McGill University
Math 370A: Algebra III
Assignment 3: due Wednesday, October 15, 1997

1. Read Text: section 8.1. Determine the lattice of subgroups of $D_4$ and $Q_8$.
2. Text: p. 53, #2.
3. Text: p. 57, #3. Show also that this group is isomorphic to the group in problem 5 on page 36.
4. Text: p. 58, #6.
5. Text: p. 58, #11.
6. Text: p. 63, #2.
7. Text: p. 63, #5.
8. Text: p. 63, #6.
9. Text: p. 63, #8.
10. Let $(H, \cdot, 1)$, $(K, \cdot, 1)$ be groups and let $\phi$ be a homomorphism of $K$ into $\mathrm{Aut}(H)$, the automorphism group of $H$.
    (a) Show that the binary operation on $G = H \times K$ defined by
    $$(h, k)(h', k') = (h\phi_k(h'), kk'),$$
    where $\phi_k = \phi(k) \in \mathrm{Aut}(H)$, is a group structure on $G$. This group is called a semi-direct product of $H$ and $K$ and is denoted by $H \rtimes_\phi K$;
    (b) If $h \in H$, $k \in K$, let $\overline{h} = (h, 1)$, $\overline{k} = (1, k)$ and let $\overline{H} = \{\overline{h} | h \in H\}$, $\overline{K} = \{\overline{k} | k \in K\}$. Show that $\overline{H}, \overline{K} \leq G$, $\overline{H} \cap \overline{K} = 1$ and $G = \overline{H} \cdot \overline{K}$;
    (c) If $h \in H$, $k \in K$, show that $\overline{k} \cdot \overline{h} \cdot \overline{k}^{-1} = \overline{\phi_k(h)}$. Deduce that $\overline{H}$ is a normal subgroup of $G$ and that $G/\overline{H} \cong \overline{K}$.
    (d) If $G$ is any group having subgroups $H$, $K$ such that $G = HK$, $H \cap K = 1$ and $H$ is normal in $G$, show that $G$ is a semi-direct product of $H$ and $K$.