McGill University
Math 370A: Algebra III
Assignment 5: due Friday, December 4, 1998

1. (a) Let K be a commutative ring with unity and let A be a K-algebra. Show that the K-module $B = A \times K$ together with the operation of multiplication defined by

$$(a_1, k_1)(a_2, k_2) = (a_1a_2 + k_1a_2 + k_2a_1, k_1k_2)$$

is an K-algebra with unit element $1_B = (0, 1)$ and that this algebra is associative if and only if A is associative. Show also that the set of elements of the form $(a, 0)$ with $a \in A$ is an ideal of B isomorphic to A and that, on identifying $a \in A$ with $(a, 0)$, every element of B can be uniquely written in the form $a + k1_B$;

   (b) Give an example of a non-commutative ring with unity having eight elements which is anti-isomorphic to itself. Give a matrix representation of this ring in $M_3(\mathbb{Z}/2\mathbb{Z})$;

2. Let R be a ring with unity.

   (a) Show that J is an ideal of $M_n(R)$ if and only if $J = M_n(I)$ for some ideal I of R. Moreover, show that $M_n(R)/M_n(I) \cong M_n(R/I)$;

   (b) Show that $C(M_n(R)) = C(R)$ if we identify $a \in R$ with a times the identity matrix I_n;

   (c) If K is a field, deduce that $M_n(K)$ is a simple ring with center K.

3. (a) Show that $(-1, -1)_{\mathbb{C}} \cong M_2(\mathbb{C})$;

   (b) Show that $(-1, -1)_{\mathbb{Q}}$ is not isomorphic to $(-1, -7)_{\mathbb{Q}}$; (Hint: Show that $x^2 = -7$ has no solution in $(-1, -1)_{\mathbb{Q}}$.)

   (c) If K is a field of characteristic $\neq 2$ and $a, b \in \mathbb{Q}^*$, show that $(a, b)_K$ is a simple ring with center K. (We identify $a \in K$ with ae_1.)

4. Let R be a ring with unity and let u be an element of R which has a right inverse.

   (a) Prove that the following conditions on u are equivalent: (1) u has more than one right inverse, (2) u is not a unit, (3) u is a left zero divisor;

   (b) Prove that, if u has more than one right inverse, it has infinitely many.

5. (a) Let $(M, \cdot)$ be a monoid and let A, B be commutative rings. If $f : A \to B$ is a homomorphism of rings, show that $f_* : A[M] \to B[M]$, where $f_*(\sum a_m m) = \sum f(a_m)m$, is a homomorphism of rings. Show also that, if $I = \ker(f)$, then $\ker(f_*) = I[M]$. Deduce that $A[M]/I[M] \cong (A/I)[M]$ for any ideal I of A;

   (b) Let $(G, \cdot)$ be a group and K a commutative ring with unity. For each conjugacy class C of G let $\Sigma_C = \sum_{g \in C} g$ in the group algebra $K[G]$ of G over K. Show that $C(K[G]))$ is a free K-module with basis the elements Σ_C.