FABULOUS PRO-p-GROUPS

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Abstract. Let p be an odd prime. A pro-p-group G is said to be fabulous if it is a mild quadratic pro-p-group that is also fab. The only known examples appear as Galois groups of maximal p-extensions number fields unramified outside a finite set S of primes with residual characteristics $\neq p$. We not have a single example of a fabulous pro-p-group having an explicit presentation. This paper is a attempt to find such examples.

1. Introduction

Let p be an odd prime. We call a quadratic pro-p-group fabulous if it is fab and mild. These groups appear often as the Galois group $G_S(p)$ of the maximal p-extension of a number field K that is unramified outside a finite set S of primes with residual characteristics $\neq p$ (the tame case), cf. [6]. [14], [9], [10], [12]. They also appear in the case of restricted ramification and prescribed decomposition in the mixed case, cf. [16], [15], [11], even for function fields in [8],[12].

In view of the importance of these groups for the Fontaine-Mazur Conjecture, cf. [2], it would be desirable to have some kind of classification of these groups. However, up to now, we do not even have an explicit presentation for a single fabulous group.

2. Definitions

Definition 1 (Fab Group). A pro-p-group G is said to be fab if $H^{ab} = H/[H,H]$ is finite for every closed subgroup H of G of finite index or, equivalently, the factors of the derived series of G are all finite.

Examples of fab pro-p-groups are finite p-groups or pro-p-groups G that are p-adic analytic with $\mathrm{Lie}(G) = [\mathrm{Lie}(G), \mathrm{Lie}(G)]$; for example, an open pro-p-subgroup of $SL_n(\mathbb{Z}_p)$. The groups $G_S(p)$ are fab for a number field K in the tame case since the ramification is tame at the primes in S. We not have a single example of an infinite non-analytic fab pro-p-group having an explicit presentation.

A fab pro-p-group G is finitely generated with minimal number of generators $d = \dim_{\mathbb{F}_p} G/G^p[G,G]$ and minimal number of relators $r \geq d$. We have

$$d = d(G) = \dim H^1(G), \quad r = r(G) = \dim H^2(G),$$

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where $H^i(G) = H^i(G, \mathbb{Z}/p/Z)$. Since $p \neq 2$ the cup product $H^1(G) \otimes H^1(G) \to H^2(G)$ yields a linear map

$$\phi : \bigwedge^2 H^1(G) \to H^2(G).$$

Definition 2 (Quadratic Group). A finitely generated pro-p-group G is said to be quadratic if ϕ is surjective.

The pro-p-group G is quadratic if and only if the dual map

$$\phi^* : H^2(G)^* \to (\bigwedge^2 H^1(G))^* = \bigwedge^2 H^1(G)^*$$

is injective. Let $V = H^1(G)^*$ and let L be the Lie algebra which is universal for linear mappings of V into Lie algebras over $\mathbb{F}_p$. If $\xi_1, \ldots, \xi_d$ is a basis for V then L is the free Lie algebra over $\mathbb{F}_p$ on $\xi_1, \ldots, \xi_d$. Then $\bigwedge^2 H^1(G)^*$ can be identified with L_2, the degree 2 component of the graded Lie algebra L.

Let $\mathfrak{r}$ be the ideal of L generated by the image W of ϕ^*. Then $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is module over $\mathfrak{g} = L/\mathfrak{r}$ via the adjoint representation. The Lie algebra $\mathfrak{g} = L/\mathfrak{r}$ is called the **holonomy Lie algebra** of G; it is an invariant of G. If U is the enveloping algebra of $\mathfrak{g}$ then $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a finitely generated U-module. If M is a free U-module on the image of one (and hence any) basis $\rho_1, \ldots, \rho_m$ for W then the Lie algebra $\mathfrak{g}$ is said to be **mild** in which case the sequence $\rho_1, \ldots, \rho_m$ is said to be **strongly free**. If $c_n = \dim_{\mathbb{F}_p} \mathfrak{g}_n$, the formal power series

$$P(t) = \sum_{n \geq 0} c_n t^n$$

is called the Poincaré series of the graded algebra $\mathfrak{g}$. This Lie algebra is mild if and only if $1/P(t) = 1 - dt + mt^2$ (cf. [6], Prop 3). The coefficients a_n of $1/1 - dt + mt^2$ are all ≥ 0 if and only if $d^2 \geq 4m^2$. Indeed, if $d^2 < 4m$ then a_n is a constant multiple of $\cos(n\delta)$ where $\delta = (1/m - d^2/4m)^{1/2}$.

Definition 3 (Mild Quadratic Group)**.** A quadratic pro-p-group G is said to be mild if its holonomy Lie algebra is mild.

Conversely, let $\rho_1, \ldots, \rho_m$ be a sequence of homogeneous elements of degree 2 in the free $\mathbb{F}_p$-Lie algebra L on $\xi_1, \ldots, \xi_d$ and let $\mathfrak{r}$ be the ideal of L generated by $\rho_1, \ldots, \rho_m$. To construct a quadratic group G whose holonomy Lie algebra is $\mathfrak{g}$ let

$$\rho_k = \sum_{i<j} \bar{a}_{ijk} [\xi_i, \xi_j]$$

with $\bar{a}_{ijk} \in \mathbb{F}_p$. Let F be the free pro-p-group on $x_1, \ldots, x_d$ and let R be the normal subgroup of F generated by $r_1, \ldots, r_m$ where

$$r_k = \prod_{j=1}^{d} x_j^{p\,a_{kj}} \prod_{i<j} [x_i, x_j]^{a_{ijk}} u_k.$$

with $a_{kj} \in \mathbb{Z}_p$, $a_{ijk} \in \mathbb{Z}_p$ a lift of $\bar{a}_{ijk}$ to $\mathbb{Z}_p$ and $u_k \in \mathbb{F}_3$, the third term of the lower p-central series (F_n) of F defined by $F_1 = F$, $F_{n+1} = F_n^p[F, F_n]$. Let $\mathfrak{L}(F)$ be the graded Lie algebra associated to the lower p-central series of F. It is a Lie algebra over $\mathbb{F}_p[\pi]$ where the action of the variable π is induced by the p-th power map in F and the Lie bracket is induced by the commutator operation. Note that the n-th homogeneous component $\mathfrak{L}_n(F) = F_n/F_{n+1}$ is denoted additively. Since $\mathfrak{L}(F)$ is the free Lie algebra over $\mathbb{F}_p[\pi]$ on $\xi_1, \dots, \xi_d$, where ξ_i is the image of x_i in $V = \mathfrak{L}_1 = F/F^p[F, F]$, we can identify the $\mathbb{F}_p$-Lie subalgebra of $\mathfrak{L}(F)$ generated by $\xi_1, \dots \xi_d$ with the free lie algebra L over $\mathbb{F}_p$ on these elements. We also have $\mathfrak{L}(F)/\pi\mathfrak{L}(F) = L$.

Then $G = F/R$ has holonomy Lie algebra $\mathfrak{g}$. To see this we use the fact that under the identification of $H^2(G)^*$ with $R/R^p[R, F]$ via the transpose of the transgression map associated to the exact sequence

$$1 \to R \to F \to G \to 1,$$

the image of r_k under ϕ is ρ_k, cf. [5], Prop. 3. This map is bijective since $R \subseteq F_2$ implies that the inflation map $H^1(G) \to H^1(F)$ is bijective. Note that G is quadratic if and only if the sequence $\rho_1, \dots, \rho_m$ is linearly independent in which case $m = r(G)$. Note also that the group G depends on the parameters $u_1, \dots, u_m$ but that the holonomy Lie algebra is the same for all choices of these parameter. We call these groups **twists** of the group corresponding the the choice $u_1 = \cdots = u_m = 1$.

Proposition 4. *If G is a mild quadratic pro-p-group then G then the cohomological dimension of G is 2 and $\mathfrak{L}(G)$, the Lie algebra associated to the lower p-central series of G, is given by*

$$\mathfrak{L}(G) = \langle \xi_1, \dots, \xi_d \mid \sigma_1, \dots, \sigma_m \rangle,$$

with $\sigma_k = \sum_j a_{kj}\pi + \rho_k$. Moreover, G is not p-adic analytic if $d > 2$ since $m \leq d^2/4$.

For the first statement cf. [6], Theorem 4.1 and [13], p. 68, Exercise (c) for the second.

There is no general algorithm for determining whether the above finitely presented pro-p-group G is mild or not. However, we do have sufficient conditions which yield a rich supply of mild groups (cf. [6], Theorem 3.3). The following invariant formulation of these conditions for quadratic groups is due to Alexander Schmidt (cf. [12], Theorem 6.2).

Proposition 5. *If $H^2(G) \neq 0$ and $H^1(G) = U_1 \oplus U_2$ with the cup-product ϕ trivial on $U_2 \wedge U_2$ and $\phi(U_1 \wedge U_2) = H^2(G)$ then G is mild.*

This is equivalent to saying that $m > 1$ and that the presentation can be chosen so that the generating set for F can be divided into two disjoint sets by a partition A, B of $\{1, \dots, m\}$ with the associated holonomy relators $\rho_1, \dots, \rho_m$ satisfying

$$\rho_k = \sum_{i \in A} a_{ijk}[\xi_i, \xi_j]$$

and, setting

$$\rho'_k = \sum_{i \in A, j \in B} a_{ijk}[\xi_i, \xi_j],$$

we have that $\rho'_1, \ldots, \rho'_m$ is a linearly independent sequence. For example, the pro-p-group

$$G = \langle x_1^p[x_1, x_2],\ x_2^p[x_2, x_3],\ x_3^p[x_3, x_4],\ x_4^p[x_4, x_1]\rangle$$

is a mild quadratic non-analytic pro-p-group with $d(G) = r(G) = 4$ since the associated holonomy relators

$$[\xi_1, \xi_2], [\xi_2, \xi_3],\ [\xi_3, \xi_4],\ [\xi_4, \xi_1]$$

satisfy this with $A = \{1, 3\}$, $B = \{2, 4\}$; here $\rho'_k = \rho_k$.

However, an algorithm for mildness exists when $d = m = 4$, cf. [3]. To state this algorithm here we will use the quadratic form $u \mapsto u \wedge u$ on $\bigwedge^2 V$ when V is 4-dimensional so that $\bigwedge^4 V = \mathbb{F}_p$ (setting $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 = 1$). The associated bilinear form is $b(u, v) = u \wedge v$. If $\xi_1, \ldots \xi_4$ is a basis of V then the elements $\xi_i \wedge \xi_j$ $(i < j)$, ordered lexicographically are a basis for $\bigwedge^2 V$ and the matrix of b with respect to this basis is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proposition 6. *Let V be a 4-dimensional vector space over $\mathbb{F}_p$ and let W be a four dimensional subspace of $\bigwedge^2 V$ spanned by $\rho_1, \ldots, \rho_4$. Then the sequence $\rho_1, \ldots, \rho_4$ is strongly free if and only if $W^\perp \cap W = 0$.*

This result follows directly from the main result of [3]. Identifying $\bigwedge^2 V$ with L_2 (so that $\xi_i \wedge \xi_j = [\xi_i, \xi_j]$), we obtain for example that

$$\begin{aligned} \rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + [\xi_1, \xi_4], \\ \rho_2 &= [\xi_2, \xi_3] + [\xi_2, \xi_4], \\ \rho_3 &= 2[\xi_3, \xi_1] + 2[\xi_3, \xi_4], \\ \rho_4 &= [\xi_4, \xi_2] + 2[\xi_4, \xi_3] \end{aligned}$$

form a strongly free sequence. In [3] it is shown that a mild quadratic algebra

$$\mathfrak{g} = \langle \xi_1, \ldots \xi_4 \mid \rho_1 \ldots, \rho_4 \rangle$$

isomorphic to precisely one of the two mild quadratic algebras

$$\begin{aligned} \mathfrak{g}_1 &= \langle \xi_1, \ldots, \xi_4 \mid [\xi_1, \xi_2],\ [\xi_2, \xi_3],\ [\xi_3, \xi_4],\ [\xi_4, \xi_1] \rangle, \\ \mathfrak{g}_2 &= \langle \xi_1, \ldots, \xi_4 \mid [\xi_1, \xi_2],\ [\xi_2, \xi_3] + [\xi_4, \xi_1],\ [\xi_3, \xi_4],\ [\xi_4, \xi_2] + g[\xi_1, \xi_3] \end{aligned}$$

with g a non-square. It is said to be of type I (resp. type II) if it is isomorphic to $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$). It is of type I if and only if the quotient $\mathfrak{g}/[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$ has an element whose centralizer is of dimension 5. The relators in our example above are of type I.

Definition 7 (Fabulous Groups). A pro-p-group G is said to be fabulous if it is a mild quadratic and fab pro-p-group.

The only known examples of non-analytic fabulous pro-p-groups are the tame Galois groups $G_S(p)$. When $K = \mathbb{Q}$ and $S = \{q_1, \dots q_d\}$ with $q_i \equiv 1 \bmod p$ we have the following presentation of $G_S(p)$ due to Koch (cf. [4], Example 11.11):

$$G_S(p) = \langle x_1, \dots x_d \mid r_1, \dots, r_d \rangle$$

with $r_i = x_i^{q_i-1}[x_i^{-1}, y_i^{-1}]$ where $y_i \equiv \prod_{j=1}^d x_j^{\ell_{ij}} \bmod F^p[F,F]$. This presentation is only partially known but ℓ_{ij} for $i \neq j$ is the residue class mod p of any integer satisfying

$$q_i = g_i^{c_{ij}} \bmod q_j$$

with g_i a fixed primitive root mod q_j. We have

$$r_i = x_i^{q_i-1} \prod_{j \neq i} [x_i, x_j]^{\ell_{ij}} u_i$$

with $u_i \in F_3$. Thus the holonomy relators $\rho_1, \dots, \rho_d$ are given by

$$\rho_i = \sum_{j \neq i} \ell_{ij} [\xi_i, \xi_j]$$

The elements ℓ_{ij} are called the linking numbers of the Koch presentation for $G_S(p)$.

If $p = 3$ and $S = \{7, 13, 31, 43\}$, we find

$$\begin{aligned}
\rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + [\xi_1, \xi_4], \\
\rho_2 &= [\xi_2, \xi_3] + [\xi_2, \xi_4], \\
\rho_3 &= 2[\xi_3, \xi_1] + 2[\xi_3, \xi_4], \\
\rho_4 &= [\xi_4, \xi_2] + 2[\xi_4, \xi_3]
\end{aligned}$$

We have seen that these relators form a strongly free sequence of type I. Hence $G_S(3)$ is mild, fab and non-analytic. After the change of basis $x_1 \mapsto x_1,\ x_2 \mapsto x_2^2,\ x_3 \mapsto x_3,\ x_4 \mapsto x_4^2$ we find that the pro-3-group

$$G = \langle x_1, \dots, x_4 \mid x_1^3[x_2, x_1][x_1, x_3][x_1, x_4], x_2^3[x_2, x_3][x_4, x_2], x_3^3[x_3, x_1][x_3, x_4], x_4^3[x_2, x_4][x_4, x_3] \rangle$$

has $G_S(3)$ as a twist. However, while G is mild and non-analytic, it is not fab; MAGMA says that it has a subgroup of index 9 which has an infinite abelianization.

3. Constructing Fabulous Groups

Let $G^{(n)}$ be the n-th derived group of the group G; we have

$$G^{(0)} = G,\ G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

Proposition 8. *Let G be a pro-p-group. The following are equivalent.*

(a) *The group G is a fab group;*
(b) *The factors of the derived series of G are finite;*
(c) *The quotient $G/G^{(n)}$ is finite for all n;*
(d) *Every solvable quotient of G is finite.*

Proof. If (a) holds then H open in G implies that $[H,H]$ open in H. This implies (b) by induction. That (b),(c) and (d) are equivalent is immediate. To prove that (c) implies (a) let H be a closed subgroup of G of finite index. Then $G^{(n)} \subseteq H$ for some n which implies $G^{(n+1)} \subseteq [H,H]$ and hence the finiteness of $H/[H,H]$. □

The n-th derived subalgebra of a Lie algebra L is defined inductively by

$$L^{(0)} = L,\ L^{(n+1)} = [L^{(n)}, L^{(n)}].$$

Definition 9 (Fab Lie Algebra). A Lie algebra L is said to be fab if $L/L^{(n)}$ is finite for all $n \geq 0$.

Let (C_n) be a central series for G; by definition, we have

$$C_1 = G,\ [C_m, C_n] \subseteq C_{m+n}.$$

Let $L(G)$ be the Lie algebra associated to this central series. Then $L(G)$ is a graded Lie algebra with n-homogeneous component $L_n(G) = C_n/C_{n+1}$ (denoted additively). If l_n is the canonical map of C_n onto $L_n(G)$, we have $l_n(xy) = l_n(x) + l_n(y)$; if $x \in C_r, y \in C_s$, we have $l_{r+s}([x,y]) = [l_r(x), l_r(y)]$.

For any closed normal subgroup H of G we have $L(G/H) = L(G)/\tilde{L}(H)$, where $\tilde{L}(H)$ is the Lie algebra associated the the central series $(\tilde{H}_n)$ of H defined by $\tilde{H}_n = H \cap C_n$. If K is a closed normal subgroup of H we also let $\tilde{L}(H/K)$ be the Lie algebra associated to the central series $(\tilde{H}_n K/K)$ of H/K. Then $\tilde{L}(H/K) = \tilde{L}(H)/\tilde{L}(K)$.

Proposition 10. $L(G)^{(n)} \subseteq \tilde{L}(G^{(n)})$.

Proof. By induction on n. This is immediate for $n = 0$. Since $G^{(n+1)}$ is the kernel of the canonical map $G^{(n)} \to G^{(n)}/G^{(n+1)}$ it follows that $\tilde{L}(G^{(n+1)})$ is the kernel of the induced homomorphism of $\tilde{L}(G^{(n)})$ onto the abelian Lie algebra $\tilde{L}(G^{(n)}/G^{(n+1)})$. Thus $[\tilde{L}(G)^{(n)}, \tilde{L}(G)^{(n)}] \subseteq \tilde{L}(G^{(n+1)})$ which implies the result since, by induction, $L(G)^{(n+1)} = [L(G)^{(n)}, L(G)^{(n)}] \subseteq [\tilde{L}(G)^{(n)}, \tilde{L}(G)^{(n)}]$. □

Corollary 11. *If $L(G)$ is fab then G is fab.*

Indeed, $L(G/G^{(n)}) = L(G)/\tilde{L}(G^{(n)})$ is a quotient of $L(G)/L(G)^{(n)}$. However, as we shall see, the converse statement is not true.

A pro-p-group G is said to be of elementary type if $G/[G,G] \cong (\mathbb{Z}/p\mathbb{Z})^d$. If G is a mild quadratic group of elementary type then an explicit presentation for the Lie algebra associated to the lower central series is known, cf. [1], Theorem 5.8. In this case we have

$$L(G) = L(F)/(p\xi_i, [ad(\lambda)\xi_i, ad(\mu)\rho_j] + [ad(\mu)\xi_j, ad(\lambda)\rho_i]),$$

for all $i \leq i, j \leq d$, λ, μ in the enveloping algebra of $L(F)$.

Proposition 12. *If G is a mild quadratic group of elementary type then $\mathfrak{L}(G)$ is fab if and only if $\mathfrak{g} = \mathfrak{L}(G)/\pi\mathfrak{L}(G)$ is fab.*

Proof. Since $\pi\mathfrak{L}(G) \subseteq [\mathfrak{L}(G), \mathfrak{L}(G)]$ it follows that $\pi^{2k}\mathfrak{L}(G)^{(k)} \subseteq \mathfrak{L}(G)^{(k+1)}$. If $\mathfrak{g}$ is fab then $M_k = \mathfrak{L}(G)^{(k)}/\mathfrak{L}(G)^{(k+1)}$ is a finitely generated $\mathbb{F}_p[\pi]$-module since $M_k/\pi M_k = \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ is finite and hence M_k is finite since it is a torsion module. Conversely, if $\mathfrak{L}(G)$ is fab then $\mathfrak{g}$ is fab since a quotient of a fab Lie algebra is fab. $\square$

If $G = G_S(p)$ with $K = \mathbb{Q}$, $p = 3$ and $S = \{7, 13, 31, 43\}$ its holonomy Lie algebra $\mathfrak{g}$ is of type I and hence isomorphic to the Lie algebra

$$\mathfrak{h} = \langle \xi_1, \ldots, \xi_4 \mid [\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], [\xi_4, \xi_1] \rangle.$$

The quotient $\mathfrak{h}/(\xi_2, \xi_4)$ is a free Lie algebra on two generators and hence is not fab. It follows that $\mathfrak{h}$ and hence $\mathfrak{g}$ is not fab. Thus the Lie algebra $\mathfrak{L}(G)$ associated to the lower 3-central series of the fab pro-3-group $G = G_S(3)$ is not fab. In this case, since $G/[G,G]$ is 3-elementary, $\mathfrak{g}$ is a quotient of $L(G) \otimes \mathbb{F}_3$, which implies that $L(G)$ is not fab.

If $\mathfrak{k} = \langle \xi_1, \ldots, \xi_4 \mid \rho_1, \ldots, \rho_4 \rangle$ is a quadratic Lie algebra over $\mathbb{F}_p$ with $\rho_1, \ldots, \rho_4$ strongly free then by [3] it is isomorphic to the Lie algebra $\mathfrak{h}$ above after possibly a quadratic extension. It follows that the Lie algebra $\mathfrak{k}$ is not fab.

More generally, the holonomy Lie algebra of a quadratic group that is mild as a consequence of Proposition 5 is not fab. We don't have an example of a mild quadratic group whose holonomy Lie algebra is fab.

The holonomy Lie algebra of the group $G = G_S(3)$ with $S = \{7, 13, 31, 61\}$ has the presentation $\langle \xi_1, \ldots, \xi_4 \mid \rho_1, \ldots, \rho_4 \rangle$ with

$$\begin{aligned}
\rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + 2[\xi_1, \xi_4], \\
\rho_1 &= [\xi_2, \xi_3] + 2[\xi_2, \xi_4], \\
\rho_1 &= 2[\xi_3, \xi_1] + [\xi_3, \xi_4], \\
\rho_1 &= [\xi_4, \xi_1] + [\xi_4, \xi_2].
\end{aligned}$$

This presentation defines a mild quadratic Lie algebra of type II. The pro-3-group $\tilde{G}$ with presentation $\langle x_1, \dots, x_4 \mid s_1, \dots, s_4 \rangle$, where

$$\begin{aligned} s_1 &= x_1^3[x_2, x_1][x_1, x_3][x_4, x_1], \\ s_2 &= x_2^3[x_2, x_3][x_2, x_4], \\ s_3 &= x_3^3[x_3, x_1][x_4, x_3], \\ s_4 &= x_4^3[x_4, x_1][x_2, x_4], \end{aligned}$$

has G as a twist. Magma reports that $\tilde{G}/\tilde{G}''$ is finite and that every subgroup of $\tilde{G}$ of index 3, 9 or 27 has a finite abelianization as well as all index 81 subgroups tested so far. We don't know if this group is fab or not. Boston [2] has found a similar example of a mild quadratic pro-2-group with 4 generators and 4 relators which is fab as far as MAGMA can tell.

Question 1. Suppose that G is a quadratic pro-p-group of elementary type and suppose that its holonomy Lie algebra is a mild quadratic algebra with 4 generators and 4 relators which is of type II. Is G fab?

Question 2. Can one find a strongly free sequence over $\mathbb{F}_p$ consisting of d quadratic Lie polynomials $\rho_1, \dots, \rho_d$ in $d \le m$ variables $\xi_1, \dots, \xi_d$ such that the Lie algebra $\mathfrak{h} = \langle \xi_1, \dots, \xi_d \mid \rho_1, \dots, \rho_d \rangle$ is mild and fab?

If the answer to this question is yes then then one can produce an explicitly presented quadratic pro-p-group G whose holonomy Lie algebra is $\mathfrak{h}$. The classification of mild quadratic Lie algebras is not known when $m = d \ge 5$. In this case we do not know even if their is more than one isomorphism class over the algebraic closure of $\mathbb{F}_p$.

Question 3. If $G_S(p)$ is quadratic and mild, can one find an explicit twist G of $G_S(p)$ such that G is fab? This would be the case if G was isomorphic to $G_S(p)$.

If the answer to any of these questions is yes, the group G in question is then a fabulous group which is non-analytic since $d(G) \ge 4$.

Remark. The above results can be extended to the case $p = 2$ when the cup-product is alternating (cf. [6], p. 175). If not, the situation is technically quite different since the map $x \mapsto x^2$ in a pro-2-group G does not induce a linear operator on $\mathfrak{L}(G)$. This case will be treated in [7].

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