Structure of the Endomorphism Ring of a Finitely Finitely Generated Module over a PID

Let M be a finitely generated module over a PID k and let $d_1|d_2|\cdots|d_s$ be the elementary invariants of M. Then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_s$ with $M_i = kz_i \cong k/(d_i)$. The endomorphisms ϕ of M can be completely described by the homomorphisms $\phi_{ij} = p_i\phi q_j : M_j \to M_i$, where $q_i : M_i \to M$ and $p_j : M \to M_j$ are respectively the canonical injections and projections. We have $\phi = \sum_{ij} q_i\phi_{ij}p_j$. The $s \times s$ matrices $[\phi_{ij}]$, with the usual definition of matrix addition and matrix multiplication form a ring isomorphic to $\mathrm{End}_k(M)$.

We have $\phi_{ij}(z_j) = a_{ij}z_i$ with $a_{ij} \in k$ determined modulo d_i. Suppose that $d_r \neq 0$ while $d_i = 0$ for $i > r$. Then, since $d_j a_{ij} z_i = \phi_{ij}(d_j z_j) = 0$, we see that $a_{ij} = 0$ for $j \leq r$, $i > r$ and that $a_{ij} = \delta_{ij} b_{ij}$ if $j < i$, where $\delta_{ij} = d_i/d_j$. The element b_{ij} is determined modulo d_j. No other conditions are put on the other a_{ij}.

If $k = \mathbb{Z}$ and M is a torsion module then M has order $\prod_{1\leq j\leq s} d_j^{2(s-j)+1}$.

Example 1 ($\mathrm{End}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})$)**.** Here the endomorphisms are represented by the matrices of the form

$$\begin{bmatrix} a & b \\ 2c & d \end{bmatrix}$$

with a, b, c taken modulo 2 and d taken modulo 4. This endomorphism is invertible iff $ad \neq 0 \bmod 2$. The endomorphism ring is of order 32 and the group of invertible elements has order 8.

If $k = k_0[X]$ with k_0 a field and d_j a polynomial of degree n_j then M_j is of dimension n_j over k_0. Since the dimension over k_0 of $\mathrm{Hom}_k(M_j, M_i)$ is n_j if $j < i$ and n_i if $j \geq i$, the dimension over k_0 of $\mathrm{End}_k(M)$ is $\sum_{1\leq j\leq s}(2(s-j)+1)n_j$. If A is an $n \times n$ matrix over k_0 having the above elementary invariants and we take $M = k_0^n$ with the module structure defined by $Xz = Az$, then the endomorphisms of M are given by $z \mapsto Bz$ with $AB = BA$. We thus have a procedure for determining the matrices which commute with A. In particular the dimension of this space of matrices is given by the formula above. This result is due to Frobenius.

Example 2. Let A be the rational matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The elementary invariants of A are X^2, X^3 and we have $M = \mathbb{Q}^5 = \mathbb{Q}[x]e_1 \oplus \mathbb{Q}[X]e_3$ with $\mathrm{ann}(e_1) = X^2$, $\mathrm{ann}(e_3) = X^2$. Then, setting $z_1 = e_1, z_2 = e_3$, we have $\phi_{11}(z_1) = (a + bX)z_1$, $\phi_{12}(z_2) = (c + dX)z_1$, $\phi_{21}(z_1) = X(e + fX)z_2$, $\phi_{22}(z_2) = g + hX + kX^2$ for any endomorphism ϕ of M. Replacing each ϕ_{ij} by its matrix relative to the bases z_1, Xz_1 and z_2, Xz_2, X^2z_2 of $\mathbb{Q}[X]z_1$ and $\mathbb{Q}[X]z_2$ respectively, we get the following general form for the matrices commuting with A:

$$\begin{bmatrix} a & 0 & c & 0 & 0 \\ b & a & d & c & 0 \\ 0 & 0 & g & 0 & 0 \\ e & 0 & h & g & 0 \\ f & e & k & h & g \end{bmatrix}.$$

We thus see that the dimension of the space of these matrices is 9 which agrees with the number computed with the formula of Frobenius.