McGill University
Math 370A: Algebra III
Assignment 3: due Friday, November 6, 1998

1. For any group G, show that $|\mathrm{Aut(G)}| = 1$ implies $|G| \leq 2$.

2. (a) For any group G, show that $G/C(G) \cong \mathrm{Inn(G)} \trianglelefteq \mathrm{Aut(G)}$
   (b) Show that $S_3 \cong \mathrm{Inn}(S_3) = \mathrm{Aut(S_3)}$. Is this true for S_4?

3. Let m and n be positive integers with $\gcd(m,n) = 1$.
   (a) Let x, y be commuting elements of a group G with $o(x) = m, o(y) = n$. Show that $o(xy) = mn$. Deduce that $< x, y >=< xy >=< x >< y >\cong< x > \times < y >$.
   (b) Show that $C_m \times C_n \cong C_{mn}$. Deduce that $\phi(mn) = \phi(m)\phi(n)$, where ϕ is Euler's function $\phi(n) = |\{k \mid 1 \leq k \leq n, \gcd(k,n) = 1\}|$.
   (c) Show that the mapping $\phi : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, where $\phi(i+mn\mathbb{Z}) = (i+m\mathbb{Z}, i+n\mathbb{Z})$, is an isomorphism of rings.

4. Let n be a positive integer. An integer ζ which is relatively prime to n is called a primitive root modulo n if the residue class $\zeta + n\mathbb{Z}$ has multiplicative order $\phi(n)$, in which case $(Z/n\mathbb{Z})^\times$ is cyclic with generator $\zeta + n\mathbb{Z}$.
   (a) Let ζ be a primitive root modulo p, where p is an odd prime. If $\zeta^{p-1} \not\equiv 1 \pmod{p^2}$ show that ζ is a primitive root modulo p^n for all $n \geq 1$. If $\zeta^{p-1} \equiv 1 \pmod{p^2}$, show that $\zeta + p$ is a primitive root modulo p^n for all $n \geq 1$. Find primitive roots modulo $25, 49, 121$.
   (b) For $k \geq 2$, show that $(\mathbb{Z}/2^k\mathbb{Z})^\times =< -1 >< 5 >\cong C_2 \times C_{2^{k-2}}$.

5. Let G be a group and let G_L be the image in S_G under the left regular representation of G. The group $\mathrm{Hol}(G) = N_{S_G}(G_L)$ is called the holomorph of G.
   (a) Show that $\mathrm{Hol}(G) = G_L\mathrm{Aut(G)} \cong \mathrm{G} \rtimes_\phi \mathrm{Aut(G)}$, where $\phi : \mathrm{Aut(G)} \to \mathrm{Aut(G)}$ is the identity mapping.
   (b) Show that $\mathrm{Hol}(\mathbb{Z}/n\mathbb{Z}) = N_{S_n}(< (123 \cdot \cdot n) >)$ if we identify each $1 \leq i \leq n$ with its image in $\mathbb{Z}/n\mathbb{Z}$. Deduce that $\mathrm{Hol}(\mathbb{Z}/n\mathbb{Z}) = D_n \iff n = 2, 3, 4, 6$.
   (c) Show that $\mathrm{Hol}(C_2 \times C_2) \cong S_4$.

6. (a) If $f : A \to H$, $g : B \to K$ are isomorphisms of groups, show that the mapping $(f,g) : A \rtimes_\rho B \to H \rtimes_\phi K$ defined by $(f,g)(a,b) = (f(a), g(b))$ is an isomorphism of groups iff for all $b \in B$ we have $\phi_{g(b)} = f\rho_b f^{-1}$.
   (b) Show that $H \rtimes_\rho K \cong H \rtimes_\phi K$ if there is $f \in \mathrm{Aut(H)}$ such that $\phi_k = f\rho_k f^{-1}$ for all $k \in K$ or if there is $g \in \mathrm{Aut(K)}$ such that $\rho_k = \phi_{g(k)}$ for all $k \in K$.
   (c) Use the above to show that, up to isomorphism, there is only one non-trivial semi-direct product of the form $(C_2 \times C_2) \rtimes_\phi C_3$.
   (d) If p, q are primes with $q|p-1$, show that, up to isomorphism, there is only one non-trivial semi-direct product of the form $C_p \rtimes_\phi C_q$.

7. Do any five problems from the problem exercises for the first three chapters of the text. Choose problems that cover different points of the theory of groups that are not touched on in the assignments.

NB: Midterm Test, Friday November 6, in class.