McGill University
Math 371B: Algebra IV
Assignment 2: due Friday February 20, 1998

Read Text: sections 2.12-2.16

1. Let $A$ be a subring of the ring $B$ and suppose that $B$ is integral over $A$. If $P$ is a prime ideal of $A$, prove that there is a prime ideal $Q$ of $B$ such that $P = Q \cap A$. **Hint**: If $S = A - P$, let $A_P = S^{-1}A$, $B_P = S^{-1}B$. First show that $B_P$ is integral over $A_P$ for the canonical homomorphism $\phi$ of $A_P$ into $B_P$. Then show that, for any maximal ideal $I$ of $B_P$, $\phi^{-1}(I) = PA_P$. Finally, show that one can take $Q$ to be the inverse image of $I$ in $B$ with respect to the canonical homomorphism of $B$ into $B_P$.

2. Text: p. 137, #5,6,7.

3. Text: p. 140, #1,2,3.

4. Text: p. 146, #7,8. **Hint:** Use the identity $(d + \sqrt{d})(d - \sqrt{d}) = d(d-1)$.

5. Text: p. 149, #2,3.

6. Text: p. 150, #11,12.

7. Text: p. 151, #17,18,19. **Hint for #18**: First prove that, if $R$ is a ring, $R^{\mathbb{N}}$ is a monoid under the operation $f * g(n) = \sum_{d|n} f(d)g(n/d)$.

8. Text: p. 151, #20.

9. Text: p. 154, #4.

10. Let $A$ be a principal ideal domain, let $Q$ be a non-zero prime ideal of the polynomial ring $B = A[X]$ and let $P = Q \cap A$.

    (a) If $P = pA$ with $p$ a prime of $A$, show that either $Q = pB$ or $Q = pB + fB$ with $f \in B$ irreducible mod $pB$.

    (b) If $P = 0$, show that $Q = fB$ with $f \in B$ irreducible of degree $\geq 1$. **Hint**: Localize $A$ and $B$ with respect to $S = A - \{0\}$.