McGill University
Math 571: Higher Algebra 2
Assignment 3: due March 30, 2007

1. (a) Let $K = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$, $d < 0$. Show that K cannot be embedded in a cyclic extension L of $\mathbb{Q}$ of degree divisible by 4. **Hint:** If $[L, K] = 2$ and $\mathrm{Gal}(L/\mathbb{Q}) =< \sigma >$, first show that $L = K(\alpha)$ where $\alpha^2 \in K$ and $\sigma(\alpha) = b\alpha$ with $b \in K$. Then show that $N_{K/\mathbb{Q}}(b) = -1$.
   (b) Let $f(X) = X^4 + 4X^2 + 2$. Show that $f(X)$ has a cyclic Galois group over $\mathbb{Q}$.

2. Let $f(X) \in \mathbb{Q}(X)$ be of degree $n > 2$ and let K be a splitting field of $f(X)$ over $\mathbb{Q}$. Suppose that the Galois group of $f(X)$ is S_n.
   (a) Show that $f(X)$ is irreducible over $\mathbb{Q}$.
   (b) If $f(\alpha) = 0$ show that the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.
   (c) If $n \geq 4$, show that $\alpha^n \notin \mathbb{Q}$.

3. Let G be a finite group and A a $\mathbb{Z}[G]$-module. Define $N : A \to A$ by $N(a) = \sum_{\sigma \in G} \sigma(a)$ so that N is left multiplication by $\sum_{\sigma \in G} \sigma$. Let A_N be the kernel of N and NA be the image of N. If $G =< \sigma >$, define $D : A \to A$ to be left multiplication by $\sigma - 1$ so that $D(a) = \sigma a - a$. Let DA denote the image of A.
   (a) Show that N and D are $\mathbb{Z}[G]$-module homomorphisms.
   (b) Let $P_n = \mathbb{Z}[G]$ for $n \geq 0$ and let $f_n : P_n \to P_{n-1}$ be D if n is odd, N if $n > 0$ is even. If $f_0 : \mathbb{Z}[G] \to \mathbb{Z}$ is defined by $f_0(\sum_{\sigma \in G} a_\sigma \sigma) = \sum_{\sigma \in G} a_\sigma$, show that the sequence
   $$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$
   defined by the maps f_n, is exact.
   (c) Using (b), show that $H^n(G, A) = A_N/DA$ if n is odd and A^G/NA if $n > 0$ is even.

4. Let k be a finite field and K a finite extension of K with $G = \mathrm{Gal}(K/k)$.
   (a) Show that the norm map $N_{K/k} : K^* \to k^*$ is surjective.
   (b) Show that $H^n(G, K^*) = 0$ for $n \geq 1$.

5. Find the Galois group over $\mathbb{Q}$ of the following polynomials.
   (a) $X^4 + 2X^2 + X + 3$;
   (b) $X^5 - 4X + 2$;
   (c) $X^6 - 12X^4 + 15X^3 - 6X^2 + 15X + 12$.