1. (a) If $\sigma, \tau \in S_n$ and $\sigma = (a_1 a_2 \cdots a_k)$ is a k-cyle, show that $\tau\sigma\tau^{-1}$ is also a k-cycle.

   (b) Prove that the dihedral group D_n is the normalizer in S_n of the subgroup generated by the n-cycle $(12 \cdots n)$ iff $n = 2, 3, 4, 6$.

2. (a) Let p be a prime and let G be a finite p-group which acts on a finite set X. If

$$X^G = \{x \in X \mid gx = x \text{ for all } g \in G\},$$

   show that $|X^G| \equiv |X| \pmod{p}$.

   (b) Let P be a Sylow p-subgroup of a finite group G. If H is subgroup of G of order p^k which is contained in the normalizer of P in G, show that H is contained in P.

3. (a) Show that $\mathrm{Aut}(C_2 \times C_2) \cong S_3$.

   (b) Find, up to isomorphism, all groups of order 21. Give a presentation for each of these groups.

4. (a) State and prove the three Isomorphism Theorems for modules over a ring R.

   (b) If the R-module M is the direct sum of submodules M_1, M_2 and if $N = N_1 + N_2$, where N_1, N_2 are submodules of M_1, M_2 respectively, show that $M/N \cong M_1/N_1 \oplus M_2/N_2$.

5. Let K be a commutative ring.

   (a) State and prove Cayley's Theorem for associative K-algebras with unity.

   (b) If a, b are elements of K, show that there is a unique associative K-algebra structure on K^4 such that, if e_1, e_2, e_3, e_4 is the standard basis of K^4, we have

$$e_1 e_i = e_i e_1 = e_i \text{ for } 1 \leq i \leq 4, \quad e_2 e_3 = -e_3 e_2 = e_4, \quad e_2^2 = a e_1,\ e_3^2 = b e_1.$$

6. (a) If I_1 and I_2 are ideals of a ring R such that $I_1 + I_2 = R$, show that $R/I_1 \cap I_2 \cong R/I_1 \times R/I_2$.

   (b) Show how (a) can be used to show that Euler's ϕ-function is multiplicative, i.e., $\phi(mn) = \phi(m)\phi(n)$ if m, n are relatively prime positive integers.