McGill University
Math 370A: Algebra III
Assignment 2: due Friday, October 9, 1998

1. (a) Show that the $C(S_n) = 1$ for $n \geq 3$;
   (b) Show that $N_{S_n}(C_n) = D_n$ for $n = 2, 3, 4$. Is this true for all n?
2. (a) Show that $\mathrm{Aut}(C_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times$; (See Text page 60, # 26.)
   (b) If G=Sym(Cube), show that $G \cong C_2 \times S_4$.
3. Let H, K be groups and let ϕ be a homomorphism of K into $\mathrm{Aut}(H)$.
   (a) Show that the binary operation on $G = H \times K$ defined by $(h, k)(h', k') = (h\phi_k(h'), kk')$, where $\phi_k = \phi(k)$, is a group structure on G. This group is called a semi-direct product of H and K and is denoted by $H \rtimes_\phi K$;
   (b) If $h \in H$, $k \in K$, let $\overline{h} = (h, 1)$, $\overline{k} = (1, k)$ and let $\overline{H} = \{\overline{h} | h \in H\}$, $\overline{K} = \{\overline{k} | k \in K\}$. Show that $\overline{H}$, $\overline{K}$ are subgroups of G that are respectively isomorphic to H, K under the mappings $h \mapsto (h, 1)$, $k \mapsto (1, k)$. Show also that if $\overline{\phi} : \overline{K} \to \mathrm{Aut}(\overline{\mathrm{H}})$ is the homomorphism obtained from ϕ by transport of structure, we have $\overline{\phi}_{\overline{k}}(\overline{h}) = \overline{\phi_k(h)} = \overline{k} \cdot \overline{h} \cdot \overline{k}^{-1}$;
   (c) Show that $G = \overline{H} \cdot \overline{K}$, $\overline{H} \cap \overline{K} = 1$, $\overline{H} \trianglelefteq G$ and that $G/\overline{H} \cong \overline{K}$.
   (d) If G is any group having subgroups H, K such that $G = HK$, $H \cap K = 1$ and $H \trianglelefteq G$, show that $G \cong H \rtimes_\phi K$, where $\phi_k(h) = khk^{-1}$.
4. (a) Show that any semi-direct product of $C_2 \times C_2$ and C_2 which is not a direct product is isomorphic to D_4.
   (b) Show that the four semi-direct products of C_8 and C_2 are non-isomorphic groups of order 16. (Hint: Find the number of elements of order $2, 4, 8$.)
5. (a) Let Q_8 be the subgroup of S_8 generated by $\sigma = (1234)(5678)$ and $\tau = (1537)(2846)$. Show that Q_8 is a non-commutative group of order 8 that is not isomorphic to D_4. This group is called the quaternion group;
   (b) Show that $Q_8 \cong C_4 \rtimes_{-1} C_4 / < (a^2, a^2) >$ if $C_4 =< a >$.