McGill University
Math 370A: Algebra III
Midterm Test (Part 2): due Monday, November 9, 1998

5. The purpose of this exercise is to prove that $\mathrm{Aut(Q_8) \cong S_4}$.

   (a) Show that that the subgroup of S_4 generated by $(14)(23), (24)(13)$ is a normal subgroup of S_4 which is isomorphic to $C_2 \times C_2$. This group is called the Klein 4-group and is denoted by V_4.

   (b) Identifying S_3 with the permutations of $\{1, 2, 3, 4\}$ which fix 4, show that $S_4 = V_4 S_3$ and that $V_4 \cap S_3 = 1$.

   (c) Deduce that $S_4 = V_4 \rtimes_\phi S_3$, where ϕ is determined by the action of S_3 on V_4 by inner automorphisms.

   (d) Using the presentation $Q_8 =< a, b \mid a^4 = b^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} >$, show that there are automorphisms σ, τ of Q_8 such that, setting $c = ab$, we have $\sigma(a) = b, \sigma(b) = c, \sigma(c) = a$, $\tau(a) = b^{-1}, \tau(b) = a^{-1}$;

   (e) Show that $K =< \sigma, \tau >$ is a subgroup of $\mathrm{Aut(Q_8)}$ of order 6 and that there is an isomorphism $g : K \to S_3$ with $g(\sigma) = (123)$, $g(\tau) = (12)$;

   (f) If $H = \mathrm{Inn}(Q_8)$ is the group of inner automorphisms of Q_8, show that $H \lhd \mathrm{Aut(Q_8)}$, that $\mathrm{Aut(Q_8) = HK}$ and that $H \cap K = 1$. Deduce that $\mathrm{Aut(Q_8) = H \rtimes_\rho K}$, where ρ is determined by the action of K on H via inner automorphisms.

   (g) If $\overline{a}, \overline{b}, \overline{c}$ are respectively conjugation by a, b, c, show that there is an isomorphism

$$f : H \to V_4 \quad \text{with} \quad f(\overline{a}) = (14)(23), f(\overline{b}) = (24)(13), f(\overline{c}) = (34)(12).$$

   (h) Show that, for all $k \in K$, we have $\phi(g(k)) = f\rho(k)f^{-1}$.

   (i) Using the results of assignment 3, deduce that $\mathrm{Aut(Q_8) \cong S_4}$.