilde{L}$ generated by $\rho_1,\ldots,\rho_m$. Now consider the exact sequence

$$0 \to \mathfrak{r}/[\mathfrak{r},\mathfrak{r}] \to \mathrm{gr}(\bar{\Gamma})^d \to \mathrm{gr}(\bar{\Gamma}) \to \mathbb{F}_2 \to 0,$$

Since $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a free $\mathrm{gr}(\bar{\Gamma})$-module of rank m, we obtain the exact sequence

$$0 \to \mathrm{gr}(\bar{\Gamma})^m \to \mathrm{gr}(\bar{\Gamma})^d \to \mathrm{gr}(\bar{\Gamma}) \to \mathbb{F}_2 \to 0.$$

This yields (f). By a result of Serre (cf. [9], V, 2.1), we obtain the exact sequence

$$0 \to \bar{\Gamma}^m \to \bar{\Gamma}^d \to \bar{\Gamma} \to \mathbb{F}_2 \to 0.$$

By a result of [2], section 5, this proves (b) and (c). If $\mathfrak{R} = (\rho_1,\ldots,\rho_m)$ is the ideal of the restricted Lie algebra $\mathrm{gr}(F(X))$ generated by $\rho_1,\ldots,\rho_m$, we have canonical homomorphisms of restricted Lie algebras

$$\mathrm{gr}(F(X))/\mathfrak{R} \to \mathrm{gr}(G) \to \mathrm{gr}'(G) \to \mathrm{gr}(\bar{\Gamma}),$$

where the first arrow is surjective and $\mathrm{gr}'(G)$ is the restricted Lie algebra associated to the Zassenhaus filtration (G'_n) of G induced by the filtration of Γ. Since $\mathrm{gr}(\bar{\Gamma})$ is the enveloping algebra of the restricted Lie algebra $\mathrm{gr}(F)/\mathfrak{R}$, the Birkhoff-Witt Theorem for restricted Lie algebras shows that all arrows are injective which yields (a) and (d). The injectivity of $\mathrm{gr}(G) \to \mathrm{gr}'(G)$ yields $G_n = G'_n$ for all n by induction which proves (e). The proof of (g) follows from (d), (e), (f) and Proposition A3.10 of [9]. □

Remark. The formula given in (g) partially answers a question of Morishita stated in [10] in a remark after Theorem 3.6.

7. Proof of Theorem 1.1

Let $(\chi_i)_{1\leq i\leq d}$ be a basis of $H^1(G)$ with $(\chi_i)_{i\in S}$ a basis of U and $(\chi_j)_{j\in S'}$ a basis of V. Let (ξ_i) be the dual basis of $H^1(G)^* = L_1(G)$ and let g_i be any lift of ξ_i to G. Let F be the free pro-2-group on $x_1,\ldots,x_d$ and let $f : F \to G$ be the homomorphism sending x_i to g_i. Then the induced mapping of $L_1(F)$ into $L_1(G)$ is an isomorphism which we use to identify these two groups. If R is the kernel of f the presentation $G = F/R$ is minimal and the transgression map tg : $H^1(R/R^2[R,F]) \to H^2(G)$ is an isomorphism. Hence $\mathrm{tg}^* : H^2(G)^* \to R/R^2[R,F]$

is an isomorphism which we use to identify these two groups. If ψ is the inverse of tg^* and $r \in R$ we let $\bar{r} = \psi(r)$.

The cup product $H^1(G) \otimes H^1(G) \to H^2(G)$ vanishes on the subspace W generated by elements of the form $a \otimes b + b \otimes a$ and so, by duality, induces a homomorphism

$$H^2(G)^* \to L_1(F) \otimes L_1(F) = (H^1(G) \otimes H^1(G))^*,$$

whose image is contained in W^0, the annihilator of the subspace W. Since $\dim W = d(d-1)/2$ we have $\dim W^0 = d(d+1)/2$. Now $L_2(F)$ can be identified with the subspace of the tensor algebra of $L_1(F)$ generated by the elements of the form ξ^2 and $[\xi, \eta] = \xi\eta + \eta\xi$. Since these elements lie in W^0 and $\dim L_2(F) = \dim W^0$ we obtain that $W^0 = L_2(F)$. If

$$H^1(G) \otimes' H^1(G) = (H^1(G) \otimes H^1(G))/W$$

is the symmetric tensor product of $H^1(G)$ with itself we have

$$H^1(G) \otimes' H^1(G) = U \otimes' U \oplus V \otimes' V \oplus U \otimes' V,$$

where $U \otimes' V$ is the image of $U \otimes' V$ in $H^1(G) \otimes' H^1(G)$. Since the cup-product vanishes on $U \otimes' U$ it induces a homomorphism

$$\varphi : V \otimes' V \oplus U \otimes' V = (H^1(G) \otimes' H^1(G))/U \otimes' U \to H^2(G)$$

which is surjective since, by assumption, the cup-product maps $U \otimes V$ onto $H^2(G)$. Since the annihilator of $U \otimes' U$ is contained in $\mathfrak{a}_2$, where $\mathfrak{a}$ is the ideal of $L(F)$ generated by the ξ_i with $i \in S'$, we get an injective homomorphism

$$\varphi^* : H^2(G)^* \to \mathfrak{a}_2.$$

Let $r_1, \ldots, r_m$ generate R as a closed normal subgroup of F. Since $r_i \in F_2$ we have

$$r_k \equiv \prod_{i=1}^{d} x_i^{2a_{ik}} \prod_{i<j} [x_i, x_j]^{a_{ijk}} \mod F_3$$

with $a_{ik} = \bar{r}_k(\chi_i \cup \chi_i)$ and $a_{ijk} = \bar{r}_k(\chi_i \cup \chi_j)$ (cf. [7], Prop. 3). Moreover, if ρ_k is the initial form of r_k, we have

$$\varphi^*(\bar{r}_k) = \rho_k = \sum_{i=1}^{d} a_{ik}\xi_i^2 + \sum_{i<j} a_{ijk}[\xi_i, \xi_j].$$

By Theorems 4.4 and 4.7, the elements $\rho_1, \ldots, \rho_m$ form a strongly free sequence if their images in $(\mathfrak{a}/\mathfrak{a}^*)_2 = \mathfrak{a}_2/\mathfrak{b}$, where $\mathfrak{b}$ is the subspace of $\mathfrak{a}_2$ generated by the elements ξ_i^2, $[\xi_i, \xi_j]$ with $i, j \in S'$, are linearly independent. If $\mathfrak{c}$ is the subspace of $\mathfrak{a}_2$ generated by the elements $[\xi_i, \xi_j]$ with $i \in S, j \in S'$ then $\mathfrak{a}_2 = \mathfrak{b} \oplus \mathfrak{c}$. The images of the ρ_i in $\mathfrak{a}_2/\mathfrak{b}$ form a linearly independent sequence if and only if the projections of the ρ_i on $\mathfrak{c}$ form an independent sequence. But this is equivalent to the composite

$$H^2(G)^* \to \mathfrak{a}_2 \to \mathfrak{c}$$

being injective. Now $\mathfrak{a}_2$ is the dual space of

$$(H^1(G) \otimes' H^1(G))/U \otimes' U = V \otimes' V \oplus U \otimes' V$$

and, with respect to this duality, we have $\mathfrak{c} = (V \otimes' V)^0$ which implies that the canonical injection

$$\iota : U \otimes' V \to V \otimes' V \oplus U \otimes' V$$

is dual to the projection of $\mathfrak{a}_2$ onto $\mathfrak{c}$. Since $\phi \circ \iota$ is surjective it dual $\iota^* \circ \varphi^*$ is injective. But the latter is the composite $H^2(G)^* \to \mathfrak{a}_2 \to \mathfrak{c}$.

8. Proof of Theorem 1.2 and Examples

Without loss of generality, we may assume $S_0 = \{q_1, \dots, q_m\}$ with $m \geq 2$, $q_1 \equiv 1 \bmod 4$ and $q_m \equiv 3 \bmod 4$. Let $q_1', \dots, q_m'$ be primes $\equiv 1 \bmod 4$ which are not in S_0 and such that

(a) q_i' is a square mod q_j' for all i, j,
(b) q_1' is not a square mod q_m and q_i' is not a square mod q_i and q_{i-1} for $1 < i \leq m$.

Let $S = \{q_1', q_1, q_2', q_2, \dots, q_m', q_m, q_{m+1}\}$ where $q_{m+1}$is a prime $\equiv 3 \bmod 4$ distinct from $q_1, \dots, q_m$ and such that q_{m+1} is not a square mod q_1' but is a square mod q_i' for all $i \neq 1$. Let

$$(p_1, \dots, p_{2m+1}) = (q_1', q_1, q_2', q_2, \dots, q_m', q_m, q_{m+1}).$$

and let $x_1, \dots, x_{2m+1}$ be generators for the inertia subgroups of $G_S(2)$ at the primes $p_1, \dots, p_{2m+1}$ respectively. Then, by [5], Theorem 11.10 and Example 11.12, the group $G = G_S(2)$ has the presentation $G = F(X)/R = \langle x_1, \dots, x_{2m+1} \mid r_1, \dots, r_{2m+1}, r \rangle$, where

$$r_i \equiv x_i^{2a_i} \prod_{j=1}^{2m+1} [x_i, x_j]^{\ell_{ij}} \bmod F_3,$$

$$r \equiv \prod_{i=1}^{2m+1} x_i^{a_i} \bmod F_2$$

with $a_i = 0$ if and only if $p_i \equiv 1 \bmod 4$ and $\ell_{ij} = 1$ if p_i is not a square mod p_j and 0 otherwise. Moreover, we can omit the relator r_{2m+1}. By construction we have

$$r \equiv \prod_{i=2}^{m-1} x_{2i}^{a_{2i}} x_{2m} x_{2m+1} \bmod F_2$$

so that $x_{2m+1} \equiv x_{2m} x_4^{a_4} \cdots x_{2m-2}^{a_{2m-2}} \bmod F_2$. Hence $G = \langle x_1, \dots, x_{2m} \mid r_1', \dots, r_{2m}' \rangle$ where

$$r_i' \equiv x_i^{2a_i} \prod_{j=1}^{2m} [x_i, x_j]^{\ell_{ij}'}$$

with $\ell_{ij}' = 0$ if i, j are odd and

$$\ell_{12}' = \ell_{23}' = \ell_{34}' = \cdots = \ell_{2m-1,2m}' = \ell_{2m,1}' = 1$$

but $\ell_{1,2m}' = 0$. The image of the initial form of r_i' in $\tilde{L}_{\mathrm{mix}}(X)$ (here $\tau_i = 1$ for all i) is

$$\rho_i' = \xi_i^{2a_i} + \sum_{j=1}^{2m} \ell_{ij}' [\xi_i, \xi_j].$$

By Corollary 4.8 the sequence $\rho'_1, \ldots, \rho'_{2m}$ is strongly free in $\tilde{L}_{\mathrm{mix}}(X)$ and therefore G is mild by Theorem 4.4.

Example 1. To illustrate the above proof, let $S_0 = \{13, 3\} = \{q_1, q_2\}$. Then $q'_1 = 41$, $q'_2 = 5$, $q_3 = 19$ satisfy the required conditions. Then

$$S = \{41, 13, 5, 3, 19\} = \{p_1, p_2, p_3, p_4, p_5\}$$

and the relators for the first presentation are

$$\begin{aligned}
r_1 &\equiv [x_1, x_2][x_1, x_4][x_1, x_5] \bmod F_3,\\
r_2 &\equiv [x_2, x_1][x_2, x_3][x_2, x_5] \bmod F_3,\\
r_3 &\equiv [x_3, x_2][x_3, x_4] \bmod F_3,\\
r_4 &\equiv x_4^2[x_4, x_1][x_4, x_3][x_4, x_5] \bmod F_3,\\
r_5 &\equiv x_5^2[x_5, x_1][x_5, x_2] \bmod F_3,\\
r &= x_4x_5 \bmod F_2.
\end{aligned}$$

Hence $G = G_S(2)$ has the presentation $< x_1, x_2, x_3, x_4 \mid r'_1, r'_2, r'_3, r'_4 >$ where

$$\begin{aligned}
r'_1 &\equiv [x_1, x_2] \bmod F_3,\\
r'_2 &\equiv [x_2, x_1][x_2, x_3][x_2, x_4] \bmod F_3,\\
r'_3 &\equiv [x_3, x_2][x_3, x_4] \bmod F_3,\\
r'_4 &\equiv x_4^2[x_4, x_1][x_4, x_3] \bmod F_3.
\end{aligned}$$

Example 2. This example is due to Denis Vogel and while it does not illustrate exactly the above proof it does contain the basic idea which led to the result. Let $S = \{5, 29, 7, 11, 3\}$. Using the above notation for a Koch presentation of $G_S(2)$ with $p_1 = 5, p_2 = 29, p_3 = 7, p_4 = 11, p_5 = 3$ we have

$$\begin{aligned}
r_1 &\equiv [x_1, x_3][x_1, x_5] \bmod F_3,\\
r_2 &\equiv [x_2, x_4][x_2, x_5] \bmod F_3,\\
r_3 &\equiv x_3^2[x_3, x_1][x_3, x_4] \bmod F_3,\\
r_4 &\equiv x_4^2[x_4, x_2][x_4, x_5] \bmod F_3,\\
r_5 &\equiv x_5^2[x_5, x_1][x_5, x_2] \bmod F_3,\\
r &\equiv x_3x_4x_5 \bmod F_2.
\end{aligned}$$

Omitting r_5 and setting $x_5 = x_3x_4 \bmod F_2$, we get

$$\begin{aligned}
r'_1 &\equiv [x_1, x_4] \bmod F_3,\\
r'_2 &\equiv [x_2, x_3] \bmod F_3,\\
r'_3 &\equiv x_3^2[x_3, x_1][x_3, x_4] \bmod F_3,\\
r'_4 &\equiv x_4^2[x_4, x_2][x_4, x_3] \bmod F_3.
\end{aligned}$$

The images of the initial forms of these relators in $\tilde{L}_{\text{mix}}(X)$ (all $\tau_i = 1$) are

$$\begin{aligned}\rho_1' &= [\xi_1, \xi_4],\\ \rho_2' &= [\xi_2, \xi_3],\\ \rho_3' &= \xi_3^2 + [\xi_3, \xi_1] + [\xi_3, \xi_4],\\ \rho_4' &= \xi_4^2 + [\xi_4, \xi_2] + [\xi_4, \xi_3].\end{aligned}$$

If $\mathfrak{a}$ is the ideal of $\tilde{L}_{\text{mix}}(X)$ generated by ξ_3, ξ_4 the ρ_i' are in $\mathfrak{a}$ and their images in $\mathfrak{a}/\mathfrak{a}^*$ are the classes of

$$[\xi_1, \xi_4],\ [\xi_2, \xi_3],\ [\xi_1, \xi_3],\ [\xi_2, \xi_4]$$

which are part of a basis for $(\mathfrak{a}/\mathfrak{a}^*)_2$. Hence $G_S(2)$ is mild. If $a_n = \dim L(G_S)$ then $a_1 = 4$ and

$$a_n = \sum_{k=2}^{n} (\frac{1}{k} \sum_{\ell | k} \mu(\frac{k}{\ell})(2^{\ell+1} + (-1)^{\ell} 4)$$

for $n \geq 2$ by Theorem 5.3.

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