McGill University
Math 371B: Algebra IV
Assignment 2: due Monday, February 1, 1999

A ring A is called a valuation ring if A is an integral domain such that for any non-zero element x of its quotient field we have $x \in A$ or $1/x \in A$. A valuation ring which is a field is called a trivial valuation ring. A discrete valuation ring is an example of a non-trivial valuation ring. Any subring A of a field K such that for any $x \in K^* = K - \{0\}$ we have $x \in A$ or $1/x \in A$ is a valuation ring with quotient field K. Such a subring of K is called a valuation ring of K.

1. Show that a valuation ring is a local ring. If F is a field, show that the localization of the polynomial ring $F[x,y]$ at the maximal ideal (x,y) is not a valuation ring.

2. Show that a ring is a discrete valuation ring if and only if it is a PID with a single non-zero maximal ideal. Show also that a discrete valuation ring is a maximal element in set of non-trivial valuation rings of its quotient field.

3. Show that the non-trivial valuation rings of $\mathbb{Q}$ are the discrete valuation rings $\mathbb{Z}_{(p)}$ where p is a prime of $\mathbb{Z}$. **Hint**: If M is the maximal ideal of the valuation ring, show that $M \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p of $\mathbb{Z}$.

4. If F is a field, find the non-trivial valuation rings of $F(x)$, the quotient field of the polynomial ring $F[x]$, which contain F. **Hint**: Use the fact that any valuation ring of $F(x)$ contains x or $1/x$ and that $F[x] \cong F[1/x]$ is a PID.

5. (a) If F is a field, show that the polynomial ring $F[x]$ has infinitely many primes (even if K is finite).

   (b) Give an example of a PID with finitely many primes.

6. Let A be a principal ideal domain, let Q be a non-zero prime ideal of the polynomial ring $B = A[X]$ and let $P = Q \cap A$.

   (a) If $P = pA$ with p a prime of A, show that either $Q = pB$ or Q is the maximal ideal $pB + fB$ with $f \in B$ of degree ≥ 1 and irreducible mod p.

   (b) If $P = 0$, show that $Q = gB$ with $g \in B$ irreducible of degree ≥ 1. **Hint**: Localize A and B with respect to $S = A - \{0\}$.

   (c) If $f \in B$ is of degree ≥ 1 and irreducible modulo a prime p of A and $g \in B$ is irreducible of degree ≥ 1, show that $gB \subseteq pB + fB$ iff f divides g mod p. Deduce that gB is maximal iff A has finitely many primes and $g = a_0 + a_1x + \cdots + a_nx^n$ with a_0 a unit of A and $a_1, \ldots, a_n$ divisible by all the primes of A.

   (d) IF f is algebraically closed, show that the non-zero prime ideals of the polynomial ring $F[x,y]$ are the ideals $(x-a, y-b)$ and (f) with $a, b \in F$ and f irreducible. Deduce that the maximal ideals of $F[x,y]$ are the point ideals $(x-a, y-b)$.