McGill University
Math 346B/377B: Number Theory

Assignment 1: due Wednesday February 7, 2001

346B: Do 6 questions.
377B: Do 8 questions.

Part A

1. Prove that $n^{13} - n$ is divisible by $2, 3, 5, 7$ and 13 for any integer n.
2. Prove that $n^5/5 + n^3/3 + 7n/15$ is an integer for any integer n.
3. What are the last two digits in the decimal representation of 3^{400}?
4. Solve the congruence $x^3 + 2x - 3 \equiv 0 \pmod{45}$.
5. Find all solutions of the system

$$2x \equiv 18 \pmod{24}, \quad 5x \equiv 11 \pmod{14}, \quad 4x \equiv 6 \pmod{21}.$$

Part B

1. If a, b are positive integers such that $a^n | b^{n+1}$ for all natural numbers n, show that $a|b$.
2. If p_n denotes the n-th prime, prove that $p_n^2 < p_1 p_2 \cdots p_{n-1}$ for $n > 4$. **Hint:** Use the fact that $p_n < 4^n$.
3. Show that 30 is the largest integer $N > 1$ such that $1 < a < N$, $(a, N) = 1$ implies that a is prime.
4. Let p be a prime.
   (a) Show that $p \mid \binom{p^n}{k}$ for $1 \leq k \leq p^n - 1$ but that p^2 does not divide $\binom{p^n}{p^{n-1}}$;
   (b) Show that p does not divide $\binom{p^n k}{p^n}$ if p does not divide k;
   (c) Find the gcd of $\binom{n}{1}, \cdots, \binom{n}{n-1}$.
5. Find all integers n such that $\phi(n)|n$.