McGill University
Math 371B: Algebra IV
Assignment 4: due Wednesday, March 24, 1999

1. (a) Show that $\mathbb{Z}[\sqrt{-2}] = \{\alpha = a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the norm $N(\alpha) = a^2 + 2b^2$. Find the units of this ring and the factorization of $3, 5, 11, 13, 17, 19, 23, 29, 31$.

   (b) Show that $\mathbb{Z}[\sqrt{-5}]$ is not a PID by showing that $(3, 2 + \sqrt{-5})$ is not a principal ideal. Show also that $2 + \sqrt{-5}$ is irreducible but not prime.

2. (a) Show that $X^n - 3$ is irrducible over $\mathbb{Z}[\sqrt{-2}]$ for $n \geq 1$.

   (b) Factor the polynomial $X^4 + X^3 - X^2 + 6$ over $\mathbb{Q}$.

3. Find the invariant factors of the integral matrix

$$\begin{bmatrix} 6 & 2 & 3 & 0 \\ 2 & 3 & -4 & 1 \\ -3 & 3 & 1 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}.$$

   Find an internal direct sum decomposition of the $\mathbb{Z}$-module $\mathbb{Z}^4/N$ where N is the submodule of Z^4 generated by the rows of A.

4. If $A = \mathbb{Z}[i]$ is the ring of Gaussian integers, find the structure of the A-module $M = A^3/N$ where N is the submodule of A^3 generated by

$$f_1 = (1, 3, 6), \;\; f_2 = (2 + 3i, -3i, 12 - 18i), \;\; f_3 = (2 - 3i, 6 + 9i, -18i).$$

   Show that M is finite of order 352512.

5. Verify that the characteristic polynomial of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{bmatrix}$$

   is a product of linear factors over $\mathbb{Q}$. Determine the rational and Jordan canonical forms for A over $\mathbb{Q}$ and find matrices P such that PAP^{-1} is in canonical form.