McGill University
Math 571: Higher Algebra 2
Assignment 1: due January 26, 2007

1. Let k be a commutative ring and let A be the free k-module with basis the subsets S of $J_n = \{1, 2, \ldots, n\}$. If S, T are disjoint subsets of J_n we let

$$\lambda(S,T) = |\{(s,t) \mid s \in S, t \in T, s > t\}|, \quad \epsilon(S,T) = (-1)^{\lambda(S,T)}.$$

   (a) Show that there is a unique structure of associative k-algebra on A such that for any subsets S, T of $\{1, 2, \ldots, n\}$ we have

$$ST = \begin{cases} \epsilon(S,T)(S \cup T) & \text{if } S \cap T = \emptyset, \\ 0 & \text{if } S \cap T \neq \emptyset. \end{cases}$$

   (b) Show that A and $\bigwedge k^n$ are isomorphic as k-algebras.

2. Let $p \geq 2$ be a natural number and let $\mathcal{S}$ be the set of natural number sequences $s = (i_1, i_2, \ldots, i_p)$ with $1 \leq i_1, i_2, \ldots, i_p \leq n$. The symmetric group S_p acts on $\mathcal{S}$ via

$$\sigma(i_1, \cdots, i_p) = (i_{\sigma^{-1}(1)}, \cdots, i_{\sigma^{-1}(p)}).$$

   Let

$$\begin{aligned} \mathcal{S}_0 &= \{s \in \mathcal{S} \mid s = (i_1, i_2, \ldots, i_p), i_j = i_k \text{ for some } j \neq k\}, \\ \mathcal{S}_1 &= \mathcal{S} - \mathcal{S}_0, \\ \mathcal{S}_2 &= \{s = (i_1, i_2, \ldots, i_p) \in \mathcal{S}_1 \text{ with } 1 \leq i_1 < i_2 < \cdots < i_p\}, \\ \mathcal{S}_3 &= \{\sigma s \mid s \in \mathcal{S}_2,\ \sigma \in S_p\,,\ \sigma \neq 1\} = \mathcal{S}_1 - \mathcal{S}_2. \end{aligned}$$

   Let M be a free k-module with basis $(e_1, \ldots, e_n)$ and for any $s = (i_1, i_2, \ldots, i_p) \in \mathcal{S}$ let $e_s = e_{i_1} \otimes \cdots \otimes e_{i_p}$. The family of elements e_s with $s \in \mathcal{S}$ is a k-basis for $\bigotimes^p M$. If $s \in \mathcal{S}_1$, $\sigma \in S_p$, let $n_{s,\sigma} = e_s - \epsilon_\sigma e_{\sigma s}$, where ϵ_σ is the sign of the permutation σ. Let $f_s = e_s$ if $s \in \mathcal{S}_0 \cup \mathcal{S}_2$ and $f_s = n_{t,\sigma}$ if $s = \sigma t$ with $t \in \mathcal{S}_2$, $\sigma \in S_p$, $\sigma \neq 1$. Finally, let I_p be the submodule of $\bigotimes^p M$ generated by the family of elements of the form $u_1 \otimes \cdots \otimes u_p$ with $u_i = u_j$ for some $i \neq j$.

   (a) Show that $(f_s)_{s \in \mathcal{S}_2 \cup \mathcal{S}_0 \cup \mathcal{S}_3}$ is a k-basis for $\bigotimes^p M$.

   (b) Show that $n_{s,\sigma} \in I_p$. (Hint: Show that $n_{s,\sigma\tau} = n_{s,\sigma} + \epsilon_\sigma n_{\sigma s, \sigma\tau\sigma^{-1}}$.)

   (c) Show that $(f_s)_{s \in \mathcal{S}_0 \cup \mathcal{S}_3}$ is a k-basis for I_p. (Hint: Show that $n_{\sigma s,\tau} = \epsilon_\sigma n_{s,\tau\sigma} - \epsilon_\sigma n_{s,\sigma}$ and use the fact that I_p is generated by the set $\{e_s \mid s \in \mathcal{S}_0\} \cup \{n_{s,\tau} \mid s \in \mathcal{S}_1, \tau \text{ a transposition}\}$.)

   (d) Deduce that $(f_s)_{s \in \mathcal{S}_2}$ is a k-basis for $\bigotimes^p M$ mod I_p.

3. Let M be a free k-module with basis $e = (e_i)_{1 \leq i \leq n}$. Let $f \in \mathrm{End}(M)$ and let $A = (a_{ij})$ be the matrix of f with respect to the basis e. If $S = \{i_1 < i_2 < \cdots < i_p\}$, $T = \{j_1 < j_2 < \cdots < j_p\}$ are p-element subsets of $\{1, 2, \ldots, n\}$ we let $A_{S,T}$ be the $p \times p$ submatrix $(b_{k\ell})$, where $b_{k\ell} = a_{i_k j_\ell}$, and set $a_{S,T} = \det(A_{S,T})$. We also let $e_S = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$.

   (a) Show that $\bigwedge^p f(e_S) = \sum_{|T|=p} a_{S,T} e_T$.

   (b) Show that $\mathrm{Tr}(\bigwedge^p f) = \sum_{|S|=p} a_{S,S}$.

   (c) If $\lambda \in k$, show that $\det(\lambda - f) = \sum_{p=0}^n c_p \lambda^{n-p}$, where $c_i = (-1)^p \mathrm{Tr}(\bigwedge^p f)$.

4. Let M, N be finitely generated k-modules with M or N projective.

   (a) Show that the canonical map of $M^* \otimes N^*$ to $(M \otimes N)^*$ which sends $\phi \otimes \phi$ to the linear form $x \otimes y \mapsto \phi(x)\psi(y)$ is an isomorphism.

   (b) Show that the canonical map of $M^* \otimes N$ into $\mathrm{Hom}_k(M, N)$, sending $\phi \otimes y$ to the k-linear map $x \mapsto \phi(x)y$ is an isomorphism.

5. Let p, q be distinct primes, let $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ and let $\alpha = \sqrt{p} + \sqrt{q}$.

   (a) Show that $[K : \mathbb{Q}] = 4$.

   (b) Show that $K = \mathbb{Q}(\alpha)$. What is the minimal polynomial of α over $\mathbb{Q}$?

   (c) Find all isomorphisms of K into $\mathbb{C}$.

   (d) Show that $\mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{q}), \mathbb{Q}(\sqrt{pq})$ are the only subfields of K which are of degree 2 over $\mathbb{Q}$.

   (e) If p, q, r are distinct primes, prove that $[\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r}) : \mathbb{Q}] = 8$.

6. Let p be a prime, let $\alpha = \sqrt[4]{p} \in \mathbb{R}$ be the positive fourth root of p and let $K = \mathbb{Q}(\alpha)$.

   (a) Show that $[K : \mathbb{Q}] = 4$.

   (b) Find all isomorphisms of K into $\mathbb{C}$.

   (c) Show that $\mathbb{Q}(\sqrt{p})$ is the only subfield of K of degree 2 over $\mathbb{Q}$.

   (d) If q is a prime distinct from p, show that $[\mathbb{Q}(\sqrt[4]{p}, \sqrt{q}) : \mathbb{Q}] = 8$.

7. Find the Galois groups of the splitting fields of the following polynomials over $\mathbb{Q}$.

   (a) $(X^2 - 2)(X^2 - 3)$.

   (b) $(X^2 - 2)(X^2 - 3)(X^2 - 5)$.

   (c) $X^4 - 2$.

   (d) $(X^4 - 2)(X^2 - 3)$.

   (e) $X^5 - 2$.