McGill University
Math 371B: Algebra IV
Assignment 3: due Friday, February 12, 1999

1. An absolute value on a ring A is a function from A to $\mathbb{R}$ such that, denoting its value at $x \in A$ by $|x|$, we have (i) $|x| \geq 0$ with equality iff $x = 0$, (ii) $|xy| = |x||y|$, (iii) $|x+y| \leq |x| + |y|$. The absolute vlaue is said to be non-archimedian if $|x+y| \leq \max(|x|, |y|)$. A valued ring is a ring together with an absolute value. A sequence (x_n) of elements of the valued ring A is said to be a Cauchy sequence if

$$(\forall \epsilon > 0)(\exists N > 0)(\forall m, n \geq N)|x_m - x_n| < \epsilon.$$

The valued ring is said to be complete if every Cauchy sequence (x_n) converges, i.e.,

$$(\exists x \in A)(\forall \epsilon > 0)(\exists N > 0)(\forall n \geq N)|x - x_n| < \epsilon.$$

The sequence is said to be bounded if $(\exists M > 0)(\forall n)|x_n| \leq M$ and a null sequence if it converges to 0, i.e.,

$$(\forall \epsilon > 0)(\exists N > 0)(\forall n \geq N)|x_n| < \epsilon.$$

A series $\sum_{n\geq 0} a_n$ is said to converge if the sequence (s_n) of partial sum $s_n = \sum_{k=0}^{n-1} a_n$ is convergent.

   (a) If A is complete, show that $\sum_{k\geq 0} |a_n|$ convergent implies that $\sum_{n\geq 0} a_n$ convergent. If A is complete with respect to a non-archimedian absolute value, show that $\sum_{n\geq 0} a_n$ is convergent iff (a_n) is a null sequence.

   (b) If C is the set of Cauchy sequences of the valued ring A and N is the set of null sequences, show that C is a subring of $A^{\mathbb{N}}$ and that N is a prime ideal of C which is maximal if A is a field.

   (c) Show that $x \mapsto (x, x, \ldots, x, \ldots) + N$ is an isomorphism of A with a subring of $\hat{A} = C/N$ and that, identifying A with its image in $\hat{A}$, the absolute value on A extends to one on $\hat{A}$. Show also that $\hat{A}$ is complete with respect to this absolute value and that A is dense in $\hat{A}$, i.e., every element of $\hat{A}$ is the limit of a sequence of elements of A. The ring $\hat{A}$ is called the completion of A with respect to the given absolute value.

   (d) If A is a discrete valuation ring with valuation v and $0 < c < 1$, show that $|x| = c^{v(x)}$ if $x \neq 0$ and $|0| = 0$ defines an absolute value on A. If $I = \{x \in A \mid |x| < 1$, show that I is a maximal ideal of A and that $\hat{A}$ is isomorphic to the inverse limit of the rings A/I^n as valued rings. This justifies the use of $\hat{A}$ for both types of completions.

2. Let A be a complete valued ring and let $f = \sum_{n\geq 0} a_n \in A[[X]]$ be a formal power series in X with coefficients in A. If $x \in A$ such that the series $\sum_{n\geq 0} a_n x^n$ is convergent, we let $f(x) = \sum_{n\geq 0} a_n x^n$ denote the limit.

   (a) If $\sum_{n\geq 0} a_n x^n$ is convergent and $|y| < |x|$, show that $\sum_{n\geq 0} a_n y^n$ is absolutely convergent. i.e., $\sum_{n\geq 0} |a_n y^n|$ is convergent.

   (b) If B is the set of $f = \sum_{n\geq 0} a_n \in A[[X]]$ such that $\sum_{n\geq 0} a_n x^n$ is absolutely convergent for $|x| < r$, show that B is a subring of $A[[X]]$ and that for every $x \in A$ with $|x| < r$, the mapping $e_x : B \to A$ defined by $e_x(f) = f(x)$ is a homomorphism of rings.

(c) With the notation of (b) let $f \in B$ and let $\tilde{f}$ be the function on $|x| < r$ defined by $\tilde{f}(x) = f(x)$. Show that $\tilde{B} = \{\tilde{f} | f \in B\}$ is a subring of the ring of A-valued functions on $|x| < r$ which is isomorphic to B if the valuation on A is not trivial. Hence, if there is $a \in A$ with $|a| \neq 0, 1$, we may identify B with a ring of functions on $|x| < r$.

(d) If the valuation on A is non-archimedean, show that (a),(b),(c) are true if the words "absolutely convergent" are replaced by "convergent".

3. Let A be any ring which contains $\mathbb{Q}$ as a subring and let

$$E(X) = \sum_{n \geq 0} \frac{X^n}{n!} \in A[[X]], \quad L(X) = \sum_{n \geq 1} (-1)^n \frac{(X-1)^n}{n} \in A[[X-1]].$$

(a) Show that $E(X+Y) = E(X)E(Y)$ in $A[[X,Y]]$.

(b) Show that $E(L(1+X)) = 1 + X$ and $L(E(X)) = X$ in $A[[X]]$. Hint: Use the fact that for $x \in \mathbb{R}$ we have $E(x) = e^x$ and $L(x) = \log(x)$ if $|x-1| < 1$.

(c) Show that, for p an odd prime, $E(X)$ and $L(1+X)$ are convergent on $p\mathbb{Z}_p$ and that $\exp(x) = E(x)$ is an isomorphism of the additive group $p\mathbb{Z}_p$ with the multiplicative group $1 + p\mathbb{Z}_p$, the inverse being the mapping $\log(x) = L(x)$. What happens if $p = 2$?

4. (a) For any $x \in \mathbb{Z}_p^\times$ show that the sequence (x^{p^n}) is convergent. If $\zeta = t(x)$ is its limit, show that (a) $\zeta^p = \zeta$, (b) $\zeta \equiv x \bmod p\mathbb{Z}_p$, (c) $x \equiv y \bmod p\mathbb{Z}_p$ implies that $t(x) = t(y)$, (d) $t(xy) = t(x)t(y)$. Show that t induces and isomorphism τ of $(\mathbb{Z}_p/p\mathbb{Z}_p)^* \cong (\mathbb{Z}/p\mathbb{Z})^*$ with a finite cyclic subgroup of $\mathbb{Z}_p^\times$ of order $p-1$. If $\bar{x} \in \mathbb{Z}_p/p\mathbb{Z}_p$ is a non-zero residue class, the representative $\tau(\bar{x})$ is called the Teichmueller representative of $\bar{x}$. Show that it is the unique $\zeta \in \bar{x}$ such that $\zeta^{p-1} = 1$.

(b) Show that the equation $x^2 = -1$ has two solutions in $\mathbb{Q}_5$. Find the first 5 terms of the 5-adic expansion of each solution.