The Dihedral Group

The group $D_n = \mathrm{Sym}(\{1,2,\ldots,\mathrm{n}\},\mathrm{s})$, where $n \geq 3$ and

$$s = \{\{1,2\},\ldots,\{i,i+1\},\ldots,\{1,n\},\{n,1\}\},$$

is called the dihedral group of degree n. This group contains the subgroup $C_n =< \sigma >$, where $\sigma = (12\cdots n)$. Since C_n is its own centralizer in S_n, any element τ of D_n which is not in C_n does not commute with σ. But then, if $\tau(1) = i$, we must have $\tau(2) = \sigma^{-1}(i)$; otherwise, $\tau(2) = \sigma(i)$ which implies inductively that $\tau(j) = \sigma^{j-1}(i)$ and hence that $\tau(j) = \sigma^{i-1}(j)$, i.e., that $\tau = \sigma^{i-1}$. It follows inductively that $\tau(j) = \sigma^{-j+1}(i) = \sigma^{i-j}(1)$. Thus $\tau(j) = i - j$ for $1 \leq j \leq i$ and $\tau(j) = n - j + i + 1$ for $i + 1 \leq j \leq n$ which imply that $\tau^2 = 1$. Moreover, any function τ defined in this way is in D_n. It follows that $|D_n| = 2n$ and hence that C_n is a normal subgroup of D_n. Thus D_n is generated by σ and any element τ not in C_n. Moreover, $\tau\sigma\tau^{-1} = \sigma^{-1}$. Hence $D_n \cong C_n \rtimes_\rho C_2$, where

$$\rho : C_2 \to \mathrm{Aut}(C_n)$$

is the homomorphism defined by $\rho(\tau)(\sigma) = \sigma^{-1}$. It follows that D_n has the presentation

$$D_n =< \sigma, \tau \mid \sigma^n = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} >$$

since any group having these generators and relations is of order at most $2n$. Indeed, the elements in such a group are of the form $\sigma^i\tau^j$ with $0 \leq i < n, 0 \leq j < 2$. The group D_n is also isomorphic to the group of symmetries of a regular n-gon.

If $a, b \in D_n$ with $o(a) = n, o(b) = 2$ and $b \notin < a >$, we have

$$D_n =< a, b \mid a^n = 1, b^2 = 1, bab^{-1} = a^{-1} >$$

and so there is an automorphism ψ such that $\psi(\sigma) = a, \psi(\tau) = b$ and any automorphism of D_n is of this form. Thus $|\mathrm{Aut(D_n)}| = \mathrm{n}\phi(\mathrm{n})$, where $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

Problem 1. *Show that* $\mathrm{Aut(D_n)} = \mathrm{C_n} \rtimes_\rho \mathrm{Aut(C_n)}$*, where* $\rho : \mathrm{Aut(C_n)} \to \mathrm{Aut(C_n)}$ *is the identity map.*

Hints. Let $A = \{\tau, \sigma\tau, \sigma^2\tau, \ldots, \sigma^{n-1}\tau\}$ be the elements of order 2 not in $< \sigma >$. Then, since A generates D_n, we obtain an injective homomorphism of $\mathrm{Aut(D_n)}$ into S_A by restricting $\psi \in \mathrm{Aut(D_n}$ to A. Show that the automorphism of D_n sending σ to σ and τ to $\sigma\tau$ yields an n-cycle λ and that, for any automorphism ψ of $\mathrm{Aut(C_n)}$, the automorphism of D_n sending σ to $\psi(\sigma) = \sigma^k$ and τ to τ yields a permutation γ such that $\gamma\lambda\gamma^{-1} = \lambda^k$. $\square$

Problem 2. *Show that, for n odd, the elements of D_n not in C_n are all conjugate and that, for n even, the complement of C_n in D_n is the union of two conjugacy classes.*